A Simple "Public Debt-Deflation" Theory: Leeper revisited*

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Abstract

In this paper, we focus on interactions between monetary and fiscal policies in a non-Ricardian economy with capital and taking into account the presence of a positivity constraint on the nominal interest rate. We demonstrate in this framework the possible coexistence of four steady state equilibria, each with the properties of one of the equilibria described by Leeper (1991). But whereas in Leeper (1991), each equilibrium corresponds to a particular configuration of the parameters which describe fiscal and monetary policies as active or passive, we get these four equilibria for a single set of parameters. We show in particular that a liquidity trap—deflationary—equilibrium, which is also characterized by a high public debt-to-GDP ratio, a low capital stock and a low consumption level, owns the usually required properties for local determinacy, as well as the more traditional equilibrium targeted by the monetary and fiscal authorities. The model is calibrated based on European annual data and simulated in order to qualitatively asses the implications of a self-fulfilling expectation shock.

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1 Introduction

Macroeconomic policy discussions recognize the intimate connection between monetary and fiscal policy. A representative example of such a connection is the Maastricht Treaty and the Stability and Growth Pact. On the other hand, the study of the interaction between monetary and fiscal policies has been the object of vigorous interest since the seminal works of Sargent and Wallace (1981), Aiyagari and Gertler (1985), and more recently, Leeper (1991), Sims (1994), Woodford (1994, 2003) and Cochrane (2005) around the particularly provocative subject of the "Fiscal Theory of the Price Level" (FTPL).

The main contribution of this literature is an explicit specification of the conditions under which the monetary and fiscal policies interact, contrasting, thus, the traditional configuration—the quantity theory of money—where no interaction of the fiscal policy with monetary policy is allowed. In fact, the fiscal policy is neutral if the following conditions are fulfilled : there is *i)* no fiscal distortions, *ii)* no wealth effects or financial constraint, and *iii)* the fiscal policy is Ricardian in the sense of Woodford (1995), *i.e.* the fiscal authority ensures the government solvency by respecting its intertemporal budget constraint for any sequence of the price level and other endogenous variables.

Accordingly, if the last condition is unfulfilled, then the fiscal policy is non-Ricardian which means that the intertemporal budget constraint of the government needs an adjustment of the price level to be balanced. In the sense of Leeper (1991), fiscal policy is said to be "active" and then monetary policy must be "passive"[1]. This interaction between monetary and fiscal policy corresponds to the FTPL.

When the condition *ii)* is not satisfied then the economy is non-Ricardian. In this case, wealth effects can emerge and make the fiscal policy non neutral. As spelled out by Leith and Wren-Lewis (2000), even when fiscal policy is passive—does not constrain the active monetary policy—it can still influence prices.

With a different objective from that of Leith and Wren-Lewis, Cushing (1999) and Bénassy (2000) study the consequences of pegging the nominal interest rate in a non-Ricardian economy. But, while Cushing (1999) tries to show that the price level is always indeterminate in the presence of these effects, Bénassy (2000) shows, more clearly according to us, that the nominal indeterminacy[2] described by Sargent and Wallace (1975) disappears around the steady state which is locally determinate. There is nevertheless, as a general rule, another stationary equilibrium locally indeterminate towards which converge the multiple trajectories emphasized by Cushing (1999). The link between these results and those of the FTPL is not immediate. The presence of wealth effects does not any more allow to consider a simple rule as satisfying or not the criteria of a fiscal Ricardian policy. The difference

[1] An active monetary policy arises when the Taylor principle is fulfilled and passive monetary policy arises at the opposite case, when the response of the nominal interest rate is less than one-for-one to inflation. Similarly, passive fiscal policy occurs when the local convergence of the government debt is guaranteed and active fiscal policy happens when taxes do not respond sufficiently to debt to cover real interest payments and public spending.

[2] Benassy explains clearly the difference between multiplicity of equilibrium and nominal indeterminacy.

between the conclusions of Cushing (1999) and Bénassy (2000) actually results from the little operational character of this concept in a non-Ricardian Economy[3].

The model developed in this paper proposes a generalization of Cushing (1999) and Bénassy (2000) to a more complex economy, including: an interest rate rule *à la* Leeper (1991)-Taylor (1993), a zero lower bound on the nominal interest rate, and the presence of capital in the production process. On the other hand, we can see our contribution as an extension of Leeper (1991) to take into account the presence of wealth effects in a non-Ricardian economy with capital and with a zero lower bound on the interest rate.

Before summarizing our results let us recall the main findings of Leeper (1991). The interaction between simple monetary and fiscal policies yields four configurations depending on the policy parameters set by monetary and fiscal authorities. A determinate equilibrium then requires one active and one passive policy.

The results we obtain are the following: we can see coexisting the four types of equilibria described by Leeper (1991), but for one set of policy parameters. This means that in our framework the determinacy region is no longer specified by the policy parameter space. We notably show that a liquidity trap equilibrium, also characterized by a high public debt-to-GDP ratio, a low capital stock and a low consumption level, possesses the usually required properties of determinacy, like a more traditional equilibrium targeted by the monetary and fiscal authorities.

Our result emerges from the double non-linearity associated to the presence of wealth effects and the zero lower bound on the nominal interest rate. In short, when the nominal interest rate rule is bounded below by zero, the presence of wealth effects emphasizes the global indeterminacy problem identified by Benhabib, Schmitt-Grohé and Uribe (2001a, 2001b).

In addition, in our model, a detailed analytical analysis is made and allows to give the dynamic characteristics of each equilibrium. We find that two equilibria are locally determinate, one equilibrium is overdeterminate and one equilibrium is locally indeterminate. To resume, the four equilibria have, locally, the same dynamic characteristics as the equilibria described by Leeper (1991). Consequently, two convergent paths coexist. This finding becomes more interesting because of the capital accumulation. Indeed, the presence of capital stock allows the wealth effects of public debt to cause significant supply and demand effects. Accordingly, the debt-liquidity trap, or "public debt-deflation" equilibrium, corresponding to a lower level of capital stock, is associated to a recessionary trajectory.

Furthermore, from the perspective of global analysis, the existence of two locally convergent paths arises the question of self-fulfilling prophecies. An expectation shock can lead the economy from a virtuous trajectory to the debt-liquidity-trap trajectory. Therefore, our model could provide an alternative or complementary explanation for some episodes of deflation—like the Japanese recession of the 90s[4]—

[3] The Cushing's result is linked to a supplementary condition that he imposes. Despite the use of a simple fiscal rule, the author does not allow the fiscal policy to be non-Ricardian in the sense of Woodford (1995).

[4] It is worth noting that, during the Japanese liquidity-trap episode, the government debt has increased from 60% of GDP, in 1993, to 160%, in 2003.

based on agents' expectations change. In order to evaluate the possibility and the implications of a self-fulfilling expectation shock, the model is calibrated on annual data and permits us to simulate the effects of such a shock. We find that the liquidity trap and the increase in the public debt level are both the consequences of an initial deflation caused by a change in expectations.

Related literature

A recent literature has focused on the interaction between monetary an fiscal policy issue in a New Keynesian framework in the presence of wealth effects and with capital accumulation. Annicchiarico (2007), Annicchiarico *et al.* (2006), among others, study the effect of shocks when fiscal policy is non neutral because of wealth effects. They do not focus, however, on the matter of multiple equilibria. This issue was already analyzed within the framework of an exchange economy by Annicchiarico *et al.* (2009) in a continuous-time model. They point out the existence of four equilibria when wealth effects and zero lower bound on the nominal interest rate are taken into account[5].

Closest to our work in ideas and motivation is the paper of Leith and von Thadden (2008). The main finding of their work is that the local determinacy region is not solely specified by the policy parameter space but also by the steady state government debt level.

Their framework differs from ours in two points. First, it ignores the lower bound on nominal interest rate and considers therefore only one source of non-linearity associated to the presence of wealth effects. Although, despite the presence of this non-linearity, they don't have multiple equilibria. The reason is that—and this is the second difference with our paper—they use a fiscal rule in which the government debt target corresponds to its endogenous steady state level. In a model à la Blanchard-Yaari[6], this particular fiscal rule leads a second equilibrium—that is likely to appear due to the non-linearity—to correspond to the golden rule, and to be associated to a negative government debt level. This second equilibrium does not present any interest to their analysis. On the other hand, the extended version of the Weil's (1987, 1991) model we use can exhibit two positive values of stationary government debt, as in more traditional OLG models.

The paper proceeds as follows. In section 2, we build the model of a non Ricardian economy with money and capital and we introduce simple monetary and fiscal rules. In Section 3, we turn our attention to steady state equilibria. Section 4 is devoted to the study of the local properties of these equilibria and proposes a discussion about the global dynamics. In section 5, the model is calibrated based on Euro area annual data and is simulated in order to qualitatively asses the implications of a self-fulfilling expectation shock. Section 6 concludes.

[5] Guillard (2004) found the same results in a discrete-time model: "Politique monétaire et fiscale dans un monde non-Ricardien: une théorie fiscale de l'inflation", mimeo *Université d'Evry val d'Essonne.*

[6] The basic version of the overlapping generation model *à la* Blanchard-Yaari is always characterized by under-accumulation.

2 The model

We use an expanded version of Weil's (1987, 1991) overlapping-generations structure. The economy consists of many infinitely-lived families of agents. Each period new and identical infinetely-lived families appear in the economy without initial wealth. The economy also consists of identical infinetely-lived firms using capital and labor to produce a unique good, of the fiscal authority (the government) and of the monetary authority (the central bank). We use a stochastic framework and we assume that markets are complete.

2.1 Households

In period t, the economy is populated by a large number N_t of agents. Each period a new dynasty appears consisting of $(N_t - N_{t-1}) = nN_{t-1}$ agents where $n \geq 0$ represents at the same time the population growth rate and the birth rate.

Each household belonging to the dynasty $j \leq t$ has preferences defined over consumption and real money balances described by the utility function:

$$E_t \sum_{s=t}^{\infty} \beta^{s-t} U\left(c_{j,s}, \frac{M_{j,s}}{P_s}\right) \tag{1}$$

where E_t denotes the mathematical expectations operator conditional on information available at time t, $\beta \in [0,1]$ represents a subjective discount factor, and $U(\cdot,\cdot)$ is a period utility index assumed to be strictly increasing in its two arguments and strictly concave. The variables, $c_{j,t}$, P_t and $M_{j,t}$, represent respectively, the consumption of the household j in period $t \geq j$, the price of consumption good, and the nominal money balances held by household j in period $t \geq j$.

At the beginning of the period t, the household $j < t$ holds the initial nominal wealth, $V_{j,t}$, defined by:

$$V_{j,t} = M_{j,t-1} + (1 - \delta + \kappa_t) P_t k_{j,t} + D_{j,t} \tag{2}$$

where $(1 - \delta + \kappa_t) P_t k_{j,t}$ is the nominal value of the capital stock, including the capital incomes net of depreciation, and $D_{j,t}$ is the beginning-of-period state-contingent value of all other financial assets, whether privately issued or claims on the government.

In each period, agents supply an inelastic and constant amount of labour and receive a real wage, $w_{j,t}$. Each agent uses his total financial wealth augmented by the wage incomes net of taxes, $P_t \tau_{j,t}$, to consume and to reconstitute his financial holdings. We can write the household's flow budget constraint as follows:

$$P_t c_{j,t} + M_{j,t} + E_t Q_{t,t+1} D_{j,t+1} + P_t k_{j,t+1} \leq V_{j,t} + P_t (w_{j,t} - \tau_{j,t}) \tag{3}$$

where $Q_{t,t+1}$, is the stochastic discount factor[7].

[7] To be more precise, $Q_{t,t+1}$ is the asset price in period t, that gives one unit of money in a given state of the world in period $t+1$, weighted by the probability (or density function) of such state.

