Public debt sustainability, exchange rates, and country-specific saving rates

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Abstract. The recent dramatic rise of government deficits in some advanced countries causes concerns both with respect to the sustainability of government debt and the stability of (real) exchange rates of highly indebted countries. This paper explores these concerns in a two-country OLG model of the world economy with hugely differing national saving rates to mimic the Japan-US example. It is found that limits for public debt levels in both countries exist and are negatively related. Moreover, if sustainable public debt is unilaterally expanded, the real exchange rate of the debt-expanding country is unaffected if capital income shares are internationally equal.

JEL Classification: H63, F41

Keywords: Sustainability, public debt, real exchange rate, two-country OLG model

1. Introduction

The recent dramatic rise of government deficits in most advanced countries to counter the adverse growth and employment effects of global financial crisis causes concerns both with respect to the sustainability of government debt (IMF 2009) and the stability of (real) exchange rates of highly indebted countries (for the US-Dollar Goyette 2009, Wiedemer et al. 2009). Still close international integration of government bond markets makes it thus imperative for macroeconomists to explore limits for national government debt levels as well as to investigate the impacts of larger national debts on the real exchange rates among interdependent economies. This paper addresses these topics within a stylized two-country overlapping generations (OLG) model, and finds that contrary to the "Wall Street view" (Nyahoho 2006, 416) a larger sustainable government debt does not impact the real exchange rate but there exist limits for national debts approaching them could lead to a sudden collapse of the world economy.

Public debt instead of public deficit sustainability (Chalk 2000) had become a matter of theoretical concern already several years ago. Rankin and Roffia (2003) find in Diamond (1965)-type overlapping generations (OLG) model "that, even with a constant stock of government debt, fiscal policy may be unsustainable because a steady state of the economy with non-degenerate values of the variables *may not exist*" (Rankin and Roffia 2003, 218; italics in original).[1] The private capital labor ratio (aggregate capital intensity) associated with this unsustainable government debt level is called an 'interior maximum' in contrast to a 'degeneracy' in which the capital intensity approaches zero as a consequence of an excessively high government debt level.

Rankin and Roffia's (2003) contribution, although invaluable, is, however, restricted to a closed (or small open) economy setting which precludes the analysis of sustainable government debt in large, open, and interdependent economies. Farmer and Zotti (2010) extended Rankin and Roffia's one-good, closed economy into a two-good, two-country overlapping generations' model, and found "that a world maximum sustainable government debt level (a weighted average of domestic and foreign debt) always exists and is reached at an interior maximum as in Rankin and Roffia's closed economy" (Farmer and Zotti 2010, 3).

To obtain this result, the authors assume that the time preference rates among domestic and foreign younger households are identical. As is well-known, equal time preference rates within a log-linear intertemporal utility function imply equal saving rates of younger households in both countries. Saving rates equality is at odds with the empirical evidence concerning Asian economies and the US economy. Moreover, in two-country models with internationally equal saving rates a relatively high-indebted country always exhibits a net foreign debtor status which contradicts the Japanese situation of both high government debt and a net foreign creditor position. To the best of this author's knowledge, there does not exist hitherto in the literature any two-country OLG model with internationally differing saving rates to address limits for national debt and exchange rate impacts of larger public debt.

To close this research gap is the main objective of the present paper. In particular, it sets out first to explore the existence and determinants of 'maximum sustainable' (Rankin and Roffia 2003) debt levels in a two-country overlapping generations' model which is also able to mimic the Japan-US example. Secondly, it intends to investigate within the proposed model context the impacts of unilateral national debt expansion below maximum sustainable debt on the real exchange rate and on private capital accumulation at home and abroad.

The public debt effects on the real exchange have been already dealt with in the established literature (Feldstein 1986, Frenkel and Razin 1986, Zee 1987, Lin 1994). However, the ambiguity of the real exchange rate effects of public debt has not been resolved so far. Zee (1987) finds that the real exchange rate of the more indebted country depreciate (appreciate) if this country is a net foreign debtor (creditor), whereas according to Lin (1994) the real exchange rate effect of public debt is independent of the net foreign asset position, and depends only on international differences with respect to production technologies. Farmer and Zotti (2010) are able to resolve the ambiguity by confirming Lin's (1994) and rejecting Zee's (1987) results albeit under the assumption of internationally equal saving rates. Again, it is an open question whether the real exchange rate effects of larger national debt change under different saving rates among countries.

To close both of these research gaps, Farmer and Zotti's (2010) two-good, two-country setting with equal saving rates is extended into a two-country OLG model with different

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[1] Thus, a constant (= time-stationary) stock of government debt is not sufficient for

saving rates among countries.[2] To lend empirical support to the two-country setting, one country in the model is depicted to represent the collection of net foreign creditor countries in reality while the other model country represents the net foreign debtor countries as a whole.[3] Since both collections of countries comprise developing and advanced countries respectively, it is assumed for simplicity that each model country represents an average of developing and advanced countries in reality which can be characterized by similar production technologies.[4]

Given the model setting with unequal saving rates, the following questions regarding maximum sustainable debt are asked:[5] Do maximum sustainable public debt levels in both countries always exist? What happens when these limits for sustainable public debt are reached? Are the national limits interdependent and which common factors determine the national debt limits? One main finding is that a domestic maximum sustainable public debt level (for some given foreign public debt level) always exists and is reached at an interior maximum as in Rankin and Roffia's (2003) closed and Farmer and Zotti's (2010) open economy.

Second, given this public debt limit it is still not clear which role it plays with respect to the existence and characteristics of non-trivial steady state solutions for private capital intensities and with respect to the real exchange rate between both countries. Extending Ono's (2002) existence analysis in a closed-economy OLG model into the two-country setting with internationally differing saving rates, it will be shown that the magnitude of the existing domestic public level relative to the corresponding maximal sustainable debt level is decisive for the existence of a unique or multiple non-trivial steady states of capital intensities.

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sustainability.

2 In particular, the most obvious type of debt instrument referred to by Rankin and Roffia as 'interest-exclusive' debt, where government debt is treated like a savings account, is investigated. The remaining two types considered by Rankin and Roffia (2003) are 'interest-inclusive' debt and interest payments on government debt alone.

3 A clear characteristic of the world economy during the fifteen years before the outburst of current global economic crisis has been the development of rising external imbalances as measured by the diverging net foreign asset positions (see IMF 2006, 74; IMF 2008, 35).

4 Cobb-Douglas production functions are defined as 'similar' if production elasticities (or, respectively, capital income shares) are internationally equal, while the scale parameter reflecting the technological level might differ across countries.

5 The first two questions closely follow Rankin and Roffia (2003, 219).

Third, multiple steady state solutions necessitate dynamic stability analysis. From the two approaches to dynamic stability of steady states found in the literature (Gandolfo 1997, 334) the approach which investigates sufficiency conditions regarding preferences, technologies and policy parameters for dynamic stability will be adopted.[6] As in the two-country OLG model with equal saving rates, only saddle path stability of the steady state associated with the higher capital intensities can be proven.[7]

Using the conditions for saddle path stability of the larger steady state, next the steady state effects of a unilateral expansion of sustainable public debt on the real exchange rate and domestic and foreign capital intensities are examined. While established OLG wisdom that a unilateral expansion of public debt crowds out private capital in both countries is confirmed, it is not found that the real exchange rate effect depends on the net foreign asset position of the more indebted country.

Finally, it is also of interest to know whether the real exchange rate along the transition path towards the new steady state also is independent of the net foreign asset position. Here Zee (1987, 611) claims that an unexpected expansion of public debt does not impact on the real exchange rate in the shock period while its adaptation thereafter depends on the net foreign asset position of the more indebted country. By thoroughly analyzing the transitional dynamics of the real exchange rate and capital intensities for similar technologies and calculating the transition path numerically for dissimilar technologies, the steady state results will be confirmed qualitatively.

The paper is organized as follows. In the next section, the set-up of the two-good, two-country OLG model with unequal saving rates is presented and the intertemporal equilibrium dynamics is derived. Section 3 is devoted to the analysis of maximum sustainable public debt and to the investigation of the existence of steady state solutions for private capital intensities under public debt levels below maximum. Section 4 is concerned with the dynamic stability of the steady state solutions and the comparative steady state effects of

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6 The other approach adopted by Zee (1987, 615) assumes asymptotic dynamic stability as a necessary condition for comparative steady state analysis. However, perfect foresight of asset holders in deterministic OLG models makes the assumption of asymptotic stability of the exchange rate dynamics questionable.

7 Saddle-path stability implies that at least one equilibrium variable represents a jump variable. It is natural to suggest that the real exchange rate acts as a jump variable which responds immediately to parameter shocks. Brecher et al. (2005) also find saddle-path stability in an infinitely lived agent (ILA) two-country model.

shocks in sustainable public debt levels. Section 5 deals with closed-form solutions of the transitional dynamics towards the steady state under similar technologies and with a numerical calculation of economic transition under dissimilar technologies. Section 6 summarizes and concludes.

2. The two-good, two-country OLG model with unequal saving rates

In the OLG model which consists of two interdependent countries, Home and Foreign, time is discrete and is indexed by $t = 0,1,2,...$. As in Zee (1987), there are two tradable goods, X and Y^*, and each country specializes in the production of a unique composite commodity, which can be used for consumption as well as for investment purposes. The commodity produced in Home is designated by X and the one produced in Foreign by Y^*.[8] In accordance with Lin (1994) we assume Cobb-Douglas production functions in both countries with different technological levels but equal production elasticities of capital (capital income shares) across countries, i.e. $M \neq M^*, \alpha = \alpha^*$ (for short, we will refer to this assumption as internationally 'similar' production technologies).

In each period, a large number of identical firms operate under perfect competition. To produce the quantity of commodity X_t in Home, firms employ two factors of production, capital services K_t and labor services N_t, scaled by $M > 0$ to account for a total productivity parameter reflecting the technological level:

(1) $$X_t = M\left(A_t\right)^{1-\alpha}\left(K_t\right)^{\alpha}, \ 0 < \alpha < 1,$$

whereby $A_t = a_t N_t$ denotes the efficiency-weighted labor input and $0 < a_t$ is labor-efficiency per employee. The corresponding growth factor of efficiency-weighted labor is equal to the product of the time-stationary growth factor of labor productivity G^a, and the population growth factor G^L: $G^A \equiv G^a G^L$. Since firms operate in a fully competitive environment, the production elasticity of capital services α with $0 < \alpha < 1$ represents the capital income share. Analogously, $1-\alpha$ is the labor income share.