Markets are supposed to be complete. This assumption implies the existence of the risk-free one-period nominal interest rate defined by:

$$1 + i_t = [E_t Q_{t,t+1}]^{-1} \tag{4}$$

Finally, the household is subject to an appropriate set of borrowing limits that rule out "Ponzi Games". Let us define:

$$h_{j,t} = \frac{1}{P_t} E_t \sum_{s=t}^{\infty} Q_{t,s} \left[P_s \left(w_{j,s} - \tau_{j,s} \right) \right] \tag{5}$$

the household j's human capital wich corresponds to the discounted value of future labor incomes net of taxes. In the absence of financial market frictions, the borrowing constraint takes the form:

$$V_{j,t+1} \geq -P_{t+1} h_{j,t+1} \qquad \forall j, \forall t. \tag{6}$$

This constraint implies that the household has to be able to reimburse his debt contracted in period t in each state of the world that may be realized at date $t+1$.

The representative household of generation j maximizes his intertemporal utility (1) subject to the budget constraint (3) and the borrowing constraint (6), where $V_{j,t}$ is defined by equation (2).

Denoting $U_x(t) = \partial U(\cdot) / \partial x_t$, the first-order conditions for this maximizing problem can be written as follows:

$$\beta \frac{U_{c_j}(t+1)}{U_{c_j}(t)} = Q_{t,t+1} \frac{P_{t+1}}{P_t} \tag{7}$$

$$\frac{U_{m_j}(t)}{U_{c_j}(t)} = \frac{i_t}{1+i_t} \tag{8}$$

$$\left(E_t Q_{t,t+1} \frac{P_{t+1}}{P_t} \right)^{-1} = 1 + \kappa_{t+1} - \delta \equiv R_t \tag{9}$$

$$E_t Q_{t,t+1} V_{j,t+1} + P_t c_{j,t} + \frac{i_t}{1+i_t} M_{j,t} = V_{j,t} + P_t \left(w_{j,t} - \tau_{j,t} \right) \tag{10}$$

$$\lim_{T \to +\infty} E_t Q_{t,T} V_{j,T} = 0 \tag{11}$$

Equation (7) is a stochastic Euler equation summarizing the intertemporal arbitrage between present and future consumptions in each possible state of the world. Equation (8) represents an arbitrage condition between real money balances and present consumption. Equation (9) is a no-arbitrage condition relative to the saving choice in terms of capital accumulation or in terms of nominal state-contingent assets. Note that the net return on capital, $\kappa_{t+1} - \delta$, is not associated to an expectation operator because we assume a risk-free production. Thus, R_t represents

$E_t Q_{t,t+1} D_{j,t+1}$ can be rewritten as $\sum q_{t,t+1} D_{j,t+1}$ (where $q_{t,t+1}$ is an asset price) and represents the state-contingent assets portofolio. We have more generally: $Q_{t,T} = Q_{t,t+1} \times Q_{t+1,t+2} \times .... \times Q_{T-1,T}$ and $Q_{t,t} = 1$.

the real risk-free gross interest rate, and it is known in period t. Equation (10) is the household j's balanced budget constraint obtained by combining equations (3), (2) and (9). Finally, Equation (11) corresponds to the transversality condition and states that the discounted value of the financial wealth (or debt) tends to zero when time goes to infinity.

Iterating Equation (10) forward, with the use of (11), leads to the following household j's intertemporal budget constraint:

$$V_{j,t} = E_t \sum_{s=t}^{+\infty} Q_{t,s} \left[P_s c_{j,s} + \frac{i_s}{1+i_s} M_{j,s} - P_s \left(w_{j,s} - \tau_{j,s} \right) \right] \tag{12}$$

In order to obtain an explicit outcome for individual consumption, one specifies the utility function as follows:

$$U\left(c_{j,t}, \frac{M_{j,t}}{P_t} \right) = \xi \ln c_{j,t} + (1-\xi) \ln \frac{M_{j,t}}{P_t}$$

Defining

$$R_{t,t+1} = \left(Q_{t,t+1} \frac{P_{t+1}}{P_t} \right)^{-1} \tag{13}$$

as the stochastic gross real interest rate corresponding to real return of the state-contingent nominal asset[8], equations (7) and (8) can then be rewritten as:

$$c_{j,t} = \beta^{-1} \frac{c_{j,t+1}}{R_{t,t+1}} \tag{14}$$

and

$$c_{j,t} = \xi \left[c_{j,t} + \frac{i_t}{1+i_t} \frac{M_{j,t}}{P_t} \right]$$

Introducing these results into equation (12) and using (5), one can easily show that the optimal consumption of agent j is a constant fraction of his consolidated wealth (financial wealth + human wealth).

$$c_{j,t} = \xi (1-\beta) \left(v_{j,t} + h_{j,t} \right) \tag{15}$$

where $v_{j,t} = V_{j,t}/P_t$.

2.2 Aggregation

Noting that the generation j is composed of $N_j - N_{j-1}$ agents, the following aggregation rule is applied to get *per capita* aggregate variables:

$$x_t = \sum_{j \leq t} \frac{(N_j - N_{j-1})}{N_t} x_{j,t} \tag{16}$$

for $x_{j,t} = c_{j,t}$, $v_{j,t}$, and $h_{j,t}$.

[8] Note that according to (9), we have: $R_t = \left[E_t \left(1/R_{t,t+1} \right) \right]^{-1}$.

We assume that the agent's inelastic supply of labor corresponds to one unit of labor, whatever the age of the agent, and we assume that taxes are independent of the age. Therefore, $w_{j,t} = w_t$ and $\tau_{j,t} = \tau_t$, $\forall j$ and so that $h_{j,t} = h_t \quad \forall j$.

Finally, notice that applying the aggregate rule (16) in period t to the variable $v_{j,t+1}$, we get:

$$\begin{aligned}
\sum_{j \leq t} \frac{(N_j - N_{j-1})}{N_t} v_{j,t+1} &= \frac{N_{t+1}}{N_t} \sum_{j \leq t} \frac{(N_j - N_{j-1})}{N_{t+1}} v_{j,t+1} \\
&= (1+n) \left[\sum_{j \leq t+1} \frac{(N_j - N_{j-1})}{N_{t+1}} v_{j,t+1} - \frac{n}{1+n} v_{t+1,t+1} \right] \\
&= (1+n)\, v_{t+1}
\end{aligned}$$

since $v_{t+1,t+1} = 0$, the dynasty $j = t+1$ having no financial wealth in period $t+1$.

Using this result and applying the aggregate rule (16) to equation (14) where we replace $c_{j,t+1}$ by its expression given by equation (15) expressed in $t+1$, we obtain:

$$c_t = \xi (1-\beta) \beta^{-1} \frac{(1+n)\, v_{t+1} + h_{t+1}}{R_{t,t+1}}$$

Finally, by incorporating (15) expressed in $t+1$ in the previous equation, it can be rewritten:

$$c_t = \beta^{-1} \frac{c_{t+1}}{R_{t,t+1}} + \Psi \frac{v_{t+1}}{R_{t,t+1}} \tag{17}$$

where $\Psi = n\xi \left(\beta^{-1} - 1\right) \geq 0$ if $n \geq 0$.

This equation is the aggregate stochastic Euler equation which differs from the individual Euler condition (14) as long as the population growth rate is different from zero[9]. It includes a real wealth effect which is characteristic of a non Ricardian economy. In each state of nature, the growth rate of individual consumption is greater than the aggregate growth rate, reflecting the heterogeneity of individual wealth. An increase in the expected beginning-of-period financial wealth in $t+1$ benefits only to currently alive consumers in period t and thus it can't be proportionally distributed amongst present and future aggregate consumptions.

2.3 Firms

It is assumed that there exists a large number of competitive firms with access to a standard neoclassical technology: $Y_t = F(K_t, L_t)$, where Y_t, K_t and L_t denote the aggregate levels of production, physical capital and labour demand, respectively. The production function is homogeneous of degree one, concave, twice continuously differentiable and satisfies the Inada conditions. Firms are price takers in input and output markets. Let $k_t = K_t/L_t$ denote the *per capita* capital stock, the *per capita* output level $y_t = Y_t/L_t$ is given by: $y_t = F(k_t, 1) \equiv f(k_t)$.

[9] Recall that in Weil's model the population growth rate couldn't be negative since the absence of death.

Competitive profit-maximizing firms leads to the standard conditions that factor prices equal their respective marginal products:

$$\kappa_t = f_k(k_t) \tag{18}$$

$$w_t = f(k_t) - k_t f_k(k_t) \tag{19}$$

Given the constant return to scale, factor payments exhaust firm revenues.

2.4 Monetary and Fiscal Authorities

The government collects lump-sum taxes in the amount of $P_t T_t$, spends $P_t G_t$, prints money M_t and issues one-period nominally risk-free bonds B_t at the nominal price of $(1+i_t)^{-1}$. Denoting:

$$\Omega_t = M_{t-1} + B_{t-1}$$

the total beginning-of-period t government debt, including money balances, the government flow budget constraint can be written:

$$\frac{\Omega_{t+1}}{(1+i_t)} + \frac{i_t}{1+i_t} M_t + P_t T_t = \Omega_t + P_t G_t \tag{20}$$

2.4.1 Fiscal Rule

We assume that in order to determine the amount of the lump-sum taxes, the fiscal authority applies the following simple rule:

$$T_t = z_t Y_t + \theta \frac{\Omega_t}{P_t} - \frac{i_t}{1+i_t} \frac{M_t}{P_t} \tag{21}$$

The first term on the right-hand side of equation (21), $z_t Y_t$, represents the part of taxes proportional to the output. z_t is a choice variable of fiscal authority, but it can be perceived by private agents as stochastic. The second component reflects the fact that the government debt is partially backed by direct taxes. It generalizes the rule proposed by Leeper (1991) to the total government debt, $\Omega_t = M_{t-1} + B_{t-1}$, instead of B_{t-1} alone. The parameter θ verifies : $0 \leq \theta \leq 1$. Finally, the government transfers all its seigniorage revenues, $\frac{i_t}{1+i_t} M_t / P_t$, to agents. The last two assumptions will considerably simplify the model by neutralizing the effects of seigniorage on the total government debt dynamic.

The government expenditures are assumed to be proportional to the output:

$$G_t = g_t Y_t \tag{22}$$

where g_t is determined by fiscal authority but it can also be perceived as stochastic by private agents.

Inserting (21) and (22) into the budget constraint (20) and using the definition of the nominal gross interest rate (4) and the definition of the stochastic real gross interest rate (13), we obtain the following equation:

$$E_t\left(\frac{\omega_{t+1}}{R_{t,t+1}}\right)=\frac{1}{1+n}\left[(1-\theta)\,\omega_t+(g_t-z_t)\,y_t\right] \tag{23}$$

which describes the dynamic of $\omega_t=\Omega_t/P_tN_t$, the total *per capita* government debt in real terms.

To simplify the analysis, we will assume that in long run the fiscal authority imposes the condition: $g=z$, in order to guarantee that the primary deficit can equal zero when the debt is entirely paid back.

2.4.2 Monetary Rule

Taking up the assumption introduced by Leeper (1991) and then generalized and popularized by Taylor (1993) we assume that monetary authority has, in the short-run, leverage over the nominal interest rate that responds to the deviation (or the ratio) of inflation from its long-run target, $\bar{\Pi}$.

In order to take into account a lower bound constraint on the nominal interest rate[10], we specify the following class of non linear monetary rules:

$$1+i_t=\Phi\left(\bar{R}_t,\Pi_t;\bar{\Pi}\right) \tag{24}$$

where $\bar{R}_t$ is a gross real interest rate target and the function $\Phi(\cdot)$ is assumed to be continuous and to have the following properties:

$$\begin{aligned}
\Phi\left(R,\bar{\Pi};\bar{\Pi}\right) &= R\bar{\Pi} \quad \forall R,\ \forall\bar{\Pi} \text{ s.t. } R\bar{\Pi}\geq 1+\underline{i}\longrightarrow 1^+,\\
\Phi_\Pi\left(R,\bar{\Pi};\bar{\Pi}\right) &> R \qquad \forall R,\ \forall\bar{\Pi} \text{ s.t. } R\bar{\Pi}\geq 1+\underline{i}\longrightarrow 1^+,\\
\Phi(\cdot) &\geq 1+\underline{i}\longrightarrow 1^+,\\
\Phi_\Pi(\cdot) &\geq 0, \quad \Phi_R(\cdot)\geq 0, \quad \Phi_{\Pi^2}(\cdot)\geq 0.
\end{aligned} \tag{H1}$$

The first condition helps to guarantee that the inflation target $\bar{\Pi}$ can be reached at stationary state when the real interest rate target, $\bar{R}$, equal the long run value of the real interest rate, R, as long as the resulting nominal interest rate is strictly positive[11]. The second condition is the Leeper's condition (Leeper, 1991) for an active monetary policy when the inflation target is reached. Note that, by combining this condition with the first one, we obtain: $\phi_{\bar{\pi}}>1$, where $\phi_{\bar{\pi}}$ is the elasticity of $\Phi(\cdot)$ with respect to Π_t when $\Pi_t=\bar{\Pi}$, that is, the more popular "Taylor Principle". The third condition generalizes the zero lower bound on the nominal interest rate constraint to all possible values of the gross real interest rate target and the gross inflation rate. The three last conditions help to preclude atypical rules.

The case where $\bar{R}_t$ represents a constant target, *i.e.* $\bar{R}_t=\bar{R}\ \forall t$, is often used in the literature, notably by Taylor (1993). Nevertheless we will analyze the case where $\bar{R}_t$ is equal to the current real gross interest rate, *i.e.* $\bar{R}_t=R_t$, that could be wise to stabilize inflation around its target when the stationary level of capital is not yet reached.

[10] This point was analysed particularly by Benhabib, Schmitt-Grohé and Uribe (2001a, 2001b).

[11] Taking into account the logarithmic form of the utility function, the zero bound on nominal interest rate can never be reached. A positive lower bound, $\underline{i}>0$, has to be defined (see Alstadheim and Henderson [2006]). The limit case $\underline{i}=0$ can be considered in a cashless economy, when $\xi=1$.

2.5 Market Clearing

In equilibrium, the surplus of state-contingent assets supplied by agents equals zero, thus their financial holdings are composed of government bonds, money and capital:

$$v_t = \frac{M_{t-1} + B_{t-1}}{P_t N_t} + R_{t-1} k_t = \omega_t + R_{t-1} k_t$$

It follows that the stochastic aggregate Euler equation (17) takes the form:

$$c_t = \beta^{-1} \frac{c_{t+1}}{R_{t,t+1}} + \Psi \frac{\omega_{t+1} + R_t k_{t+1}}{R_{t,t+1}}$$

Using (9) and (18), we define the function $\tilde{R}(k_{t+1})$ that determines the value of the gross real interest rate according to the capital accumulated in t :

$$R_t = 1 - \delta + f_k(k_{t+1}) \equiv \tilde{R}(k_{t+1}) \tag{25}$$

We can then describe an equilibrium by the following set of equations:

$$c_t = \beta^{-1} \frac{c_{t+1}}{R_{t,t+1}} + \Psi \frac{\omega_{t+1} + \tilde{R}(k_{t+1}) k_{t+1}}{R_{t,t+1}} \tag{26}$$

$$k_{t+1} = \frac{1}{1+n} \left[(1-\delta) k_t + (1-g_t) \cdot f(k_t) - c_t \right] \tag{27}$$

$$E_t \left(\frac{\omega_{t+1}}{R_{t,t+1}} \right) = \frac{1}{1+n} \left[(1-\theta) \omega_t + (g_t - z_t) f(k_t) \right] \tag{28}$$

$$E_t \left(\frac{1}{R_{t,t+1}} \right) = \frac{1}{\tilde{R}(k_{t+1})} \tag{29}$$

$$E_t \left(\frac{1}{R_{t,t+1} \Pi_{t+1}} \right) = \frac{1}{1+i_t} \tag{30}$$

$$1 + i_t = \Phi\left(\bar{R}_t, \Pi_t; \bar{\Pi}\right) \tag{31}$$

Equation (27) is the good market clearing condition. (28) is the real *per capita* government budget constraint (23). Equation (29) comes from (9), (13) and (25). Finally, equation (30) is the risk-free one-period nominal interest rate equation (4) where we have used (13).

If the period $t+1$ is characterized by S_{t+1} possible states of the world then the later system of equations is composed by $5 + S_{t+1}$ equations allowing to find the values of c_t, k_{t+1}, ω_t, Π_t, i_t and the S_{t+1} values of $R_{t,t+1}$, subject to equilibrium existence and uniqueness. Notice that it is possible, in theory at least, to eliminate the variables i_t and $R_{t,t+1}$—both non-predetermined and non-dynamic—in order to reduce the size of the system. So we can consider a representation[12] composed of four dynamic equations where two variables, c_t and Π_t, are non-predetermined and two variables, k_t and x_t, are predetermined, with $x_t = \Pi_t \omega_t = (M_{t-1} + B_{t-1}) / N_t P_{t-1}$.

[12] Appendix 4 gives details of such a representation.

This choice would theoretically permit to solve the problem posed by the dynamic status of $\omega_t = (M_{t-1} + B_{t-1}) / N_t P_t$ and $\Pi_t = P_t / P_{t-1}$, whose values can jump but not independently of each other. This representation is more satisfactorily from a conceptual point of view but is not sufficiently malleable on a technical level. Later on we will take a roundabout way to analyze the previous model.

3 Steady State Equilibria

A deterministic steady state equilibrium is a vector (c, k, ω, Π) verifying a four-equations system which is obtained from equations (26) to (31) where we delete the indications of time and uncertainty and we merge the deterministic version of (30) with (31). In addition, assuming $g = z$, we obtain:

$$\left[\tilde{R}(k) - \beta^{-1}\right] c = \Psi \left[\tilde{R}(k) k + \omega\right] \tag{32}$$

$$\left[\frac{1+n}{\tilde{R}(k)} - (1-\theta)\right] \omega = 0 \tag{33}$$

$$c = (1-g) f(k) - (n+\delta) k \tag{34}$$

and:

$$\tilde{R}(k) \Pi = \Phi\left(\bar{R}, \Pi\right) \tag{35}$$

The first three equations are independent of Π. The system is then dichotomous and allows to find (c, k, ω) independently of the monetary policy. For a given value of k, equation (35) allows to find the equilibrium value(s) of Π according to the target, $\bar{R}$ which can (or cannot) be chosen to be equal to the actual steady state value of $\tilde{R}(k)$.

Notice that this long run dichotomy is not a fundamental characteristic of such a model. It is the consequence of *i)* the simple monetary and fiscal rules that we use, and *ii)* the adoption of the variable ω, the beginning-of-period real debt, rather than $x = \Pi\omega$, the end-of-period real debt.

3.1 Equilibrium Inflation

We begin this subsection by analyzing the monetary part of the steady state. According to assumption (H1), and when the real interest rate target coincides with the long run real interest rate : $\bar{R} = \tilde{R}(k)$, equation (35) has at least one solution corresponding to the inflation target, $\bar{\Pi}$.

Benhabib, Schmitt-Grohé and Uribe (2001a, 2001b) show that the possibility of the existence of a second steady state equilibrium is one of the unexpected consequences of the zero lower bound on the nominal interest rate. It is notably the case when the rule is active in the sense of Leeper (1991) around the inflation target $\bar{\Pi}$, as we have supposed in (H1). In this case, a second equilibrium appears, corresponding to a lower inflation rate, potentially negative and reminding the Keynesian

liquidity trap. We illustrate this case by the following figure where we assume that $\bar{R} = \tilde{R}(k)$:

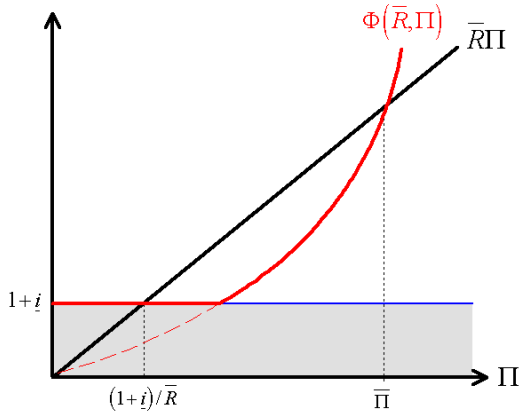

Figure 1

The figure 1 corresponds to the case where the function $\Phi(\cdot)$ crosses the horizontal axis defined by $1+\underline{i}$ for a value of Π greater than $(1+\underline{i})/\bar{R}$, which determines the lower equilibrium value in $\Pi^{**} = (1+\underline{i})/\bar{R}$. The associated nominal interest rate, i, is *at* its minimum value, $\underline{i}$, and then the liquidity trap is reached.

3.2 Debt, Capital, and Interest

We now analyze the real part of the deterministic steady state. Equation (33) admits two evident solutions, $\omega^* = 0$ and $\tilde{R}(k^{**}) = \frac{1+n}{(1-\theta)}$, corresponding to two stationary equilibrium vectors of the variables c, k, and ω.

3.2.1 "Autarkic Equilibria"

First, we study the solution corresponding to a zero public debt in the steady state. Equation (32) together with equation (34), allow to obtain the value of the per capita capital stock and the per capita consumption in an implicit form. We get:

$$\omega^* = 0 \tag{36}$$

$$R^* = \tilde{R}(k^*) = \frac{\beta^{-1}}{1-\Psi k^*/c^*} \tag{37}$$

$$c^* = (1-g)\, f(k^*) - (n+\delta)\, k^* \tag{38}$$

The second equation allows to make sure that the equilibrium gross interest rate, R^*, verifies:

$$R^* > \beta^{-1}$$

where β^{-1} is the gross interest rate in the Ricardian economy, obtained by assuming that $\Psi = n = 0$.

Because the parameter Ψ, given by $n\xi\left(\beta^{-1}-1\right)$, is small, the gap between R^* and β^{-1} is likely to be low. Besides, the value of the equilibrium-debt level equals

zero, as in the Ricardian case[13]. With reference to the standard OLG model, where this kind of equilibria corresponds to the absence of exchange among generations, we call these first equilibria: "autarkic equilibria".

3.2.2 "Debt Equilibria"

The second solution of equation (33) allows to compute the equilibrium value of the real public debt according to equation (32). One obtains:

$$\omega^{**} = \Psi^{-1}\left(\frac{1+n}{1-\theta} - \beta^{-1}\right)c^{**} - \frac{1+n}{1-\theta}k^{**} \tag{39}$$

where the values of k^{**} and c^{**} are respectively given by (40), implicitly, and by (41):

$$R^{**} = \tilde{R}\left(k^{**}\right) = \frac{1+n}{1-\theta} \tag{40}$$

$$c^{**} = (1-g)\, f\left(k^{**}\right) - (n+\delta)\, k^{**} \tag{41}$$

Comparing autarkic and debt equilibria we obtain the following proposition whose proof is provided in appendix 1:

Proposition 1 *The real value of the per capita public debt is positive in a debt equilibrium if and only if the associated real interest rate is greater than the autarkic real interest rate, i.e.*:

$$R^{**} = \frac{1+n}{1-\theta} \geq R^{*} \Longleftrightarrow \omega^{**} \geq 0.$$

The intuition of this proposition is straightforward. In presence of wealth effects, an increase in the real public debt level increases the net wealth of the agents and leads them to increase their level of consumption. Accordingly, their savings does not grow sufficiently to absorb the new issued debt, which led to an increase in the real interest rate. The reverse is also true and a higher interest rate is necessarily associated to a higher level of the real public debt.