Profit maximization in Home implies:

(2) $$q_t = \alpha M\left(k_t\right)^{\alpha-1}, \ k_t \equiv K_t / A_t \ ,$$

(3) $$w_t = (1-\alpha) M\, a_t (k_t)^{\alpha},$$

whereby k_t is Home's capital efficiency-labor ratio (= aggregate capital intensity), q_t denotes the real price of capital services and w_t is the real wage rate in Home.

The production sector in Foreign is described by the following equations:

(1*) $$Y_t^* = M^* \left(A_t^*\right)^{1-\alpha} \left(K_t^*\right)^{\alpha},\ A_t^* = a_t N_t^*,$$

(2*) $$q_t^* = \alpha M^* \left(k_t^*\right)^{\alpha-1}, k_t^* \equiv K_t^* / A_t^*,$$

(3*) $$w_t^* = (1-\alpha) M^*\, a_t \left(k_t^*\right)^{\alpha}.$$

Denoting real investment in capital in Home by I_t and in Foreign by I_t^*, respectively, and assuming that both capital stocks depreciate completely within one period, real capital in Home and in Foreign accumulate over time as follows:

(4) $$K_{t+1} = I_t,$$

(4*) $$K_{t+1}^* = I_t^*.$$

As usual in the Diamond-type OLG framework, two generations of homogeneous individuals overlap in each period t. At date t, a new generation of size L_t enters the economy in Home, and in Foreign the new generation is of size L_t^*. In each period *t*, the population in both countries grows according to the common, exogenously fixed factor G^L. Each generation lives for two periods, working during the first when young and retiring in the second when old. In the following, the young generation is indexed by superscript 1 (indicating the first period of life) and the old generation is indexed by superscript 2 (indicating the second period of life). Each member of the young generation in Home (Foreign) supplies one unit of labor inelastically to firms and receives the wage rate w_t (w_t^*) in return. There is no labor-leisure choice.

Both countries are open to international trade in goods and assets (government bonds). As in Zee (1987, 605), only "the domestically produced commodity can be purchased and

[8] Henceforth all variables referring to Foreign are denoted by an asterisk.

stored by domestic residents as capital to be used in home-country production in the following period." Physical capital is therefore internationally immobile. The population does not migrate between countries.

In this framework, domestic as well as foreign households choose between consumption of domestic, x_t^1 ($y_t^{*,1}$) and of foreign commodities, y_t^1 ($x_t^{*,1}$).

The budget constraint (in real and per-capita terms) of the household living in Home, when young is

(5) $$x_t^1 + e_t y_t^1 + s_t = w_t - \tau_t ,$$

whereby $$s_t \equiv K_{t+1}/L_t + B_{t+1}^H/L_t + e_t\, B_{t+1}^{*,H}/L_t .$$

Here e_t denotes the real exchange rate (units of the Home good per unit of the foreign good), s_t is real per-capita savings and τ_t is the lump-sum tax in period t, while B_{t+1}^H/L_t and $B_{t+1}^{*,H}/L_t$ denote the stocks of domestic and of foreign government bonds which the household in Home plans to hold at the beginning of period $t+1$. Clearly, domestic real capital, domestic bonds and also foreign bonds are perfect substitutes from the perspective of Home's younger household.

When old the budget constraint of period-t young household in Home is:

(6) $$x_{t+1}^2 + e_{t+1} y_{t+1}^2 = \left(1+i_{t+1}\right)\left(K_{t+1}/L_t + B_{t+1}^H/L_t\right) + \left(1+i_{t+1}^*\right) e_{t+1}\left(B_{t+1}^{*,H}/L_t\right),$$

whereby i_t (i_t^*) denotes the real interest rate in Home (Foreign).

Home households preferences are represented by the following intertemporal log-linear utility function:

(7) $$U_t = \zeta \ln x_t^1 + \left(1-\zeta\right)\ln y_t^1 + \beta\left[\zeta \ln x_{t+1}^2 + \left(1-\zeta\right)\ln y_{t+1}^2\right],$$

whereby β ($0<\beta<1$) denotes the future discount factor and 0< ζ <1 ($1-\zeta$) is the expenditure share for domestic (foreign) commodities. Each household maximizes the utility function (7) subject to the budget constraints defined by equations (5) and (6). The optimal consumption and savings quantities of the household in Home are given in the appendix (see equations (A.1)-(A.5)).

The corresponding budget constraints for the household in Foreign are:

(5*) $$\frac{1}{e_t}x_t^{*,1}+y_t^{*,1}+s_t^*=w_t^*-\tau_t^*,$$

whereby $$s_t^*\equiv K_{t+1}^*/L_t^*+B_{t+1}^{*,F}/L_t^*+\frac{1}{e_t}\left(B_{t+1}^F/L_t^*\right),$$

(6*) $$\left(\frac{1}{e_{t+1}}\right)x_{t+1}^{*,2}+y_{t+1}^{*,2}=\left(1+i_{t+1}^*\right)\left(K_{t+1}^*/L_t^*+B_{t+1}^{*,F}/L_t^*\right)+\left(\frac{1}{e_{t+1}}\right)\left(1+i_{t+1}\right)\left(B_{t+1}^F/L_t^*\right).$$

As before, B_{t+1}^F/L_t^* ($B_{t+1}^{*,F}/L_t^*$) denotes the stock of domestic (foreign) government bonds which Foreign households hold at the beginning of period $t+1$.

The utility function of the household in Foreign is:

(7*) $$U_t^*=\zeta\ln x_t^{*,1}+\left(1-\zeta\right)\ln y_t^{*,1}+\beta^*\left[\zeta\ln x_{t+1}^{*,2}+\left(1-\zeta\right)\ln y_{t+1}^{*,2}\right],$$

whereby β^* ($0<\beta^*<1$) denotes the future discount factor of the younger household in Foreign.

Since government bonds are assumed to be perfectly mobile across Home and Foreign, a real international interest parity condition holds between the two countries:

(8) $$\left(1+i_{t+1}\right)=\left(e_{t+1}/e_t\right)\left(1+i_{t+1}^*\right).$$

The government in Home collects lump sum tax τ_t to finance the costs of public debt per-efficiency capita, $b_t\equiv B_t/A_t$:

(9) $$G^Ab_{t+1}+\tau_t/a_t=\left(1+i_t\right)b_t,\ G^A\equiv G^aG^L.$$

As in Diamond (1965, 1137), it is assumed that the government runs a ‘constant-stock’ fiscal policy (for more details see Azariadis 1993, 319 or De la Croix and Michel 2002, 216-226): $b_{t+1}=b_t=b,\ \forall t$. The lump sum tax becomes endogenous and is determined by

(10) $$\tau_t=b\left[\alpha M\left(k_t\right)^{\alpha-1}-G^A\right]a_t.$$

Equation (10) transpires that τ_t is larger than zero (i.e. it is a tax and not a subsidy) if the interest factor is larger than the natural growth factor or in other words that dynamic effi-

ciency holds. To ensure dynamic efficiency in all periods it is assumed that the parameters are appropriately chosen.

The lump-sum tax rate in Foreign is determined as follows:

(10*) $$\tau_t^* = b^* \left[\alpha M^* \left(k_t^* \right)^{\alpha - 1} - G^A \right] a_t \, .$$

Because of the competitive nature of the economy, markets clear in each period.

In equilibrium, the demand for labor in Home is equal to the total number of agents born at time *t* in Home:

(11) $$N_t = L_t \, .$$

Clearing of the labor market in Foreign requires:

(11*) $$N_t^* = L_t^* .$$

Furthermore, without loss of generality, we assume that the supply of labor is equal across countries, i.e. $L_t = L_t^*$.

The product market clearing condition of Home reads as follows:

(12) $$x_t = \left(1/a_t\right) x_t^1 + \left(1/G^L\right)\left(1/a_t\right) x_t^2 + G^A k_{t+1} + \left(1/a_t\right) x_t^{*,1} + \left(1/G^L\right)\left(1/a_t\right) x_t^{*,2} \, ,$$

whereas foreign product market clearing demands:

(12*) $$y_t^* = \left(1/a_t\right) y_t^{*,1} + \left(1/G^L\right)\left(1/a_t\right) y_t^{*,2} + G^A k_{t+1}^* + \left(1/a_t\right) y_t^1 + \left(1/G^L\right)\left(1/a_t\right) y_t^2 \, .$$

The world market for Home bonds clears according to:

(13) $$B_t = B_t^H + B_t^F \, ,$$

and symmetrically for Foreign bonds we have:

(13*) $$B_t^* = B_t^{*,H} + B_t^{*,F} \, .$$

The world asset market clearing condition requires that the total amount of savings in the world equals the total world supply of assets from Home and Foreign:

(14) $$s_t / a_t + e_t \left(s_t^* / a_t \right) = G^A \left[k_{t+1} + b + e_t \left(k_{t+1}^* + b^* \right) \right].$$

3. Intertemporal equilibrium dynamics and existence of steady states

This section derives the intertemporal equilibrium dynamics of the two-country model and investigates the existence and multiplicity of steady states. Moreover, the existence of a maximum sustainable public debt level in Home given the foreign debt level is analyzed.

From the international interest parity condition (8), the equation of motion of the real exchange rate follows, and we have:

$$e_{t+1} = e_t \left(\frac{M}{M^*} \right) \frac{\left(k_{t+1}^*\right)^{1-\alpha}}{\left(k_{t+1}\right)^{1-\alpha}}. \tag{15}$$

By inserting the optimal saving function for Home (A.5) and the analogous function for Foreign into the world asset market clearing condition (14) and considering the profit maximizing conditions (3) and (3*) as well as the equations for lump taxes (10) and (10*), the following difference equation describing the law of motion of the international asset market is obtained:

$$k_{t+1} + e_t k_{t+1}^* = \sigma_0 \left(k_t\right)^{\alpha} - b\left\{\sigma\left[\left(1+i_t\right)/G^A - 1\right]+1\right\} + e_t \sigma_0^* \left(k_t^*\right)^{\alpha} - e_t b^* \left\{\sigma^*\left[\left(1+i_t^*\right)/G^A - 1\right]+1\right\}, \tag{16}$$

$$\sigma_0 \equiv \left(1-\alpha\right)\sigma\left(M/G^A\right), \sigma_0^* \equiv \left(1-\alpha\right)\sigma^*\left(M^*/G^A\right),\ 1+i_t \equiv \alpha M \left(k_t\right)^{\alpha-1}, 1+i_t^* \equiv \alpha M^* \left(k_t^*\right)^{\alpha-1}.$$

From the two national product market clearing conditions (12) and (12*), the third dynamic equation is obtained:

$$k_{t+1} - e_t\left[\zeta/\left(1-\zeta\right)\right]k_{t+1}^* = \left(M/G^A\right)\left(k_t\right)^{\alpha} - e_t\left(M^*/G^A\right)\left[\zeta/\left(1-\zeta\right)\right]\left(k_t^*\right)^{\alpha}. \tag{17}$$

Equations (15)-(17) represent the three-dimensional dynamic system of the two-good, two-country OLG model with similar technologies across countries.