[13] Since the equation (33) has to be verified when $\Psi = 0$, the stationary debt level is: $\omega^{R} = 0$ in the ricardian case.

3.2.3 A graphical representation

The two kinds of equilibria can easily be represented in a $(k; c)$ plan:

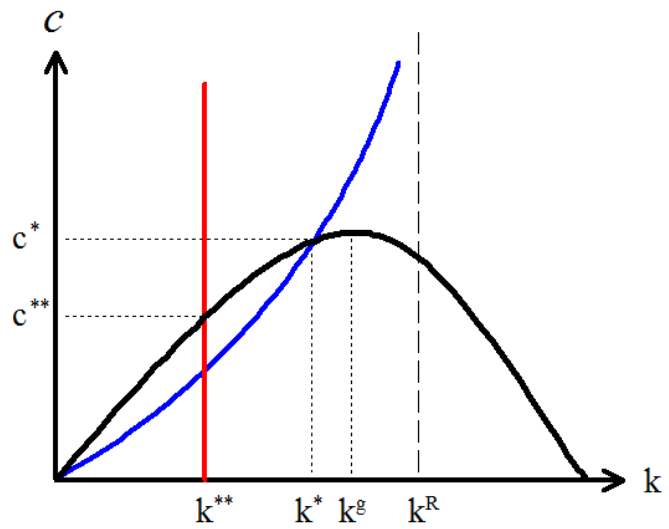

Figure 2

The hump-shaped curve in Figure 2 is the steady state resources constraint of the economy, given by equation (38) or (41). The top of this curve corresponds to a modified golden rule which is reached for a *per capita* capital stock k^g implicitly defined by: $(1-g) f_k(k^g) - (n+\delta) = 0$, or equivalently, by using (25) :

$$R^g \equiv \tilde{R}(k^g) = \frac{(1+n) - (1-\delta) g}{1-g} \tag{42}$$

The Ricardian *per capita* capital stock k^R is given by equation (37) when $\Psi = n = 0$, *i.e.* $\tilde{R}(k^R) = \beta^{-1}$. The upward sloping curve corresponds to equation (37) and intersects with (38) to give the locus (k^*, c^*). The vertical k^{**} is the steady state *per capita* capital stock in a debt equilibrium. It is implicitly defined by (40). Figure 2 represents the case where: $k^{**} < k^* < k^g$ (and $k^g < k^R$) or, equivalently, $R^g < R^* < R^{**}$ (and $\beta^{-1} < R^g$). In this case, a debt equilibrium is necessarily characterized by a positive level of public debt (Proposition 1) and a lower steady state consumption level c^{**} than the one obtained in the autarkic equilibrium, c^*. Afterward, we will assume a weaker assumption:

$$R^{**} = \frac{1+n}{1-\theta} \geq \max(R^*; R^g) \tag{H2}$$

which nevertheless guarantees the positivity of ω^{**}.

3.3 Multiple Equilibria

Recall that in Section 3.1, we noted that for a given level of real interest rate, there were two stationary state values for the rate of inflation, one of the two being associated to an active monetary policy and the other to a passive policy. Accordingly, each of the two gross interest rates R^* and R^{**} can be associated with two possible inflation rates, and our economy potentially admits four equilibria. These equilibria are represented on the figures $3a$ and $3b$, each representing a particular version of the monetary rule:

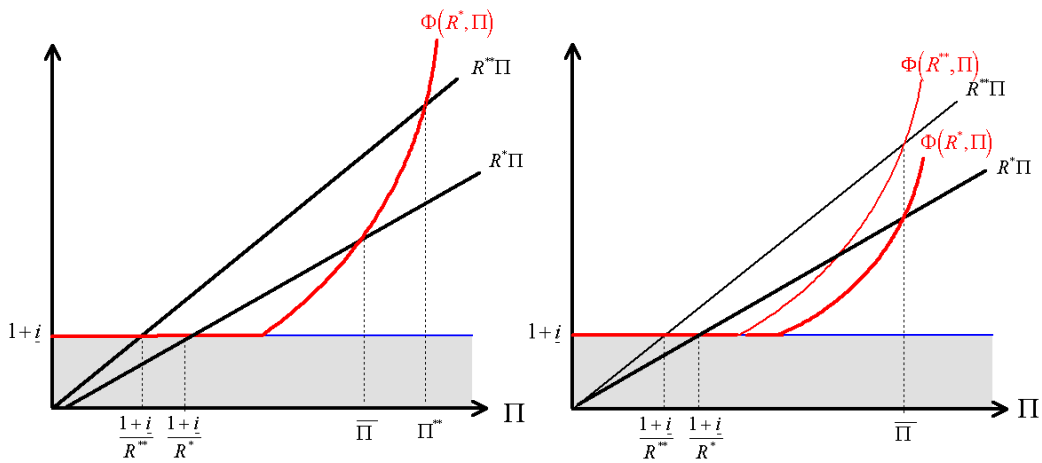

Figure 3a Figure 3b

In the first case, represented on figure 3a, the monetary rule depends on a constant real interest rate target[14], corresponding to the autarkic equilibrium: $\bar{R}_t = R^*$. If the actual real gross interest rate is R^{**} then the target $\bar{\Pi}$ is not reached and an inflationary bias appears.

In the second case, represented on figure 3b, the monetary rule depends on the current real interest rate: $\bar{R}_t = R_t$. As we can note, this rule presents the advantage of not making the long term inflation rate depending on the equilibrium value of the real interest rate, except in the liquidity trap. So, the inflation target $\bar{\Pi}$ can be reached for R^* and for R^{**}. On the other hand, the assumption adopted about the representation of a liquidity trap does not allow us to obtain the uniqueness of the lower inflation rate.

4 From Local to Global Dynamics

As we will verify in this section, the four steady state equilibria we obtained correspond to—and have, locally, the same dynamics properties as—those analyzed by Leeper (1991), but unlike the case of the Ricardian economy without liquidity trap considered by Leeper, these four equilibria coexist for a given set of the policy parameters.

[14] The rule used in this case is similar to the Leeper-Taylor's rule.

4.1 The linearized model

In order to study the dynamics of the economy, we start by analyzing the local stability around each stationary equilibrium. As we have already noted, the most relevant linearized model would be the one constituted of the predetermined variables k_t and $x_t = (M_{t-1} + B_{t-1}) / N_t P_{t-1}$ and the non predetermined variables c_t and Π_t. Then the Blanchard and Kahn (1981) conditions would theoretically allow to characterize the local dynamics of the four stationary equilibria. We shall adopt this procedure in the last section in order to simulate numerically the model, but the dimension of the system does not allow us to characterize analytically the equilibria. On the other hand, the system composed of the variables c_t, k_t, ω_t and Π_t offers some interesting possibilities that we are going to investigate.

Because one of the variables, ω_t, could equal zero in the long run, we linearize the equations (26) to (31) around any stationary equilibrium, by defining each variable in difference : $\hat{u}_t = u_t - u$, where u represents the variable u_t evaluated in one of the stationary equilibria. We obtain :

$$\hat{c}_t = \frac{\beta^{-1}}{R} E_t \hat{c}_{t+1} + \frac{\Psi}{R} E_t \hat{\omega}_{t+1} + \left(\Psi - \frac{\beta^{-1} c + \Psi \omega}{R^2} f_{kk} \right) \hat{k}_{t+1} \tag{43}$$

$$\hat{k}_{t+1} = \frac{1}{1+n} \left([R - g f_k] \hat{k}_t - \hat{c}_t - f \cdot \hat{g}_t \right) \tag{44}$$

$$E_t \hat{\omega}_{t+1} = \frac{R}{R^{**}} \hat{\omega}_t + \frac{\omega f_{kk}}{R^{**}} \hat{k}_{t+1} + \frac{R}{1+n} f \cdot (\hat{g}_t - \hat{z}_t) \tag{45}$$

$$E_t \hat{\pi}_{t+1} = \phi_\pi \hat{\pi}_t + (\phi_R - 1) \frac{\Pi}{R} f_{kk} \hat{k}_{t+1} \tag{46}$$

where $\phi_\pi = \Pi \Phi_\Pi / \Phi$ and $\phi_R = R \Phi_R / \Phi$ are the elasticity of the function $\Phi(\cdot)$ and where we have used $\frac{1+n}{1-\theta} = R^{**}$.

Denoting $\hat{Y}_t = \begin{bmatrix} \hat{k}_t & \hat{\omega}_t & \hat{c}_t & \hat{\pi}_t \end{bmatrix}'$, the vector of the endogenous variables, using the long run equations (32) to (35) and neglecting the shocks $\hat{g}_t$ and $\hat{z}_t$, the equations (43) to (46) could be combined in order to get the following state-space form :

$$E_t \hat{Y}_{t+1} = J_4 (k, \omega, c, \pi) \cdot \hat{Y}_t \tag{47}$$

where the Jacobian matrix $J_4 (k, \omega, c, \pi)$ is given by:

$$J_4(\cdot) = \begin{pmatrix} \frac{R - g f_k}{1+n} & 0 & -\frac{1}{1+n} & 0 \\ \frac{R - g f_k}{1+n} f_{kk} \frac{\omega}{R^{**}} & \frac{R}{R^{**}} & -\frac{1}{1+n} f_{kk} \frac{\omega}{R^{**}} & 0 \\ \frac{R - g f_k}{1+n} \left(c \frac{f_{kk}}{R} - \Psi \beta R \right) & -\Psi \beta \frac{R}{R^{**}} & \beta R - \frac{c \frac{f_{kk}}{R} - \Psi \beta R}{1+n} & 0 \\ \frac{R - g f_k}{1+n} (\phi_R - 1) \frac{\Pi}{R} f_{kk} & 0 & -(\phi_R - 1) \frac{\Pi}{R} \frac{f_{kk}}{1+n} & \phi_\pi \end{pmatrix} \tag{48}$$

The vector $\hat{Y}_t$ is composed of a predetermined variable, $\hat{k}_t$, a non predetermined variable, $\hat{c}_t$, and the two variables $\hat{\omega}_t$ and $\hat{\pi}_t$, both potentially non predetermined but linked to one another by the relation:

$$\hat{\omega}_t = \frac{1}{\Pi} \hat{x}_t - \frac{\omega}{\Pi} \hat{\pi}_t$$

where $\hat{x}_t$ is predetermined. It is therefore necessary, in order to apply the Blanchard and Kahn conditions, to consider one of the two variables ($\hat{\pi}_t$ or $\hat{\omega}_t$) as predetermined and the other one ($\hat{\omega}_t$ or $\hat{\pi}_t$) as non predetermined. The matrix $J_4(k,\omega,c,\pi)$, evaluated in one of the stationary states, has to possess two eigenvalues inside the unit circle and two eigenvalues outside in order to let the associated equilibrium locally determinate.

The interest of the matrix $J_4(k,\omega,c,\pi)$, with regard to the Jacobian matrix $\mathcal{J}_4(k,x,c,\pi)$ which would be associated to the vector variables $\hat{Z}_t = \begin{bmatrix} \hat{k}_t & \hat{x}_t & \hat{c}_t & \hat{\pi}_t \end{bmatrix}'$, lies in its decomposition property. The three first lines of the last column are composed of zero, what means that we can study the properties (the eigenvalues) of the sub-systems $J_3(k,\omega,c)$ and $J_1(\pi)$ independently of each other, with :

$$J_3(k,\omega,c) = \begin{pmatrix} \frac{R-gf_k}{1+n} & 0 & -\frac{1}{1+n} \\ \frac{R-gf_k}{1+n} f_{kk}\frac{\omega}{R} & \frac{R}{R^{**}} & -\frac{1}{1+n} f_{kk}\frac{\omega}{R} \\ \frac{R-gf_k}{1+n}\left(c\frac{f_{kk}}{R} - \Psi\beta R\right) & -\Psi\beta\frac{R}{R^{**}} & \beta R - \frac{c\frac{f_{kk}}{R} - \Psi\beta R}{1+n} \end{pmatrix} \tag{49}$$

and :

$$J_1(\pi) = \phi_\pi$$

The eigenvalue associated to $J_1(\pi)$ is its unique component, ϕ_π. If the function $\Phi(\cdot)$ is of the form used in the figures $3a$ or $3b$, we have $\phi_\pi > 1$ in $\bar{\Pi}$, as well as in Π^{**}, and $\phi_\pi = 0$, around the liquidity trap equilibria in $(1+\underline{i})/R^*$ and in $(1+\underline{i})/R^{**}$.

The sub-system $J_3(k,\omega,c)$ is easier to study when the type of the considered steady state is specified.

4.2 Autarkic Equilibria

In an autarkic steady state equilibrium, the real debt equals zero, which allows to simplify the matrix $J(k,\omega,c)$:

$$J_3(k^*,\omega^*,c^*) = \begin{pmatrix} \frac{R^*-gf_k^*}{1+n} & 0 & -\frac{1}{1+n} \\ 0 & \frac{R^*}{R^{**}} & 0 \\ \frac{R^*-gf_k^*}{1+n}\left(c^*\frac{f_{kk}^*}{R^*} - \Psi\beta R^*\right) & -\Psi\beta\frac{R^*}{R^{**}} & \beta R^* - \frac{c^*\frac{f_{kk}^*}{R} - \Psi\beta R^*}{1+n} \end{pmatrix}$$

Rearranging the variables, it is once again possible to decompose this matrix into two sub-systems $J_2(k^*,c^*)$ and $J_{1'}(\omega^*)$, with :

$$J_2(k^*,c^*) = \begin{pmatrix} \frac{R^*-gf_k^*}{1+n} & -\frac{1}{1+n} \\ \frac{R^*-gf_k^*}{1+n}\left(c^*\frac{f_{kk}^*}{R^*} - \Psi\beta R^*\right) & \beta R^* - \frac{c^*\frac{f_{kk}^*}{R^*} - \Psi\beta R^*}{1+n} \end{pmatrix}$$

and:

$$J_{1'}(\omega^*) = R^*/R^{**}$$

Under assumption (H2), the eigenvalue R^*/R^{**} is strictly less than 1 and we show, in appendix 2, that the condition :

$$(1-g)\left(R^*-R^g\right)\left(\beta R^*-1\right)-\Psi\beta R^*<-c^*\frac{f_{kk}^*}{R^*} \tag{H3}$$

is necessary and sufficient for the matrix $J_2\left(k^*,c^*\right)$ to admit one and only one eigenvalue less than unity in absolute value[15]. These results are summarized by the following proposition whose proof is provided in appendix 2:

Proposition 2 *Under assumptions (H1), (H2) and (H3), the autarkic equilibrium associated to the inflation target,* $\bar{\Pi}$, *is locally determinate and the autarkic liquidity trap equilibrium is locally indeterminate.*

Equivalent results are obtained for Ricardian economies by putting $n=0$ and by replacing R^* with β^{-1}.

4.3 Debt Equilibria

The matrix $J_3^{**}=J_3\left(k^{**},\omega^{**},c^{**}\right)$ corresponding to the debt equilibria is obtained by putting $R=R^{**}$ in (49). One obtains :

$$J_3^{**}=\begin{pmatrix} \frac{R^{**}-gf_k^{**}}{1+n} & 0 & -\frac{1}{1+n} \\ \frac{R^{**}-gf_k^{**}}{1+n}f_{kk}^{**}\frac{\omega^{**}}{R^{**}} & 1 & -\frac{1}{1+n}f_{kk}^{**}\frac{\omega^{**}}{R^{**}} \\ \frac{R^{**}-gf_k^{**}}{1+n}\left(c^{**}\frac{f_{kk}^{**}}{R^{**}}-\Psi\beta R^{**}\right) & -\Psi\beta & \beta R^{**}-\frac{c^{**}\frac{f_{kk}^{**}}{R^{**}}-\Psi\beta R^{**}}{1+n} \end{pmatrix}$$

In appendix 3, we analyze the characteristic polynomial $\mathcal{P}^{**}(\lambda)$ associated to the matrix J_3^{**} which allows us to show that it admit one eigenvalue in absolute value less than unity and two eigenvalues, greater than unity. One can deduce the following proposition whose proof is provided in appendix 3:

Proposition 3 *Under the assumption (H1) and (H2), the debt equilibrium associated to the higher inflation rate,* $\bar{\Pi}$ *or* Π^{**}, *is locally overdeterminate and the debt-liquidity-trap equilibrium is locally determinate*

4.4 From Local Determinacy to Global Indeterminacy

Based on propositions 2 and 3, we can conclude that the four potential stationary equilibria of our economy have, locally, the properties of the four equilibria associated to the four configurations of fiscal and monetary policies identified by Leeper (1991).

Within the framework considered by Leeper, monetary and fiscal policies simultaneously passive lead to indeterminacy and active policies[16] to overdeterminacy

[15] Notice that assumption (H3) is always verified in a Ricardian economy, when $\beta R=1$ (and $\Psi=0$). An other case verifying (H3) is when $R^*<R^g$ which is a stronger version of assumption (H2) but which *excludes* the representation adopted in Figure 2.

[16] Recall that for Leeper, a fiscal policy is called active when the fiscal authority pays no attention to the debt stabilization objective. The rule is then not very reactive to the level of debt.

(instability). Only the configurations where one of the two policies is active and the other passive provide the determinacy—*i.e.* the local uniqueness—of the equilibrium.

If we are to interpret the local stability properties of our equilibria with the same concepts, we must use a local definition of active vs passive monetary and fiscal policies, and we must explain why a policy cannot be globally active or passive.

The difficulty with the definition of an interest rate policy corresponding to a globally active monetary rule was already noted by Benhabib, Schmitt-Grohé and Uribe (2001b). The zero bound on the nominal interest rate (the liquidity trap) fails to ensure the application of an interest rate rule sufficiently reactive to inflation (active) when the rates are low. We have seen that the required non-linearity of the monetary rule doubled the number of stationary equilibria and no longer ensured the determinacy of the autarkic equilibrium when the fiscal policy was locally passive (reactive to the level of debt), which is the main result of Benhabib, Schmitt-Grohé and Uribe (2001b).

The second source of difficulty arises from the accumulation of debt. The exchange economy considered by Leeper permits to characterize a simple fiscal rule whose properties do not depend on the level of the initial real public debt[17]. The mere presence of production and capital accumulation is not sufficient to modify this result. In a Ricardian economy, the Barro-equivalence (Barro, 1974) insulates the real interest rate from the real public debt level. However, in a non-Ricardian economy, the presence of wealth effects results in the dependence of the real interest rate level on the public debt. In this case, a too simple fiscal policy (linear) is not sufficient to offset the increased debt burden associated to a high real public debt, even if it is sufficiently responsive (passive in terms of Leeper) for a lower level of debt. This finding has already been reported by Cushing (1999) and Benassy (2000) when the monetary authorities set the nominal interest rate to a constant value.

Ironically, the characteristics of both rules which explain the multiplicity of equilibria are diametrically opposed. The monetary rule would not be a problem if it was linear[18]. On the other hand, a non-linear fiscal rule, becoming more responsive to the level of debt as the real interest rate rises, would easily allow to ensure the convergence of debt to its targeted value. From another point of view, the double multiplicity of equilibria results from a double non-linearity: *i)* an interest rate rule which respects a lower bound, and *ii)* the existence of wealth effects with a too simple fiscal rule.

The most original result of our model lies probably in the coexistence of two steady state equilibria locally determinate, *i.e.* associated with saddle *i.e.*, locally unique, trajectories[19]. The first one is the targeted autarkic equilibrium. Since the Taylor principle is verified around this equilibrium, the associated monetary policy is said to be active. On the other hand, the fiscal policy is locally passive. This last

[17]The term "initial" can be misleading, insofar as the general level of prices can jump so that the real government debt is just covered by expected income, as in the highly controversial "Fiscal Theory of the Price Level".

[18]But this would require the possibility of negative nominal interest rates...

[19]It is in fact about a stable variety of dimension 2, *i.e.* "saddle plans".

point is easy to verify by rewriting the linearized government constraint (45) when $R = R^*$, $\omega = 0$, and $\hat{g}_t = \hat{z}_t$. One obtains: $E_t\hat{\omega}_{t+1} = (R^*/R^{**})\,\hat{\omega}_t$. Then, according to assumption (H2), the real public debt converges to 0, its long run value. The second considered equilibrium is the debt-liquidity trap (or public debt-deflation) equilibrium. Because the nominal interest rate is stuck at its zero lower bound, the monetary policy is forced to be passive. On the other hand, the fiscal policy is locally active because the low value of θ does not compensate for the public debt burden associated with a high real interest rate. This configuration corresponds to the FTPL, but the properties of this equilibrium also recall those of a "Samuelson equilibrium" in a traditional OLG model. We return to this point in the next section.

According to the existence of two saddle trajectories the issue of self-fulfilling expectations becomes particularly interesting. The economy by being situated on one of these two saddle trajectories could jump, thanks to a likely important shock affecting agents' expectations, on the other saddle trajectory.

5 Self-fulfilling Expectations: the Peril of Public Debt-Deflation

In this section, we will verify our last conjecture by simulating an expectation shock. For this purpose, we calibrate the structural parameters of the model based on Euro Area data. Then, assuming that the predetermined variables are in halfway between the two considered steady states, we investigate the effects of an expectation shock bringing the economy—that we suppose to be initially on a virtuous trajectory towards the autarkic equilibrium—on a public debt-deflation trajectory.

5.1 Functional Forms and Calibration

We assume that the production function is of the kind: $f(k_t) = Ak_t^{\alpha}$, where α is the capital share and A, a scaling parameter. We define the monetary rule as

$$\Phi\left(\bar{R}_t, \Pi_t; \bar{\Pi}\right) = \max\left\{R_t\bar{\Pi}\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi}; 1+\underline{i}\right\} \tag{50}$$

where $\underline{i} > 0$ and $\phi \geq 1$. This rule respects (H1).

We assume that each period corresponds to a year. The parameter values we use in the numerical analysis are shown in Table 1. Most of them are taken from Smets and Wouters (2002) and Fagan *et al* (2001). We set the discount factor β to 0.96 implying an annual discount rate approximately 4%. The capital share α is chosen to be equal to 0.3 and the depreciation of capital, δ, to 0.1. Public expenditure share, g, is set equal to 0.2. In order to have zero primary public deficit at the zero-debt steady state, we calibrate taxes-to-GDP ratio z to 0.2. The consumption weight in utility function, ξ, is set equal to 0.95. The scaling parameter, A, is calibrated such that we get, at the autarkic steady state, a value of the output equals to 100[20].

[20] Our calibration leads to $A = 20.1467$.

While Melitz (2000) estimates the weight of public debt in the fiscal rule at 0.03, Gali and Perotti (2003) estimate this parameter at 0.05. In order to obtain reasonable values both for the public debt-to-GDP ratio and the real interest rate at the debt equilibrium, we follow Melitz (2000) and we set $\theta = 0.03$. The population growth rate, n, is set equal to 0.014 in order to get a steady state value of government debt-to-GDP ratio at the debt equilibrium, ω^{**}, approximately equal to 160%. This parameter value is slightly larger than the value observed in the data but it is assumed to capture all the wealth effects which would affect the real economy.