The first step needed when analyzing the system dynamics of world market equilibrium is to investigate the existence of steady state solutions, i.e. $k_{t+1} = k_t = k$, $k_{t+1}^* = k_t^* = k^*$, $e_{t+1} = e_t = e$. Lemma 1 provides a characterization of non-trivial steady state solutions for the case of similar production technologies in both countries, i.e. $\alpha = \alpha^*$ but $M \neq M^*$.

LEMMA 1. *At a fixed point of the dynamic system* (15)-(17), *Foreign and Home capital intensities are related by* $k^* = \mu k, \mu \equiv \left(M^*/M\right)^{1/(1-\alpha)}$; *the real exchange rate is*

$e=\mu^{-1}\left[(1-\zeta)/\zeta\right]$; *and Home's capital intensity is found by solving the equation* $k=\xi(1-\alpha)\left(M/G^A\right)k^{\alpha}-\left[(1-\sigma)/\sigma\right]\vartheta-(1-\zeta)b^*\mu^{-1}\left[1-\left(\sigma^*/\sigma\right)\right]-\vartheta\alpha\left(M/G^A\right)k^{\alpha-1}$,

$\xi\equiv\left[\zeta\sigma+(1-\zeta)\sigma^*\right]$, $\vartheta\equiv\zeta b\sigma+(1-\zeta)b^*\mu^{-1}\sigma^*$.

To investigate thoroughly the existence of non-trivial steady state solutions and the existence of a maximum sustainable public debt level in Home, let $\omega\equiv\left(\alpha,\beta,\beta^*,\zeta,b,b^*,G^A,M,M^*\right)$ be the parameter vector and $\Omega=[0,1]^4\times\mathbb{R}_+^5$ be the parameter space in the world market equilibrium. Moreover, define the following parameter combinations: $\nu\equiv\alpha\left\{\left[(1-\sigma)/\sigma\right]\vartheta^2+(1-\zeta)\mu^{-1}\left(1-\sigma^*/\sigma\right)\vartheta b^*\right\}\Big/\left[\xi(1-\alpha)\right]$,

$\eta\equiv\alpha\left\{(2-\alpha)\vartheta+(1-\alpha)\left[(1-\sigma)/\sigma\right]\xi\vartheta+(1-\zeta)(1-\alpha)\mu^{-1}\xi\left(1-\sigma^*/\sigma\right)b^*\right\}\Big/\left[2\xi(1-\alpha)^2\right]$,

$\kappa_+\equiv\eta+\sqrt{\eta^2+\nu}$. Using the parameter combinations let there be three mutually exclusive parameter restrictions:

ER1: $\alpha(1-\alpha)\left(M/G^A\right)(\xi\kappa_++\vartheta)(\kappa_+)^2>1$, or

ER2: $\alpha(1-\alpha)\left(M/G^A\right)(\xi\kappa_++\vartheta)(\kappa_+)^2=1$, or

ER3: $\alpha(1-\alpha)\left(M/G^A\right)(\xi\kappa_++\vartheta)(\kappa_+)^2<1$

Finally, define $\bar{k}$ such that $H(\bar{k})\equiv\left(M/G^A\right)\bar{k}^{\alpha}-\bar{k}=0$.

PROPOSITION 1 (Existence of steady state solutions).

For any $\omega\in\Omega$ *and if either*

(i) ER1 *holds, there are one trivial* ($k=0$) *and two non-trivial steady states* k^L *and* k^H *with* $0<k^L<k^H<\bar{k}$,

or if

(ii) ER2 *holds, there are one trivial and one non-trivial steady state,*

or if

(iii) ER3 *holds, there is only the trivial steady state.*

PROOF. See the appendix.■

Cases (i) and (ii) of Proposition 1 characterized by ER1 and ER2, respectively, are illustrated in Figure 1.a and Figure 1.b respectively. The steady-state line defined as

$$F(k) \equiv \xi(1-\alpha)\left(M/G^A\right)k^{\alpha} - \left[(1-\sigma)/\sigma\right]\vartheta - (1-\zeta)b^{*}\mu^{-1}\left[1-\left(\sigma^{*}/\sigma\right)\right] - \vartheta\alpha\left(M/G^A\right)k^{\alpha-1}$$

in case (i) results from the following parameter set $G^A = 1.4,\ \beta = 0.6,\ \beta^* = 0.9,\ \zeta = 0.5,\ M = 4.5,\ M^* = 5,\ b^* = 0.3,\ b = 0.15$.[9] As claimed in Proposition 1 (i), the two non-trivial steady state solutions are depicted in Figure 1.a as k^L (= lower capital labor-ratio) and k^H (= higher capital labor ratio). It is worth mentioning that the chosen parameters imply that despite of a relatively high public debt level in Foreign this country is a net foreign creditor due to her relatively high saving rate. Hence, the situation of the USA (Home) and of Japan (Foreign) can be reproduced within our two-country model.

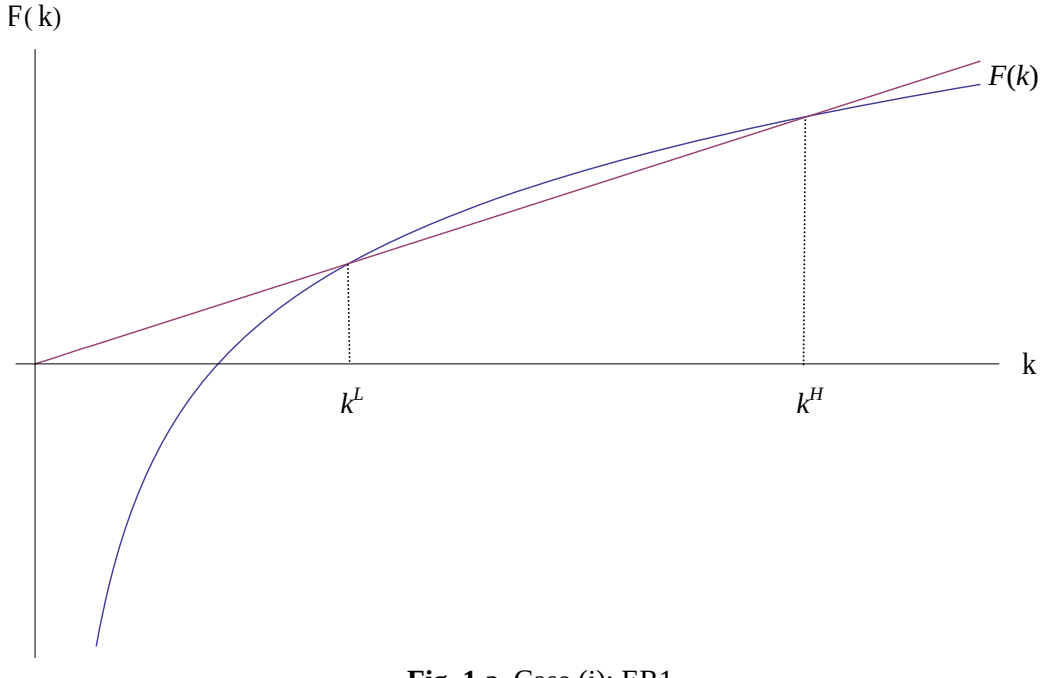

Fig. 1.a. Case (i): ER1

Case (ii) of Proposition 1 emerges if the parameter set of Case (i) is preserved with the exception of domestic public debt level which is increased from $b = 0.15$ up to $b = 0.226284$. As Figure 1.b transpires, the steady state line $F(k)$ moves downwards such that it is tangential to the 45° line above the unique non-trivial value k^{Max}.

[9] The steady state line is plotted by using MATHEMATICA 7.0. All following calculations are performed by using this software package.

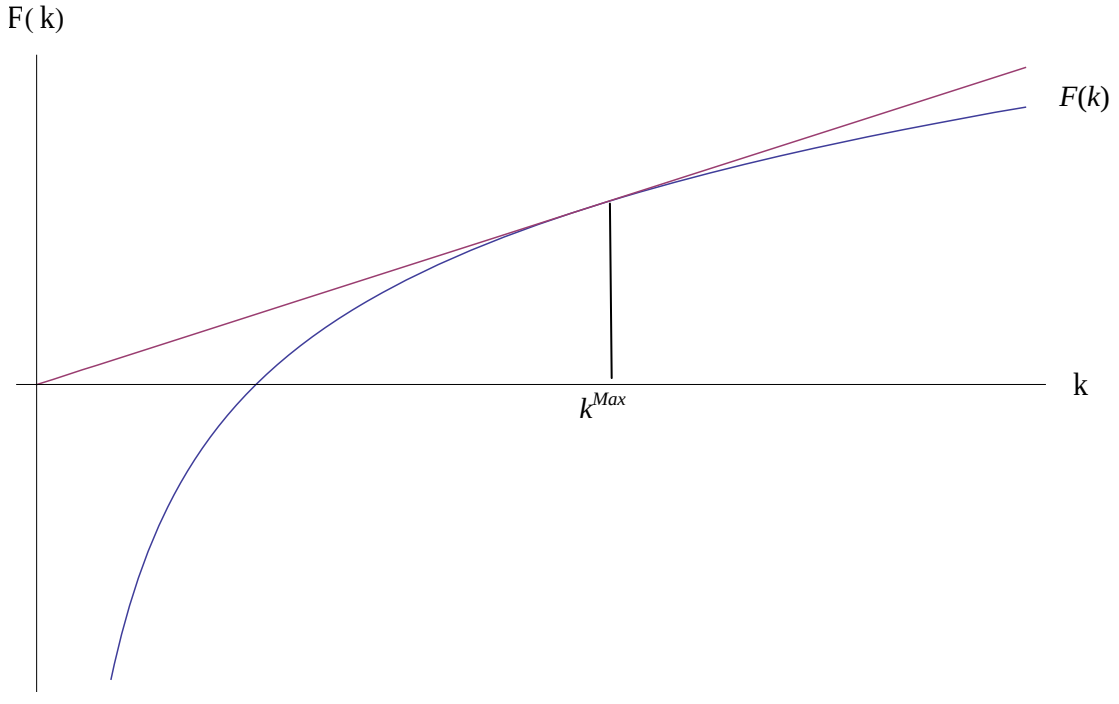

Fig. 1.b. Case (ii): ER2

Several comments with respect to Proposition 1 and the graphical illustrations in Figures 1.a and 1.b are in order. First, although the parameter restrictions comprise all $\omega \in \Omega$, the difference between ER1 and ER2 is generated by varying only the domestic public debt level b. Second, in contrast to case (i), in case (ii) the two steady state solutions k^L and k^H of case (i) collapse onto the unique steady state capital-labor ratio k^{Max}. The reason for indicating the unique steady-state capital-labor ratio as k^{Max} is that here the sustainable domestic debt level is maximal. Indeed, as in the closed economy case or in the two-country model with identical saving rates there exists a finite maximum of the domestic public debt level $\bar{b}$ implied by ER2, and it too occurs at an interior maximum rather than at a degeneracy. This claim is illustrated through Figure 2 in which b is plotted as function of k using the determining equation for k in Lemma 1.[10]

[10] The explanation for the shape of the curve plotted in Figure 2 is similar to that used by Rankin and Roffia (2003, 224) for their closed economy.

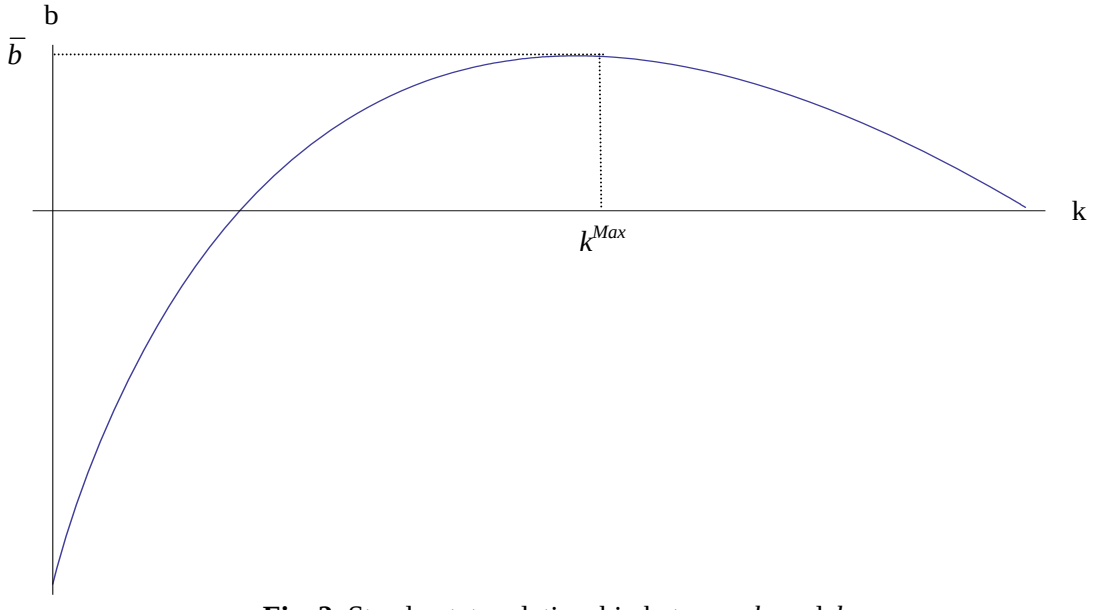

Fig. 2. Steady-state relationship between b and k.

Mathematically spoken, if $b = \bar{b}$ holds, the dynamic system undergoes a 'saddle-node bifurcation' (Azariadis 1993, 152) which represents the two-dimensional analogue to the fold bifurcation in the one-dimensional dynamic system of the closed-economy equilibrium. If b approaches $\bar{b}$, a process of unstable capital decumulation sets in, leading to a sudden implosion of the world economy.[11]

Third, the determinants of maximum sustainable public debt in Home or in Foreign also need to be identified. Focusing on the home country, we obtain immediately from Lemma 1:

(18)

$$b = \left\{ \xi(1-\alpha)\left(M/G^A\right)k^{\alpha} - k - \alpha(1-\zeta)\mu^{-1}\sigma^* b^* \left(M/G^A\right)k^{\alpha-1} - (1-\zeta)\mu^{-1}\bar{b}^*\left(1-\sigma^*\right) \right\} / \left\{ \zeta\left[\alpha\sigma\left(M/G^A\right)k^{\alpha-1} + 1 - \sigma\right] \right\}.$$

[11] It can be argued that the domestic government is unable to set b as high as $\bar{b}$ because the purchasers of government bonds would foresee the economic catastrophe of the world economy and would not buy the bonds supplied by the government. On the other hand, as Rankin and Roffia (2003, 233) rightly remark this buyer refusal "places a huge strain on the perfect foresight assumption, since the otherwise inevitable default could be many generations into the future."

Differentiating with respect to k and setting the result equal to zero yields that k which maximizes b for given b^*, and which is denoted by $k^{\max}$. In particular, $k^{\max}$ can be determined by solving the following equation:

(19)

$$\alpha(1-\alpha)\xi\sigma\left(\left(M/G^A\right)\left(k^{\max}\right)^{\alpha-1}\right)^2-\alpha\left[(2-\alpha)\sigma-(1-\alpha)\xi(1-\sigma)\right]\left(M/G^A\right)\left(k^{\max}\right)^{\alpha-1}$$
$$-1+\sigma+\alpha(1-\alpha)\left(M/G^A\right)(1-\zeta)\mu^{-1}b^*\left(\sigma^*-\sigma\right)\left(k^{Max}\right)^{a-2}=0.$$

For $\sigma^*\neq\sigma$, there is no algebraic solution of equation (19) for k^{Max}. However, for identical saving rates the last term on the right hand side of equation (19) vanishes, and $\xi=\sigma$. As a consequence, the equation (19) collapses on a quadratic equation for $\left(k^{Max}\right)^{\alpha-1}$ which exhibits a unique real and positive root. Note that for identical saving rates k^{Max} is independent of b^*. Insertion of $k^{\max}$ into the equation determining b yields for equal saving rates

(20)

$$\bar{b}=\left\{\sigma(1-\alpha)\left(M/G^A\right)\left(k^{Max}\right)^{\alpha}-k-\alpha(1-\zeta)\mu^{-1}\sigma b^*\left(M/G^A\right)\left(k^{Max}\right)^{\alpha-1}-(1-\zeta)\mu^{-1}b^*(1-\sigma)\right\}$$
$$/\left\{\zeta\left[\alpha\sigma\left(M/G^A\right)\left(k^{Max}\right)^{\alpha-1}+1-\sigma\right]\right\}.$$

Thus, for given structural parameters α, β, G^A, M, M^*, and ζ, this equation reveals a negative relationship between maximum sustainable public debt in Home and a given debt level in Foreign. In other words, the higher the public debt in Foreign, the lower the maximum public debt level has to be in Home.

As mentioned above, this argument is strictly true only for internationally identical saving rates. However, it can be generalized for the case of saving rates differing between the two countries. Suppose first that σ^* only slightly differs from σ. Since equation (18) depends continuously on σ^*, there is some small neighborhood of σ such that for all σ^* that are elements of this neighborhood the negative relationship between b^* and $\bar{b}$ remains true. Suppose next that that σ^* diverges significantly from σ. For this case we resort to numerical calculation of the relationship between b^* and $\bar{b}$ under the assumption that the structural parameters to calculate the steady-state line in Figure 2 apply. The 'scatter plot' in

Figure 3 with b^* on the abscissa and with $\bar{b}$ on the vertical axis reveals that maximal sustainable public debt in Home depends negatively on the level of public debt in Foreign even if saving rates differ significantly between both countries. Inspection of the numerical calculation reveals that although k^{Max} depends positively on b^*, the negative relationship between maximum sustainable public debt in Home and public debt in Foreign results overwhelmingly from the last negative term in the numerator of equation (18).

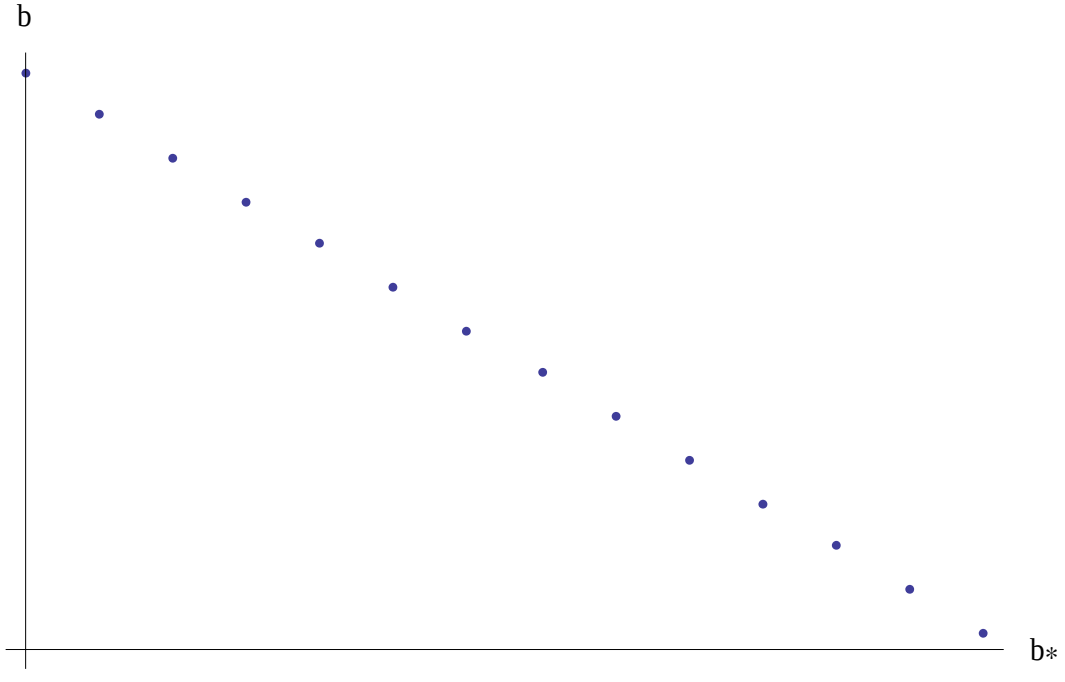

Fig. 3. The negative relationship between b^* and $\bar{b}$

In closing this section, let us look at the effects of changes in the structural parameters α, β, G^A and M on maximum sustainable public debt in Home given the foreign public debt level $b^* = 0.45$. As regards the effects of α and β on $\bar{b}$, we refer to Table 1 which shows that an increase in α and a decrease in β reduce $\bar{b}$ and the associated levels of capital intensities.[12] Moreover, a higher α and a lower β reduce the maximum sustainable debt as a