The model's parameters are summarized in the following table :

Definition	*Parameter*	*Value*
Discount factor:	β	0.96
Weight of consumption in the utility function:	ξ	0.95
Capital share of output:	α	0.3
Depreciation rate of capital:	δ	0.1
Population growth rate:	n	0.014
Public expenditure-to-GDP ratio:	g	0.2
GDP parameter in the fiscal rule:	z	0.2
Debt parameter in the fiscal rule:	θ	0.03

Table 1: Parameters values

They imply: $\Psi = n\xi\left(\beta^{-1} - 1\right) \simeq 0.55 \times 10^{-3}$. We easily obtain the values of the real gross interest rate in the Ricardian, autarkic and debt equilibria, and the value of the modified golden rule interest rate :

Real Interest Rate	*Parameter*	*Value*
Ricardian Equilibria Interest Rate:	$\beta^{-1} - 1$	4.17%
Modified-Golden Rule Interest Rate	$R^g - 1$	4.25%
Autarkic Equilibria Interest Rate:	$R^* - 1$	4.38%
Debt Equilibria Interest Rate:	$R^{**} - 1$	4.54%

Table 2: Real interest steady state values

This ranking is consistent with the representation in Figure 2.

5.2 Simulation and Discussion

Now, we assume that both capital stock k_t and real public debt x_t (the predetermined variables) are at a half distance between the targeted autarkic equilibrium and the public debt-deflation equilibrium. We then study the convergence of the economy towards each steady state. We have made this exercise both with the linearized version of the model and with a non-linearized one, using the DYNARE package for Matlab (see Juillard [2004]). We found identical results with the two versions, which signifies that the linearization does not affect the convergence towards the steady

state. We report on figure 6 the results of the non linearized model.

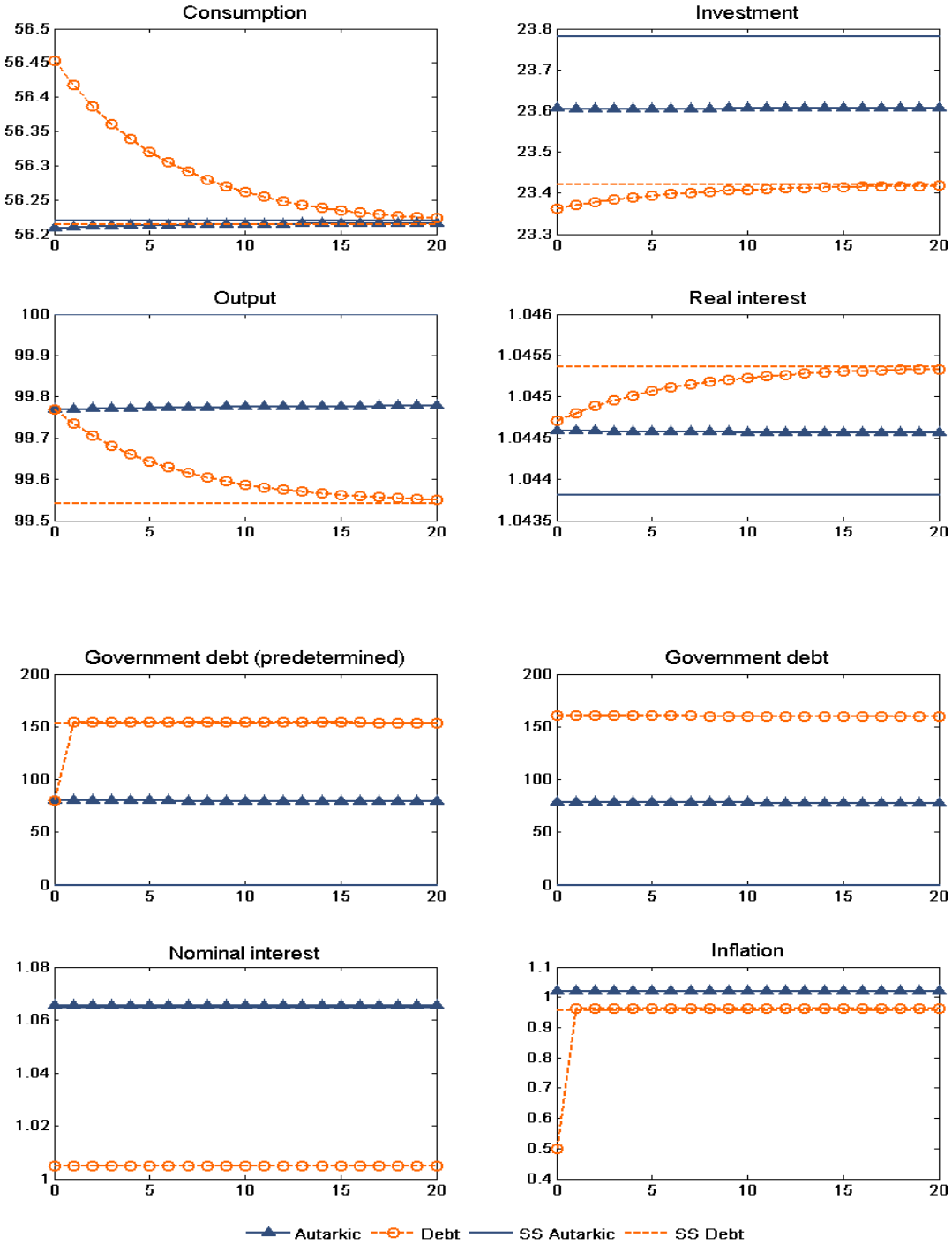

Figure 6

The convergence toward the autarkic equilibrium takes a very long time to reach the steady state. This is due to the relatively low value of θ. The adjustment of investment, output, and the real interest rate is expected since the initial value of the

capital stock is below its steady-state level. Nevertheless, consumption converges faster towards the targeted autarkic equilibrium.

Now assume that the expectations lead the economy towards the public debt-deflation equilibrium. We note first an increased consumption, a reduced investment and a sharp decline in the rate of inflation which leads the nominal interest rate to the zero bound. Then, output and consumption declines and the real interest rate rises toward its higher steady state value.

Since the economy is non Ricardian, an expected increase in public debt entails a positive wealth effect. Consequently, agents reduce their savings so as to increase their consumption. As a result, the real interest rate increases and investment decreases.

The sharp deflation can be explained by the FTPL that Leeper (1991), Sims (1994) and Woodford (1994) presented and analyzed, but with a slight extension. Referring to the "stock analogy" used by Cochrane (2005), the per capita government budget constraint (28) can be rewritten as a valuation equation:

$$\omega_t = \frac{z_t - g_t}{1-\theta} y_t + \frac{1+n}{1-\theta} E_t \left(\frac{\omega_{t+1}}{R_{t,t+1}} \right)$$

where $\omega_t = \Omega_t / P_t N_t$. Using again the assumption $z_t = g_t \ \forall t$, made in the simulation, and supposing, for sake of simplicity, that there is no more uncertainty after the expectation shock, this equation becomes:

$$\omega_t = \frac{R^{**} \omega_{t+1}}{R_t} \tag{51}$$

where we have used $R^{**} = \frac{1+n}{1-\theta}$. Because the exogenous component of the primary public surplus is zero, the right-hand term of the valuation equation is only constituted by the bubble, *i*.e. the unbacked part of the public debt. The OLG structure of our model permits the existence of an equilibrium where this right-hand term is positive at the steady state. This is the case at the public debt-deflation equilibrium.

In order to understand this last point, it can be useful to consider a simplified version of our model by supposing an endowment economy. Let y be the *per capita* endowment and $c = (1-g)\, y$, the equilibrium consumption. In absence of capital, the equilibrium aggregate Euler equation (26) can be rewritten as:

$$R_t = \beta^{-1} + \frac{\Psi}{c} \omega_{t+1} \tag{52}$$

Combining (51) and (52), we easily obtain the dynamic equation:

$$\omega_{t+1} = \frac{\beta^{-1} \omega_t}{R^{**} - \frac{\Psi}{c} \omega_t}$$

which accepts $\omega^* = 0$ and $\omega^{**} = \left(R^{**} - \beta^{-1} \right) c / \Psi$ as steady state values and which

can be represented on the following figure:

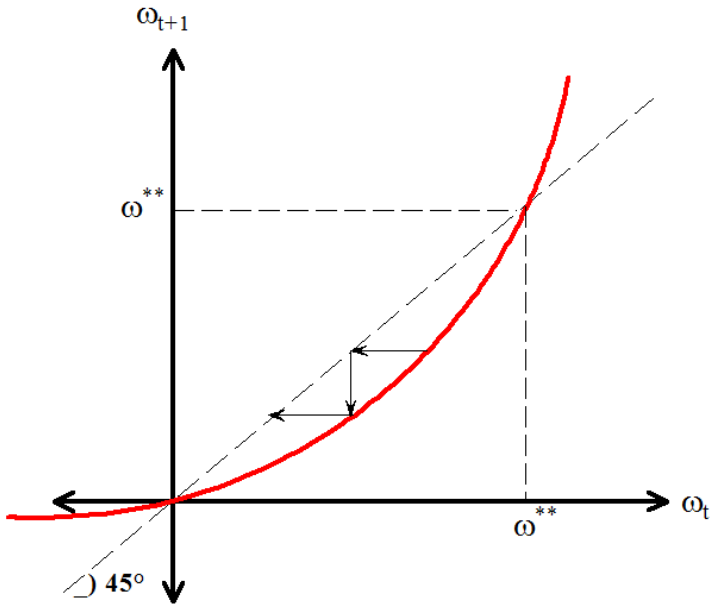

Figure 4

Starting from a value $\omega_0 = \Omega_0/P_0N_0$ between ω^* and ω^{**}, the initial value of P_0 being determined by an active monetary policy, the economy converges towards the Autarkic equilibrium ω^*. Thanks to an expectation shock, the economy can then jump to the public debt-deflation equilibrium ω^{**}, without any transition in this simple exchange economy. Since the nominal debt Ω_t is predetermined, the real value of the public debt adjusts by a price fall which throw the economy into the liquidity trap and the deflation. Interestingly, this equilibrium resembles to the Samuelson equilibrium of a standard OLG economy. In the case[21]: $\theta = 0$, this equilibrium would be a pure bubble. More generally, the steady state real public debt ω^{**} is a growing function of R^{**} and thus of θ and this is a characteristic of a FTPL equilibrium when part of the primary surplus is function of the level of the real public debt. The sole difference is that the standard FTPL needs the existence of a positive exogenous primary surplus that is not necessary in our economy, this role being played by the bubble component of the debt.

In the non Ricardian economy with capital, the public debt-deflation steady state is not immediately reached. In particular, consumption and investment take time—around 20 years—to reach theirs long run lower values. Unfortunately, we have to recognize that the adjustment of the real public debt is much more—actually too—sharp. In our simulation, the initial deflation is around 50% of the initial price level which, of course, is unrealistic. Without this last disadvantage and the initial increase of the consumption level, our model could offer an alternative—or, at least, a complementary—explanation to the more traditional reading brought by

[21] Which would require $1 + n > \beta^{-1}$.

Krugman (1998), Svensson (2001), and Eggertson and Woodford (2004) of some deflation episodes like the Japanese recession of the 90s. These authors argue that the Japanese liquidity trap was the consequence of a very negative shock on the natural interest rate in a context of inflation stabilization around a maybe too low target. The hypothesis of the liquidity trap of Benhabib, Schmitt-Grohé and Uribe (2001a, 2001b) explains the weakness of the nominal interest rates and the incapacity of the monetary authority to stabilize the economy in such a context, but does not allow to explain the entrance in recession and the persistence of this one[22]. Our public debt-deflation equilibrium does not have this flaw. The higher level of the real interest rate provokes a crowding out effect of the private investment and reduces the level of the production, which seems to be a strong characteristic of a deflationary episode. For this reason, the main objective of further researches would be to make this scenario more plausible, by both reducing the initial price jump and the initial increased consumption, but without affecting our long run results. Among other assumptions, the introduction of a learning process could be an interesting perspective.

6 Conclusion

The focus of this paper is the study of the interaction between monetary and fiscal policy in the presence of non Ricardian consumers and with capital accumulation. In the economic environment considered, the Ricardian equivalence breaks down and government debt spawn wealth effects.

To this end, we develop an extended version of Weil's(1987, 1991) overlapping generations model in which government debt affects the behavior of consumers that is the fiscal policy is no longer neutral. This model has the Ricardian equivalence as a special case, when the population is constant. Assuming a simple fiscal policy, in the spirit of Leeper (1991) and a non linear monetary rule namely a Taylor rule taking into account the zero lower bound on the nominal interest rate, in the spirit of Benhabib, Schmitt-Grohé and Uribe (2001a), our analysis of the steady state exhibits the presence of four equilibria. We analyze local steady states dynamics. Comparing the four equilibria with the four configurations described by Leeper (1991) yield the same dynamics characteristics. As a consequence, the determinacy region is no longer specified by the policy parameter space. In short, in the presence of wealth effects, a nominal interest rate rule bounded below by zero emphasizes the global indeterminacy problem identified by Benhabib, Schmitt-Grohé and Uribe (2001a, 2001b). Accurately, four equilibria are founded regardless of the policy parameter space.

Our results show that a liquidity trap, also characterized by a higher real interest rate and a higher level of real debt, possesses the usually required properties of determinacy, like the more traditional equilibrium targeted by the monetary and fiscal authorities.

[22] We could even expect a more important level of activity in a model where the weakness of the nominal interest rates reduces the level of the monetary distortions and increases the labor supply.

Furthermore, from the perspective of global analysis, the existence of two paths locally convergent arises the question of a self-fulfilling expectations shock. Indeed a self-fulfilling expectations shock can lead the economy from one trajectory to another.

To this end, the model is calibrated on annual data and allows to evaluate the implications of a self-fulfilling expectations shock. Our results show that for a given initial level of predeterminate variables, the convergence towards the debt-liquidity-trap equilibrium is fulfilled by an initial deflation and an increase in the real interest rate, caused by a change in agents' expectations. We thus give an other endogenous explanation of the liquidity trap where the public debt play a crucial role.

If these results can be empirically supported, then some deflationary episodes—like the Japanese recession of the 90s—could be reinterpreted as the consequences of a change in agents' expectations. But this proposition needs a deeper empirical arguments which are not treated in the present paper. Thereby, in our simulation, the initial deflation accounts for 50% that is not realistic. For this reason, the main objective of further researches would be to make this scenario more plausible, by both reducing the initial price jump and the initial increased consumption, but without affecting our long run results. Another question arises: how to avoid deflation? Extrapolating the results of Bénassy and Guillard (2005) who study the case of a non-Ricardian exchange economy, the control of the growth rate of nominal debt should simultaneously ensure the uniqueness and determinacy of the equilibrium.

Appendix 1

Proposition 1: *The real value of the per capita public debt is positive in the "debt equilibrium" if and only if the associated real interest rate is greater than the "autarkic" real interest rate, i.e.:*

$$R^{**} \geq R^{*} \Longleftrightarrow \omega^{**} \geq 0.$$

Proof: Using the concavity of the production function $f(k)$, and (consequently) the decrease of the function $\tilde{R}(k)$, equation (34): $c = (1-g) f(k) - (n+\delta) k$, permits us to verify that :

$$R^{**} \geq R^{*} \Longrightarrow k^{**}/c^{**} \leq k^{*}/c^{*}$$

By using (37) that we remind:

$$R^{*} = \tilde{R}(k^{*}) = \frac{\beta^{-1}}{1 - \Psi k^{*}/c^{*}}$$

we easily find :

$$R^{**} \geq R^{*} \Longleftrightarrow \frac{\beta^{-1}}{1 - \Psi k^{**}/c^{**}} \leq \frac{\beta^{-1}}{1 - \Psi k^{*}/c^{*}} \tag{A1.1}$$

Now, by using the equation (32) rewritten under the following form:

$$\tilde{R}(k) = \frac{\beta^{-1} + \Psi \omega/c}{1 - \Psi k/c}$$

and evaluated in the autarkic equilibrium and in the debt equilibrium, one observes that :

$$R^{**} \geq R^{*} \Longleftrightarrow \frac{\beta^{-1} + \Psi \omega^{**}/c^{**}}{1 - \Psi k^{**}/c^{**}} \geq \frac{\beta^{-1}}{1 - \Psi k^{*}/c^{*}} \tag{A1.2}$$

By collecting the inequalities (A1.1) and (A1.2), we finally obtain:

$$R^{**} \geq R^{*} \Longleftrightarrow \frac{\beta^{-1}}{1 - \Psi k^{**}/c^{**}} \leq \frac{\beta^{-1}}{1 - \Psi k^{*}/c^{*}} \leq \frac{\beta^{-1} + \Psi \omega^{**}/c^{**}}{1 - \Psi k^{**}/c^{**}}$$

Or, more simply:

$$R^{**} \geq R^{*} \Longleftrightarrow \omega^{**} \geq 0$$

||

Appendix 2

Proposition 2: *Under assumptions (H1), (H2) and (H3), the autarkic equilibrium associated to the inflation target,* $\bar{\Pi}$, *is locally determinate and the autarkic liquidity trap equilibrium is locally indeterminate.*

Proof: *a)* We show, at first, that the condition (H3) is sufficient so that the matrix $J_2(k^*, c^*)$ admits one and a single eigenvalue lower than the unity in absolute value. Let us remind, by convenience, $J_2(k^*, c^*)$:

$$J_2(k^*, c^*) = \begin{pmatrix} \frac{R^* - gf_k^*}{1+n} & -\frac{1}{1+n} \\ \frac{R^* - gf_k^*}{1+n}\left(c^* \frac{f_{kk}^*}{R^*} - \Psi\beta R^*\right) & \beta R^* - \frac{c^* \frac{f_{kk}^*}{R^*} - \Psi\beta R^*}{1+n} \end{pmatrix}$$

Its characteristic polynomial is given by :

$$\begin{aligned} P^*(\lambda) &= \left(\frac{R^* - gf_k^*}{1+n} - \lambda\right)\left(\beta R^* - \frac{c^* \frac{f_{kk}^*}{R^*} - \Psi\beta R^*}{1+n} - \lambda\right) \\ &\quad + \frac{R^* - gf_k^*}{1+n}\left(\frac{c^* \frac{f_{kk}^*}{R^*} - \Psi\beta R^*}{1+n}\right) \end{aligned}$$

Let us calculate the critical values of $P^*(\lambda)$. We find:

$$\begin{aligned} P^*(-1) &= (1+\beta R^*)\left(1 + \frac{R^* - gf_k^*}{1+n}\right) - \frac{\left(c^* \frac{f_{kk}^*}{R^*} - \Psi\beta R^*\right)}{(1+n)} > 0 \\ P^*(0) &= \frac{R^* - gf_k^*}{1+n}\beta R^* > 0 \\ P^*(1) &= \left(1 - \frac{R^* - gf_k^*}{1+n}\right)(1 - \beta R^*) + \frac{\left(c^* \frac{f_{kk}^*}{R^*} - \Psi\beta R^*\right)}{(1+n)} \end{aligned}$$

The signs of $P^*(-1)$ and of $P^*(0)$ are evident. Using the fact that : $R^* = 1 - \delta + f_k^*$, and remembering that $\frac{(1+n)-(1-\delta)g}{1-g} = R^g$, a necessary and sufficient condition to guarantee that $P^*(1)$ is negative is given by :

$$(1-g)(R^* - R^g)(\beta R^* - 1) - \Psi\beta R^* < -c^* \frac{f_{kk}^*}{R^*} \tag{H3}$$

The polynomial $P^*(\lambda)$ is of degree 2, the condition $P^*(1) < 0$ implied by (H3), jointly with $P^*(-1) > 0$ and $P^*(0) > 0$ is sufficient to guarantee the uniqueness of the eigenvalue inside the unit circle.

b) Notice that, by (H2), the eigenvalue of $J_{1'}(\omega^*)$ verifies : $R^*/R^{**} < 1$, one conclude that the initial matrix $J_4^* = J_4(k^*, \omega^*, c^*, \pi^*)$ has at least two eigenvalues less than the unit and one eigenvalue greater than the unit (in absolute value). According to the sign of $\phi_\pi - 1$, the equilibrium is either locally determinate $(\phi_\pi > 1)$,

or locally indeterminate $(\phi_\pi < 1)$. By (H1), the autarkic equilibrium associated to the inflation target, $\bar{\Pi}$, is locally determinate and the autarkic liquidity-trap equilibrium is locally indeterminate. $\|$

Appendix 3

Proposition 2: *Under the assumption (H1) and (H2), the debt equilibrium associated to the higher inflation rate, $\bar{\Pi}$ or Π^{**}, is locally overdeterminate and the debt-liquidity-trap equilibrium is locally determinate.*

Proof: The proof is twofold. First, we give a sufficient condition for the matrix $J_3^{**} = J_3(k^{**}, \omega^{**}, c^{**})$ to admit one eigenvalue in the absolute value less than the unit and two eigenvalues greater than the unit. Second, we deduce from (J_1^{**}) the dynamic characteristics of this equilibrium.

a) We show that the characteristic polynomial $\mathcal{P}^{**}(\lambda)$ of the matrix $J_3^{**} = J_3(k^{**}, \omega^{**}, c^{**})$ has one root in the interval $[-1, 1]$ and two outside the interval $[-1, 1]$. Let us remind, by convenience, J_3^{**} :

$$J_3^{**} = \begin{pmatrix} \frac{R^{**} - gf_k^{**}}{1+n} & 0 & -\frac{1}{1+n} \\ \frac{R^{**} - gf_k^{**}}{1+n} f_{kk}^{**} \frac{\omega^{**}}{R^{**}} & 1 & -\frac{1}{1+n} f_{kk}^{**} \frac{\omega^{**}}{R^{**}} \\ \frac{R^{**} - gf_k^{**}}{1+n} \left(c^{**} \frac{f_{kk}^{**}}{R^{**}} - \Psi\beta R^{**} \right) & -\Psi\beta & \beta R^{**} - \frac{c^{**} \frac{f_{kk}^{**}}{R^{**}} - \Psi\beta R^{**}}{1+n} \end{pmatrix}$$

Its characteristic polynomial is given by :

$$\mathcal{P}^{**}(\lambda) = -\lambda^3 + T^{**}\lambda^2 - S^{**}\lambda + D^{**}$$

where T^{**}, S^{**} and D^{**} represent the trace, the sum of the principal minors of order two and the determinant of the matrix J^{**}, respectively, which are given by:

$$\begin{aligned} T^{**} &= \frac{R^{**} - gf_k^{**}}{1+n} + 1 + \beta R^{**} - \frac{c^{**} \frac{f_{kk}^{**}}{R^{**}} - \Psi\beta R^{**}}{1+n} > 0 \\ S^{**} &= \beta R^{**} - \frac{c^{**} \frac{f_{kk}^{**}}{R^{**}} - \Psi\beta R^{**}}{1+n} + \frac{R^{**} - gf_k^{**}}{1+n}(1 + \beta R^{**}) - \Psi\beta \frac{f_{kk}^{**}}{(1+n)} \frac{\omega^{**}}{R^{**}} > 0 \\ D^{**} &= \frac{R^{**} - gf_k^{**}}{1+n} \beta R^{**} > 0 \end{aligned}$$