[12] The results reported in Table 1 are, qualitatively speaking, largely similar to those which Rankin and Roffia's (2003, 228) report in their Table 1. The only difference is that maximum debt as a ratio to maximum capital intensity in our model changes as β rises, which is due to our slightly different specification of the intertemporal utility function.

ratio to Home's maximal capital intensity. "The reason is that [an increase in] α and [a decrease in] β both lower the incentive to save: α because, by increasing the profit share and reducing the wage share in income, it shifts income from the first to the second period of life; and β because it raises the degree of 'impatience' in the consumers' intertemporal preferences" (Rankin and Roffia 2003, 227).

α	$\beta = 0.6$			$\beta = 0.7$		
	$\bar{b}$	k^{Max}	$\bar{b}/k^{\max}$	$\bar{b}$	k^{Max}	$\bar{b}/k^{\max}$
0.3	0.226284	0.354322	0.638638	0.264589	0.377431	0.701026
0.35	0.0865823	0.325758	0.265787	0.118213	0.34684	0.340829

Table 1. The effects of the capital income share and the future discount factor on domestic maximum sustainable debt and associated maximal capital intensity.

Table 2 shows that for given numerical values of α and β ($\alpha = 0.3, \beta = 0.6$) a higher G^A reduces $\bar{b}$ and maximal capital intensity while a higher M raises $\bar{b}$ and maximal capital intensity, but that maximum sustainable debt as a ratio to Home's capital intensity is unaffected by the latter parameter change. The intuition is that a larger natural growth factor reduces savings per efficiency capita and hence the supply on the international asset market leaving less room for public debt, while a larger factor productivity in Home raises Home production and Home wage income and hence increases the supply on asset markets enabling more public debt.

	$M = 4.5$			$M = 4.8$		
G^A	$\bar{b}$	k^{Max}	$\bar{b}/k^{\max}$	$\bar{b}$	k^{Max}	$\bar{b}/k^{\max}$
1.4	0.226284	0.354322	0.638638	0.248139	0.388543	0.638638
1.5	0.179094	0.322161	0.555915	0.196391	0.353276	0.555915

Table 2. The effects of the natural growth factor and of the level of total factor productivity on maximum sustainable domestic debt and on associated maximal capital intensity.

4. Stability of steady states and comparative steady state analysis of an increase in Home's sustainable public debt

Given that the domestic public debt level is sustainable that is it remains below the maximum sustainable debt level, this section is devoted to the investigation of dynamic stability of the steady state solutions and to comparative steady state analysis based on the stability analysis.

To analyze the dynamic stability properties in the neighborhood of non-trivial steady state solutions, the equilibrium dynamics is linearly approximated in a small neighborhood of each of the steady states The Jacobian matrix of the dynamic system (15)-(17) can be written as follows:[13]

(21) $$J=\begin{bmatrix} \partial e_{t+1}/\partial e_t & \partial e_{t+1}/\partial k_t & \partial e_{t+1}/\partial k_t^* \\ \partial k_{t+1}/\partial e_t & \partial k_{t+1}/\partial k_t & \partial k_{t+1}/\partial k_t^* \\ \partial k_{t+1}^*/\partial e_t & \partial k_{t+1}^*/\partial k_t & \partial k_{t+1}^*/\partial k_t^* \end{bmatrix} \equiv \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{bmatrix} \text{ with}$$

$$j_{11}=1+(1-\alpha)(H/k),\ j_{12}=-\left[(1-\alpha)(1+i)e\right]/\left(G^A k\right),\ j_{13}=\left[(1-\alpha)(1+i)e\right]/\left(G^A k^*\right),$$

$$j_{21}=-e^{-1}\left[(1-\zeta)H-\zeta\Phi\right], j_{22}=\left[(1+i)/G^A\right]\left[1-\zeta+\zeta\sigma(1-\alpha)(1+b/k)\right],$$

$$j_{23}=-(1-\zeta)\mu^{-1}\left[(1+i)/G^A\right]\left[1-\sigma^*(1-\alpha)\left(1+b^*/k^*\right)\right],$$

$$j_{31}=\zeta\mu(H-\Phi)/e, j_{32}=-\zeta\mu\left[(1+i)/G^A\right]\left[1-\sigma(1-\alpha)(1+b/k)\right],$$

$$j_{33}=\left[(1+i)/G^A\right]\left[\zeta+(1-\zeta)\sigma^*(1-\alpha)\left(1+b^*/k^*\right)\right], \text{whereby}$$

$$H\equiv\left(M/G^A\right)k^\alpha-k=\left[(1+i)/\left(G^A\alpha\right)-1\right]k \text{ and } \Phi\equiv k+b\left\{\sigma\left[(1+i)/G^A-1\right]+1\right\}-\sigma_0 k^\alpha .$$

To correctly evaluate local dynamic stability of non-trivial steady states in the world market equilibrium, information on the eigenvalues of the Jacobian matrix (21) denoted by $\lambda_i, i=1,2,3$ is needed. Lemma 2 provides this information.

LEMMA 2. *The three eigenvalues of the Jacobian evaluated at non-trivial steady states read as follows:*

$$\lambda_1=(1+i)/\left(G^A\alpha\right), \lambda_2=\alpha, \lambda_3=\left[(1+i)(1-\alpha)/G^A\right]\left\{\zeta\sigma(1+b/k)+(1-\zeta)\sigma^*\left(1+b^*/k^*\right)\right\}.$$

PROOF. See the proof of Lemma 2 in the appendix.

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[13] For hints on how to derive the elements of the Jacobian matrix J see the appendix.

Knowledge of the three eigenvalues of the Jacobian enables us to state Proposition 2, which deals with the dynamic stability of both steady states.

PROPOSITION 2 (Dynamic stability of steady states).

Suppose that ER1 *holds.*[14] *Then, the steady state solution* $\left(e, k^H, k^{*,H}\right)$ *is saddle-path stable, i.e.* $\lambda_1 > 1,\ \lambda_2 < 1, \lambda_3 < 1$, *while the steady state solution* $\left(e, k^L, k^{*,L}\right)$ *is saddle-path unstable* ($\lambda_1 > 1,\ \lambda_2 < 1, \lambda_3 > 1$).

PROOF. From Lemma 2 we know that $\lambda_2 = \alpha$ and by assumption $\alpha < 1$ holds. Second, $\lambda_1 = (1+i)/\left(G^A \alpha\right) > 1$ follows from $k^L < k^H < \bar{k}$, since $k < \bar{k} \Leftrightarrow 1 < \left(M/G^A\right)k^{\alpha-1} = (1+i)/\left(G^A\alpha\right)$. $\lambda_3 < 1 \Leftrightarrow$

$\left[\left(1+i^H\right)(1-\alpha)\right]/\left(G^A\right)\left\{\zeta\sigma\left(1+b/k^H\right)+(1-\zeta)\sigma^*\left(1+b^*/\left(k^*\right)^H\right)\right\} < 1$. This follows from the following facts: (i) $\left[\left(1+i^H\right)(1-\alpha)\right]/\left(G^A\right)\left\{\zeta\sigma\left(1+b/k^H\right)+(1-\zeta)\sigma^*\left(1+b^*/\left(k^*\right)^H\right)\right\} < 1$ is equal to the derivative of the function $F(k)$ with respect to *k* evaluated at the higher steady state, (ii) function $F(k)$ is strictly concave and (iii) its graph cuts the 45° line at the higher steady state from above (see Figure 1 above). On the other hand, for $\lambda_3 > 1$ we have $\left[\left(1+i^L\right)(1-\alpha)\right]/\left(G^A\right)\left\{\zeta\sigma\left(1+b/k^L\right)+(1-\zeta)\sigma^*\left(1+b^*/\left(k^*\right)^L\right)\right\} > 1$ which is implied by the fact that the graph of $F(k)$ cuts the 45° line at the lower steady state from below. ■[15]

Now that we know that in the case of ER1 only the higher steady state solution k^H is saddle-path stable, let us turn to comparative steady state analysis in a small neighborhood of k^H. We first investigate how the capital intensities of Home and Foreign and the real exchange rate responds to an infinitesimal shock in Home's sustainable public debt.

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14 The eigenvalues of the Jacobian evaluated at the single steady state associated with ER2 are as follows: the first is larger than one, the second eigenvalue equals α and the third is equal to unity. As in the case of ER1 not all eigenvalues of the Jacobian are less than one, as Zee (1987) would have it!

15 The proof of Proposition 2 confirms the conjecture mentioned in footnote 7 above.

To clarify the role of the net foreign asset position of Home for the steady state effects of sustainable public debt in our two-country OLG model, we follow Zee (1987, 609) and write the condition for the international asset market equilibrium in the following equivalent form:

(22) $$e = -\Phi/\Phi^*,\ \Phi \equiv k + b\left\{\sigma\left[\left(1+i\right)/G^A - 1\right] + 1\right\} - \sigma_0 k^\alpha\ ,$$

$$\Phi^* \equiv k^* + b^*\left\{\sigma^*\left[\left(1+i\right)/G^A - 1\right] + 1\right\} - \left(\sigma_0\right)^*\left(k^*\right)^\alpha,$$

whereby Φ $\left(\Phi^*\right)$ denotes the net foreign asset position of Home (Foreign).

PROPOSITION 3 (Steady state effects of unilateral sustainable debt policy). *Suppose there is an infinitesimal change of* b *while* b^* *remains unchanged* (*unilateral sustainable debt policy*). *Then, there is a negative relationship between the change of capital intensities in Home and Foreign and per-capita debt, i.e.* $dk/db < 0$ and $dk^*/db = \mu\, dk/db < 0$, *while the real exchange rate is not affected at all, i.e.* $de/db = 0$.

PROOF. $de/db = 0$ is obvious from Lemma 1. To prove $dk/db < 0$, differentiate totally (22) with respect to k, k^*, e and b. Since $de = 0$ and $dk^* = \mu dk$ hold, we obtain: $\left\{\left[1 - \left[\sigma\left(1-\alpha\right)\left(1+i\right)\right]/G^A\left(1 + b/k^H\right)\right] + e\left[1 - \left[\sigma^*\left(1-\alpha\right)\left(1+i\right)\right]/G^A\left(1 + b^*/\left(k^*\right)^H\right)\right]\mu\right\}dk +$ $\left(1 + \sigma\left[\left(1 + i - G^A\right)/G^A\right]\right)db$. After inserting $e = \mu^{-1}\left(1-\zeta\right)/\zeta$ and collecting terms, we get: $\left\{1 - \left[\left[\left(1-\alpha\right)\left(1+i^H\right)\right]/G^A\right]\left[\zeta\sigma\left(1 + b/k^H\right) + \left(1-\zeta\right)\sigma^*\left(1 + b^*/\left(k^*\right)^H\right)\right]\right\}dk =$ $-\zeta\left(1 + \sigma\left[\left(1 + i^H - G^A\right)/G^A\right]\right)db$ or: $dk/db = -\zeta\left(1 + \sigma\left[\left(1 + i - G^A\right)/G^A\right]\right)/\left[1 - \lambda_3\right]$. Since $\lambda_3 < 1$ from Proposition 2, it follows that $dk/db < 0$.■

Proposition 3 claims that a larger public debt in Home always crowds out private capital in Home *and* in Foreign, while the real exchange rate does not respond at all. The intuition behind this result is as follows. In the steady state an increase of domestic sustainable public debt causes lump-sum taxes to rise in order to pay for the additional interest. As a consequence, net wage and savings of the young household decline. To restore equilibrium, capital intensities in both countries decrease according to a fixed proportion, and hence the real interest in Home and Foreign increases equivalently thus leaving, in accordance with inter-

national interest parity condition following from (15) in steady state, the real exchange rate unaffected.

An important implication of Proposition 3 is that the net asset position of the home country is not decisive at all for the steady state response of the real exchange rate to a larger (or lower) sustainable public debt. This implication plainly contradicts Zee's (1987, 617) claim that a "higher level of domestic public debt leads to a fall (rise) in the terms of trade [= inverse of our real exchange rate, K. F.] if, at the initial steady state, the home country is a net debtor (creditor)."

To emphasize, in contrast, the independence of steady state real exchange rate from the foreign net asset position of the more indebted country, we present in Figure 4 and Figure 5, respectively, the phase lines (steady-state lines) of the dynamic system (15)-(17) when Home is a net foreign debtor or a net foreign creditor.

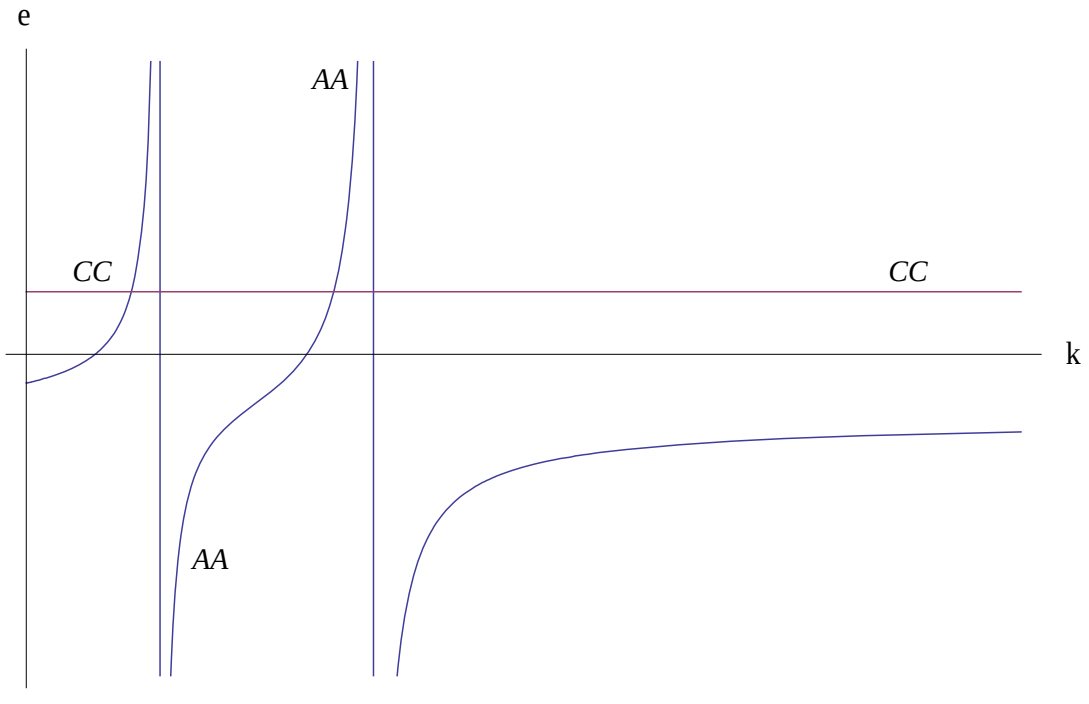

Fig.4. Steady-state lines for Home being a net foreign debtor.

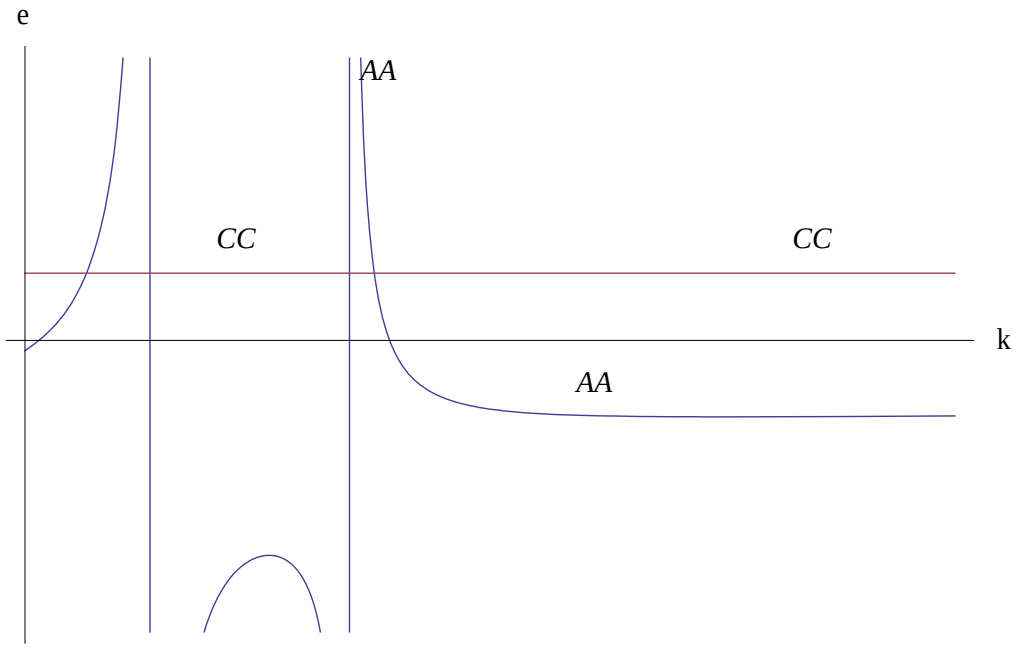

Fig.5. Steady-state Lines for Home being a net foreign creditor.

There are two steady state lines in a (k_t, e_t) diagram, termed the AA- and the CC-curve. The AA-curve (= equation (22)) can be interpreted as geometrical locus of all pairs (k_t, e_t) which assures international asset market clearing. The CC-curve ($e = \mu^{-1}\left[\zeta/(1-\zeta)\right]^{-1}$) represents all (k_t, e_t) combinations which induce equilibrium in the combined commodity markets of Home and Foreign. Clearly, the CC-curve is horizontal in the (k_t, e_t) diagram. Upon differentiating (22) with respect to k while taking into account $k^* = \mu k$, it is not difficult to show that the slope of the AA-curve is determined as follows: $de/dk_{|AA} = -\left(\partial\Phi/\partial k + e\mu\left(\partial\Phi^*/\partial k^*\right)\right)/\Phi^*$. Since the numerator on the right hand side of this expression is always larger than zero,[16] the slope of the AA-curve depends on the nega-

[16] To show that the numerator on the right hand side is larger than zero notice that $\partial\Phi/\partial k = 1-(1-\alpha)(1+i)\sigma(1+b/k)$ and a similar expression for Foreign hold. Averaging over both expressions and taking into account the stability condition $\lambda_3 < 1$ (at $k = k^H$) we see that the claim is true.

tive sign of Φ^*: if Home is a net foreign debtor ($\Phi > 0$) (Figure 4), the AA-curve is positively sloped, and its slope is negative, if Home is a net foreign creditor ($\Phi < 0$) (Figure 5).

Let us now consider a marginal change of *b*. It is already clear that a *b* shock has no impact on the CC-curve. It does, however shift the AA-curve. Analytically, this shift is determined as follows: $de/db_{|CC} = -(\partial\Phi/\partial b)/\Phi^*$. Since $\partial\Phi/\partial b = \left[\sigma(1+i)+(1-\sigma)G^A\right]/G^A > 0$, the sign of the foreign asset position of Home governs the shift: if Home is a net foreign creditor, $de/db_{|CC} < 0$, hence the AA-curve shifts downwards. The opposite is true when Home is a net foreign debtor. But whatever the shift of the AA-curve, the real exchange rate does not change whether Home is a net foreign creditor or a net foreign debtor.[17]

5. Transitional impacts of shocks in sustainable public debt

Knowing that the steady state real exchange rate effect of public debt is independent of the net foreign asset position, it is natural to ask whether the transition path of the real exchange rate towards the new steady state is also independent of the net foreign asset position or not. Here again Zee (1987, 611) claimed that in the period after the expansion of public debt the real exchange rate would rise (fall) if Home is a net debtor (creditor), and that in the shock period itself there is no impact on the real exchange rate at all.