Let us calculate the critical values and the derivative of $\mathcal{P}^{**}(\lambda)$. We find:

$$\begin{aligned} \mathcal{P}^{**}(-1) &= 1 + T^{**} + S^{**} + D^{**} > 0 \\ \mathcal{P}^{**}(0) &= D^{**} > 0 \\ \mathcal{P}^{**}(1) &= \Psi\beta \frac{f_{kk}^{**}}{(1+n)} \frac{\omega^{**}}{R^{**}} < 0 \end{aligned}$$

and,

$$\mathcal{P}_\lambda^{**}(\lambda) = -3\lambda^2 + 2T^{**}\lambda - S^{**}$$

We show, at first, that $\mathcal{P}^{**}(\lambda)$ does not admit a root in $[-\infty, 0]$. Then we prove that it admits only odd roots in $[0,1]$; either 1 or 3. Finally, we give a sufficient condition to preclude the three-roots' case.

i) $P_{\lambda}^{**}(\lambda)$ is strictly negative in $[-\infty, 0]$, accordingly the polynomial $\mathcal{P}^{**}(\lambda)$ is strictly decreasing in $[-\infty, 0]$. Given the sign of $\lim_{\lambda \to -\infty} \mathcal{P}^{**}(\lambda)$, and $\mathcal{P}^{**}(0)$ we deduce that $\mathcal{P}^{**}(\lambda) \neq 0$ in $[-\infty, 0]$. Therefore, the polynomial $\mathcal{P}^{**}(\lambda)$ does not admit a root in $[-\infty, 0]$.

ii) According to $\mathcal{P}^{**}(0) > 0$, and $\mathcal{P}^{**}(1) < 0$ the polynomial $\mathcal{P}^{**}(\lambda)$ changes of sign between 0 and 1, thus it can have, either one, or three roots in $[0,1]$.

iii) If $\mathcal{P}^{**}(\lambda)$ admits three roots in $[0,1]$, then its derivative should cancel twice in $[0,1]$. A sufficient condition to preclude the later case, is to show that the polynomial of degree two $\mathcal{P}_{\lambda}^{**}(\lambda)$ admits a positive maximum outside the interval $[0,1]$ involving that $\mathcal{P}_{\lambda}^{**}(\lambda)$ has at most one root inside the interval $[0,1]$. Now, we have:

$$\mathcal{P}_{\lambda}^{**}(0) = -S^{**} < 0$$

and

$$\mathcal{P}_{\lambda\lambda}^{**}(\lambda) = -6\lambda + 2T^{**}$$

that equals zero when $\lambda = \frac{T^{**}}{3}$. Using $R^{**} = 1 - \delta + f_k^{**}$ and $R^g = \frac{(1+n)-(1-\delta)g}{1-g}$, the condition for $T^{**} > 3$ can be written:

$$(1-g)(R^{**} - R^g) + (1+n)(\beta R^{**} - 1) > c^* \frac{f_{kk}^*}{R^*} - \Psi \beta R^*$$

that is easily verified using $H2$. In fact, according to $H2$, we have $R^{**} > R^* > \beta^{-1}$ and $R^{**} > R^g$ both involving that the left-hand term of the previous inequality is positive. This sufficient condition guarantees that $\mathcal{P}_{\lambda}^{**}(\lambda)$ cancels only once in $[0,1]$ and therefore the polynomial $\mathcal{P}^{**}(\lambda)$ admits only one root in $[0,1]$. We deduce that the matrix J_3^{**} has one eigenvalue in the absolute value less than the unit and two, greater than the unit.

b) Finally, According to the sign of $\phi_\pi - 1$, the equilibrium is either locally determinate $(\phi_\pi < 1)$, or locally overdeterminate $(\phi_\pi > 1)$. By (H1), the debt equilibrium associated to the higher inflation rate, $\bar{\Pi}$ or Π^{**}, is locally overdeterminate and the debt-liquidity-trap equilibrium is locally determinate.‖

Appendix 4

In this appendix, we derive the state-space form of the model composed of the variables $\hat{c}_t$, $\hat{\pi}_t$, $\hat{k}_t$ and $\hat{x}_t$ (rather than $\hat{\omega}_t$). We remind, by convenience, the equations (26) to (31):

$$
\begin{aligned}
c_t &= \beta^{-1}\frac{c_{t+1}}{R_{t,t+1}} + \Psi\left[\frac{\omega_{t+1}}{R_{t,t+1}} + \frac{\tilde{R}\left(k_{t+1}\right)}{R_{t,t+1}}k_{t+1}\right] && \text{(A4.1)}\\
k_{t+1} &= \frac{1}{1+n}\left[\left(1-\delta\right)k_t + \left(1-g_t\right)\cdot f\left(k_t\right) - c_t\right] && \text{(A4.2)}\\
E_t\left(\frac{\omega_{t+1}}{R_{t,t+1}}\right) &= \frac{1}{1+n}\left[\left(1-\theta\right)\omega_t + \left(g_t - z_t\right)f\left(k_t\right)\right] && \text{(A4.3)}\\
E_t\left(\frac{1}{R_{t,t+1}}\right) &= \frac{1}{\tilde{R}\left(k_{t+1}\right)} && \text{(A4.4)}\\
E_t\left(\frac{1}{R_{t,t+1}\Pi_{t+1}}\right) &= \frac{1}{1+i_t} && \text{(A4.5)}\\
1+i_t &= \Phi\left(\bar{R}_t, \Pi_t\right) && \text{(A4.6)}
\end{aligned}
$$

From (A4.1), we express the value of $R_{t,t+1}$:

$$
R_{t,t+1} = \beta^{-1}\frac{c_{t+1}}{c_t} + \Psi\left[\frac{\omega_{t+1}}{c_t} + \frac{\tilde{R}\left(k_{t+1}\right)}{c_t}k_{t+1}\right]
$$

that we inject in (A4.3), (A4.4) and (A4.5). By using the value of $1+i_t$ given by (A4.6), defining the predetermined variable $x_t = \Pi_t\omega_t = \frac{M_{t-1}+B_{t-1}}{N_t P_{t-1}}$, and rearranging the equations, we get:

$$
\begin{aligned}
c_t &= \left[E_t\left[\beta^{-1}\frac{c_{t+1}}{\tilde{R}\left(k_{t+1}\right)} + \Psi\left(\frac{x_{t+1}}{\tilde{R}\left(k_{t+1}\right)\Pi_{t+1}} + k_{t+1}\right)\right]^{-1}\right]^{-1}\\
k_{t+1} &= \frac{1}{1+n}\left[\left(1-\delta\right)k_t + \left(1-g_t\right)\cdot f\left(k_t\right) - c_t\right]\\
x_{t+1} &= \frac{\Phi\left(\bar{R}_t, \Pi_t\right)}{(1+n)}\left[\left(1-\theta\right)\frac{x_t}{\Pi_t} + \left(g_t - z_t\right)f\left(k_t\right)\right]\\
\Phi\left(\bar{R}_t, \Pi_t\right) &= \frac{\tilde{R}\left(k_{t+1}\right)E_t\left[\beta^{-1}c_{t+1} + \Psi\left(\frac{x_{t+1}}{\Pi_{t+1}} + \tilde{R}\left(k_{t+1}\right)k_{t+1}\right)\right]^{-1}}{E_t\left[\Pi_{t+1}\right]^{-1}\left[\beta^{-1}c_{t+1} + \Psi\left(\frac{x_{t+1}}{\Pi_{t+1}} + \tilde{R}\left(k_{t+1}\right)k_{t+1}\right)\right]^{-1}}
\end{aligned}
$$

which constitute a system of four dynamics, stochastic and non linear equations, with 2 predetermined variables, k_t and x_t, and non predetermined variables, c_t and Π_t. It is necessary to clarify the processes followed by g_t and z_t—2 additional predetermined variables—as well as the form of the function $\Phi\left(\cdot\right)$ and the value held for the real interest target $\bar{R}_t$ to obtain a completely specified system.

By linearizing the previous system around a some steady state, one obtains :

$$
\begin{aligned}
\hat{c}_t &= \beta^{-1}\frac{E_t\hat{c}_{t+1}}{R} + \Psi\frac{E_t\hat{x}_{t+1}}{\Pi R} - \Psi\omega\frac{E_t\hat{\pi}_{t+1}}{\Pi R} + \left(\Psi R - \frac{\beta^{-1}c + \Psi\omega}{R}f_{kk}\right)\frac{E_t\hat{k}_{t+1}}{R} \\
\hat{k}_{t+1} &= \frac{1}{1+n}\left(\left[R - gf_k\right]\hat{k}_t - \hat{c}_t - f\cdot\hat{g}_t\right) \\
E_t\hat{x}_{t+1} &= \omega E_t\hat{\pi}_{t+1} + \frac{\Pi\omega f_{kk}}{R^{**}}\hat{k}_{t+1} + \frac{R}{R^{**}}\hat{x}_t - \frac{R}{R^{**}}\omega\hat{\pi}_t + \frac{\Pi R}{1+n}f\cdot(\hat{g}_t - \hat{z}_t) \\
E_t\hat{\pi}_{t+1} &= \phi_\pi\hat{\pi}_t + (\phi_R - 1)\frac{\Pi}{R}f_{kk}\hat{k}_{t+1}
\end{aligned}
$$

where $\phi_\pi = \Pi\Phi_\Pi/\Phi$ and $\phi_R = R\Phi_R/\Phi$ are the elasticity of the function $\Phi(\cdot)$ and where we used $R^{**} = \frac{1+n}{1-\theta}$.

By denoting $\hat{Y}_{x,t} = \begin{bmatrix} \hat{k}_t & \hat{x}_t & \hat{c}_t & \hat{\pi}_t \end{bmatrix}'$, the vector of the endogenous variables and $\epsilon_t = \begin{bmatrix} \hat{g}_t & \hat{z}_t \end{bmatrix}'$, the vector of shocks, the previous equations can be combined to obtain the state-space form as follows :

$$
E_t\hat{Y}_{x,t+1} = J_x\cdot\hat{Y}_{x,t} + J_\varepsilon\cdot\epsilon_t
$$

where the Jacobian matrix J_x is given by:

$$
J_x = \begin{pmatrix}
\frac{R-gf_k}{1+n} & 0 & -\frac{1}{1+n} & 0 \\
\frac{\omega\Pi}{R^{**}}f_{kk}\frac{R-gf_k}{1+n}\phi_R & \frac{R}{R^{**}} & -\phi_R\frac{\Pi}{R}\omega\frac{f_{kk}}{(1+n)} & \left(\phi_\pi - \frac{R}{R^{**}}\right)\omega \\
\frac{R-gf_k}{(1+n)}\left(c\frac{f_{kk}}{R} - \Psi\beta R\right) & -\Psi\beta\frac{R}{R^{**}} & \beta R - \frac{c\frac{f_{kk}}{R} - \Psi\beta R}{1+n} & \beta\Psi\frac{R}{R^{**}}\frac{\omega}{\Pi} \\
\frac{R-gf_k}{(1+n)}(\phi_R - 1)\frac{\Pi}{R}f_{kk} & 0 & -(\phi_R - 1)\frac{\Pi}{R}\frac{f_{kk}}{(1+n)} & \phi_\pi
\end{pmatrix}
$$

and J_ε by :

$$
J_\varepsilon = \begin{pmatrix}
-\frac{f(k)}{1+n} & 0 \\
\Pi\left(R - \omega\frac{\phi_R}{R}f_{kk}\right)\frac{f(k)}{(1+n)} & -\Pi\frac{R}{1+n}f(k) \\
-\frac{f_{kk}f(k)}{R(1+n)}c & R\beta\frac{\Psi}{1+n}f(k) \\
-(\phi_R - 1)\frac{\Pi}{R}\frac{f_{kk}}{(1+n)}f(k) & 0
\end{pmatrix}
$$

The evolution of the variable $\hat{\omega}_t$ is obtained in a residual way:

$$
\hat{\omega}_t = \frac{1}{\Pi}\hat{x}_t - \frac{\omega}{\Pi}\hat{\pi}_t
$$

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