In investigating the transitional effects of marginal changes in Home's public debt on Home and Foreign capital intensities and on the real exchange, we have to take into account that the initial steady state is only saddle-path and not asymptotically stable (as assumed by Zee). This implies that one of the endogenous dynamic variables is a 'jump' variable not exogenously determined by initial conditions. A natural conjecture is that the real exchange rate represents the jump variable which immediately responds to a policy shock while the 'sluggish' capital stocks (per efficiency capita) in Home and Foreign do not adapt because their values are historically fixed.

To get more information about the analytical structure of the three-dimensional equilibrium dynamics around the stable steady state solution, we approximate (15)-(17) in a small neighborhood of $\left(e, k^H, k^{*,H}\right)$:

(23) $$\begin{bmatrix} e_{t+1} \\ k_{t+1} \\ k^*_{t+1} \end{bmatrix} = \left[J\left(e, k^H, k^{*,H}\right) \right] \begin{bmatrix} e_t \\ k_t \\ k^*_t \end{bmatrix} + \left[I - J\left(e, k^H, k^{*,H}\right) \right] \begin{bmatrix} e \\ k^H \\ k^{*,H} \end{bmatrix}.$$

The general solution of the first-order linear difference equation system (23) takes the following form:

(24) $$\begin{aligned} e_t &= e + \kappa_2 \upsilon_2^e \left(\lambda_2\right)^t + \kappa_3 \upsilon_3^e \left(\lambda_3\right)^t, \\ k_t &= k + \kappa_2 \upsilon_2^k \left(\lambda_2\right)^t + \kappa_3 \upsilon_3^k \left(\lambda_3\right)^t, \\ k_t^* &= k^* + \kappa_2 \upsilon_2^* \left(\lambda_2\right)^t + \kappa_3 \upsilon_3^* \left(\lambda_3\right)^t. \end{aligned}$$

Here κ_i, $i = 2,3$ denote constants determined by initial conditions for capital intensities in Home and Foreign, while $\upsilon_i = \left(\upsilon_i^e, \upsilon_i^k, \upsilon_i^*\right)^T$, $i = 2,3$ is the eigenvector associated with the eigenvalues within the unit circle λ_i, $i = 2,3$. Note that the eigenvector associated with the eigenvalue larger than unity is excluded from (24) by setting $\kappa_1 = 0$. However, this exclusion implies that the equilibrium dynamics must not start from any feasible combination $\left(e_0, k_0, k_0^*\right)$ in the neighborhood of $\left(e, k^H, k^{*,H}\right)$, and that the initial combination of dynamic variables has to be located on the stable submanifold in the $\left(e_t, k_t, k_t^*\right)$-space. If $\left(e_0, k_0, k_0^*\right)$ belongs to the stable submanifold, the economy converges on $\left(e, k^H, k^{*,H}\right)$, otherwise the system dynamics strays in finite time. Before expanding these informal claims thoroughly in Proposition 4, Lemma 3 describes the eigenvectors associated with the less than unity eigenvalues.

LEMMA 3. *The eigenvectors associated with the eigenvalues of the Jacobian* (21) *within the unit circle* υ_i, $i = 2,3$ *read as follows:*

$$\upsilon_2 = \left(-e/k, 1/\alpha + \gamma, \gamma\mu\right)^T, \upsilon_3 = \left(0, 1, \mu\right)^T, \gamma \equiv \left[\zeta b\left(1 - \sigma + \sigma\lambda_1\right)\right] / \left[k\left(\alpha - \lambda_3\right)\right].$$

PROOF. See the Appendix.

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17 This is another way to show that the real exchange rate effect of public debt variations is independent of the net foreign asset position, and this remains true whether similar or dissimilar production technologies are assumed.

The eigenvalues of Lemma 2 and the eigenvectors in Lemma 3 enable us to present in Proposition 4 a closed form solution of the transitional dynamics around the higher (saddle path stable) steady state $\left(e, k^H, k^{*,H}\right)$.

PROPOSITION 4 (Transitional dynamics around the higher steady state).
A linear approximation of the two-country, two-good equilibrium dynamics (20)-(22) *evaluated at* $\left(e, k^H, k^{*,H}\right)$ *takes the following form:*

$$e_t = e\left\{1 + \left(\alpha / k^H\right)\left[\mu^{-1}\left(k_t^* - k^{*,H}\right) - \left(k_t - k^H\right)\right]\right\}, \tag{25}$$

$$k_{t+1} = k_t + \alpha\left\{\left[\left(1-\alpha\right)/\alpha - \left(\lambda_3 - \alpha\right)\gamma\right]\left(k^H - k_t\right) + \left(1/\alpha + \gamma\right)\left(\lambda_3 - \alpha\right)\mu^{-1}\left(k_t^* - k^{*,H}\right)\right\}, \tag{26}$$

$$k_{t+1}^* = k_t^* + \alpha\left[\left(1-\lambda_3\right)/\alpha - \gamma\left(\lambda_3 - \alpha\right)\right]\left(k^{*,H} - k_t^*\right) + \alpha\gamma\left(\alpha - \lambda_3\right)\mu\left(k_t - k^H\right), \tag{27}$$

whereby k_0 and k_0^* are exogenously given.

PROOF. See the appendix.

The equilibrium dynamics depicted by equations (25)-(27) enable us to evaluate the immediate effects of a shock in the per-capita public debt of Home. Suppose that the shock occurs in period $t = 0$, it is unannounced and permanent. In view of Proposition 4, Corollary 2 below describes the immediate impacts of a small government-debt shock in Home if the economy starts on the stable submanifold in the neighborhood of $\left(e, k^H, k^{*,H}\right)$.

COROLLARY 2. *Suppose there is a finite but small change of* b *(while* b^* *remains unchanged) in period* $t = 0$*, such that* ER1 *holds even after the* b*-shock. Then, the real exchange rate of the shock period* e_0 *remains unchanged* ($de_0/db = 0$)*, while Home and Foreign capital intensities one period later exhibit a negative response to the policy shock as follows*: $dk_1^*/db = \mu\left(dk_1/db\right) < 0$.

PROOF. We know from Lemma 1 that steady-state capital intensities in Home and Foreign are related as follows: $k^{*,H} = \mu k^H$. Since the economy starts in a steady state and initial capital intensities do not respond to the policy shock, it must be true that $k_0^* = \mu k_0$ holds. Moreover, $de/db = 0$ holds. Hence, in view of (25) for $t = 0$, $de_0/db = 0$ follows

immediately. In order to show that $dk_1^*/db = \mu\left(dk_1/db\right) < 0$ is true, we consider (26) and (27) for $t = 0$, and after slight manipulations the following equations are obtained:

$k_1 = k_0 + \alpha\left\{\gamma\left(\lambda_3 - \alpha\right)\left[k^H - k_0 + \mu^{-1}\left(k_0^* - k^{*,H}\right)\right]\right\} + \left(1-\alpha\right)\left(k^H - k_0\right) + \left(\lambda_3 - \alpha\right)\mu^{-1}\left(k_0^* - k^{*,H}\right)$,

$k_1^* = k_0^* + \alpha\gamma\left(\lambda_3 - \alpha\right)\left[\mu\left(k_0 - k^H\right) + \left(k^{*,H} - k_0^*\right)\right] + \left(1-\lambda_3\right)\left(k^{*,H} - k_0^*\right)$. On account of $k_0^* = \mu k_0$ and $k^{*,H} = \mu k^H$, the equations for k_1 and k_1^* collapse to the following equations for both variables: $k_1^* = \mu k_1 = \mu\left[k_0 + \left(1-\lambda_3\right)\left(k^H - k_0\right)\right]$. Since $dk^H/db < 0$ and $\lambda_3 < 1$, it follows that $dk_1^*/db = \mu dk_1/db < 0$.■

Thus, an increase in Home's sustainable public debt in period $t = 0$ unambiguously reduces the capital intensities of Home and Foreign in period 1, but has no immediate impact on the real exchange rate in the shock period.

The insensitivity of the initial real exchange rate with respect to permanent changes in Home's public debt is most easily explained if we focus in the shock period on the Golden Rule case (i.e. $1 + i_0 = G^A$) in which lump sum taxes do not respond to $b-$variations (see equation (10)). As a consequence, per-capita savings in Home do not respond to the policy shock because the net wage of Home's young household is unchanged. The reason is that k_0 and therefore the gross wage income and lump sum taxes remain unaffected. In Foreign there is no policy change, hence per-capita savings in Foreign do not adapt to the policy shock in Home. Moreover, assume for simplicity that $\zeta = 1/2$, $G^A = 1$ and $M = M^*$ hold. Under these assumptions Corollary 2 implies that $\Delta k_1 = \Delta k_1^*$. Evaluating equation (17) for $t = 0$, we obtain an equation determining the initial real exchange rate: $e_0 = \left[\left(1-\zeta\right)/\zeta\right]\left[k_1 - \left(M/G^A\right)\left(k_0\right)^\alpha\right] \Big/ \left[k_1^* - \left(M^*/G^A\right)\left(k_0^*\right)^\alpha\right]$. Hence, since $\Delta k_1 = \Delta k_1^*$, $\Delta e_0 = 0$.

If the initial real exchange remains unchanged and the initial steady state is Golden Rule, the question remains open of what causes the crowding out of private capital in period 1 in Home and in Foreign. To answer this question we have to investigate which domestic and foreign endogenous variables in the shock period are adapting to the expansion of public debt in Home, and which endogenous variables remain unchanged. Starting with the latter,

it is clear that the production of Home's and Foreign's commodity as well as the consumption demand for both goods by the younger households in Home and Foreign do not adapt to the policy shock. What is true for the consumption demand of Home and Foreign younger households, is, however, not true with respect to the consumption demand of Home's old household for the domestic and the foreign commodity.

To see this, let's look at the consumption demand of Home's old household for the domestic good in the shock period, $x_0^2 = \zeta G^L a_0 (1+i_0)[k_0 + b - \Phi_0]$ (equation A.3 in the appendix). On the right hand side of this equation a_0, i_0, k_0 and Φ_0 are historically fixed while b is the policy parameter. When sustainable public debt per capita b in Home increases, then x_0^2 rises because a larger sustainable public debt raises the wealth of the old household in Home, and her larger consumption crowds out k_1 on account of the market clearing condition (12) in the shock period. The larger wealth induces the old household in Home also to increase the consumption of the Foreign commodity $y_0^2 = (1-\zeta) G^L a_0 (1+i_0)(e_0)^{-1}[k_0 + b - \Phi_0]$ (equation A.4 in the appendix) which crowds out k_1^* in view of the market clearing condition (12*). In this way the old household in Home, enriched by the interest-inclusive repayment of the larger debt of Home's government, transmits the domestic debt policy shock from Home to Foreign.

Knowing that there is no immediate impact on the real exchange rate in the shock period, while in the after shock period both capital intensities decline, it remains to be analyzed what happens thereafter. Corollary 3 claims that the real exchange rate still remains unaffected while capital intensities continue to fall towards their lower steady state values.

COROLLARY 3. *Suppose there is in period* $t = 0$ *a finite but small change of* b *(while* b^* *remains unchanged) such that* ER1 *holds even after the* b*-shock. Then, the real exchange rate of the periods following the shock,* $e_t, t = 1, 2, ..$ *remain unchanged* ($de_t / db = 0$, $t = 1, 2, ...$), *while Home and Foreign capital intensities in all periods later exhibit a proportional negative response to the policy shock* ($dk_{t+1}^* / db = \mu dk_{t+1} / db < 0$, $t = 1, 2, ..$).

PROOF. We know that $de/db = 0$ and from the proof of Corollary 2 we know that $k_1^* = \mu k_1$ holds. Evaluating (25) at $t = 1$ we see immediately that $de_1 / db = 0$ since again

$k^{*,H} = \mu k^H$ holds. Similarly as in the proof of Corollary 2, we obtain the following equations: $k_2^* = \mu k_2 = \mu\left[k_1 + (1-\lambda_3)\left(k^H - k_1\right)\right]$. Reiterating this procedure for $t = 2,3...$, and calculating the derivatives of the capital intensities in Home and Foreign of each period with respect to b the proof is completed. ■

As a final step, we extend our analysis to unequal capital income shares (dissimilar technologies) across countries, i.e. $\alpha \neq \alpha^*$. Due to analytical complexity, we resort to numerical illustrations of four typical cases. They follow from possible combinations of net foreign asset positions in Home and in Foreign ($\Phi < 0 \wedge \Phi^* > 0$ or $\Phi > 0 \wedge \Phi^* < 0$) and larger-smaller relationships between the magnitudes of domestic and foreign capital income shares ($\alpha < \alpha^*$ or $\alpha > \alpha^*$). In case 1, Home is a net foreign creditor ($\Phi < 0$) and her capital income share is less than in Foreign ($\alpha < \alpha^*$), while in case 2 the same holds for capital income shares and Home is a net foreign debtor ($\Phi > 0$). Analogously, in case 3 Home is a net foreign creditor ($\Phi < 0$) with a larger capital income share than in Foreign while in case 4 the same holds for capital income shares with Home being a net foreign debtor.

	Pre shock steady state			Shock period			Post shock steady state		
	e	k	k^*	e_0	k_1	k_1^*	e	k	k^*
Case 1: $\Phi < 0$ $\alpha < \alpha^*$	0.98272	0.89455	1.15150	0.98120	0.87554	1.12661	0.98379	0.85103	1.09158
Case 2: $\Phi > 0$ $\alpha < \alpha^*$	0.98288	0.88742	1.14167	0.97825	0.82926	1.06545	0.98777	0.73358	0.93101
Case 3: $\Phi < 0$ $\alpha > \alpha^*$	1.017407	1.14167	0.88742	1.01899	1.11680	0.86841	1.016466	1.08238	0.84433
Case 4: $\Phi > 0$ $\alpha > \alpha^*$	1.01758	1.15150	0.89455	0.97867	1.021794	0.83465	1.01284	0.94597	0.74458

Table 3: Effects of an increase in debt per capita ($\Delta b/b = 0.2$) on capital intensities in Home and Foreign and on the real exchange rate.

Table 3 reports on the results of an increase in *b* by 20% for the impact in the shock period and the new steady state. [18] For the numerical analysis we set for $\alpha < \alpha^*$ $\alpha = 0.25, \alpha^* = 0.3$ (with values being reversed for $\alpha > \alpha^*$) and for $\Phi < 0$ we set $b = 0.05, b^* = 0.3$ ($b = 0.15, b^* = 0.3$ for $\Phi > 0$).

As the results reported in Table 3 show, the most obvious difference between dissimilar and similar technologies concerns the real exchange rate. First, the exchange rate does adapt to a policy shock *both* in the shock period *and* thereafter. Second, the new steady state values are lower (higher) compared to the pre-shock steady state when $\alpha > \alpha^*$ ($\alpha < \alpha^*$), regardless of whether Home is a net foreign debtor or net foreign creditor. Third, in cases 1 and 2 the real exchange rate in the shock period, e_0, is smaller than their old steady state value, and they converge from below towards the higher, new steady state value, while in cases 3 and 4 the opposite holds.

In spite of these differences one basic characteristic of the transition path of the real exchange rate under similar technologies generalizes to the case of dissimilar technologies: the transition path is independent of the sign and magnitude of the net foreign asset position of the more indebted country. This is in clear contrast to Zee's (1987, 611) claim mentioned above.[19] Given that the two countries in our model truly represent the collection of net foreign creditor and net foreign debtor countries, the result implies that the transitional real exchange rate effect (if any) is independent of whether sustainable public debt is expanded in a net foreign creditor country or in a net foreign creditor country.

6. Summary and conclusions

This paper is concerned both with the sustainability of public debt and the real exchange rate effects of sustainable public debt shocks in a two-good, two-country OLG model with internationally diverging saving rates. We explore analytically the limit for sustainable national public debt levels at which a saddle-node bifurcation of the equilibrium dynamics

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18 The transitional dynamics were calculated numerically using the NLP solver of GAMS 2.5, version 21.5.

19 Zee's (1987) conclusion appears to be based on his presumption of asymptotic stability of the real exchange rate dynamics. Both the irresponsiveness of the initial real exchange rate and the dependence of the exchange rate dynamics thereafter on the sign of the net foreign asset position (Zee 1987, 611) can be traced back to this presumption.

could occur (maximum sustainable public debt). Moreover, we investigate the existence and dynamic stability of steady states for private capital intensities and the real exchange as well as their transitional dynamics when public debt levels remain below these limits (are sustainable).

Regarding maximum sustainable public debt, we find that an upper limit for the domestic level of public debt analogous to Rankin and Roffia's (2003) maximum sustainable public debt always exists, and it occurs at a non-trivial steady state (interior maximum) rather than at a trivial steady state solution for capital intensities. The determinants of this maximum in the two-country world economy are similar to those found in the closed economy: a higher capital income share, more impatience, a higher natural growth factor and less factor productivity limit the range for maximum sustainable debt. Moreover, national maximum debt levels are negatively related.

As regards the second main topic, we find that if ER1 holds (where domestic public debt is sustainable), two non-trivial (non-degenerate) steady states for private capital intensities exist in both countries. The existence condition for non-trivial steady state solutions also implies that the steady state with the higher capital intensity is saddle path stable while the other is saddle path unstable - a result which contrasts with the presumption of Zee (1987) that the real exchange rate dynamics in his two-good, two-country OLG model are asymptotically stable. At the higher, saddle-point stable steady state the foreign country can be a net foreign creditor country despite the relatively high public debt while Home can be a net foreign debtor country in spite a relatively low public debt. Hence, the two-country OLG model with markedly higher saving rate in Foreign than in Home is capable to mimic the Japan-US example.

The proof of the saddle-path stability of the steady state with the higher level of domestic and foreign capital intensities provides us with a methodological justification reexamining the effects of a unilateral expansion of sustainable public debt on steady state real exchange rate as well as on steady state domestic and foreign capital intensities. We are able to confirm Lin's (1994) result that the real exchange rate in steady state is unaffected by a unilateral public debt shock when capital income shares are equal across countries. Capital income share equality accords exactly with our assumption of internationally similar production technologies. We have to reject Zee's (1987) conclusion that the steady state effect of public debt shocks on real exchange rate depends on the net foreign asset position of the

more indebted country. As regards the steady state effects of unilateral debt policy on private capital intensities in Home and Foreign, we affirm the negative relationship as already stated by Zee (1987) and Lin (1994). Here, both approaches to dynamic stability in two-country OLG models apparently lead to the same results.

The knowledge that the steady state with the larger capital intensity is saddle-path stable also enables a thorough investigation of the transitional effects of small variations in domestic sustainable public debt. Although our transitional analysis takes the jump character of the real exchange rate fully into account, we find for the case of internationally similar technologies that the initial real exchange rate is not affected by an unexpected and unannounced shock in Home's sustainable public debt, while Home and Foreign capital intensities in the after-shock period decline in a fixed proportion, absorbing in this way alone the full policy shock. We also show that under similar technologies the real exchange rate does not respond to public debt shocks along the transition path towards the new steady state too. Again, the international borrower-lender status of a country does not matter for this result.

Finally, a numerical analysis of four typical parameter constellations under dissimilar production technologies shows that unilateral debt policy also affects the real exchange rate along the transition path towards the steady state. In accordance with the notion of saddle-path stability the real exchange rate in the shock period not only immediately adapts to the policy shock but the exchange rate usually overshoots its new steady state value. Moreover, while the transition path of the real exchange rate depends on the relative magnitude of domestic, in comparison to foreign, capital income shares, it is independent of the sign of the net foreign asset position in Home. The numerical results, obtained from typical parameter sets, contradict two main conclusions which Zee (1987) derived analytically: Namely that (1) the real exchange rate in the shock period is unimpaired by a public debt shock, and (2) the rise or fall of the real exchange rate along the transition path towards the new steady state depends on the sign of Home's net foreign asset position.

What can we conclude about the effects of a unilateral expansion of the stock of public debt on the real exchange rate and capital accumulation in a world economy consisting of two groups of countries characterized by opposite net foreign asset positions but equal capital income shares? First, there is a finite limit to the public debt level (maximum sustainable debt) in one group of countries given the public debt level in the other group. Hence, the often heard claim that an internationally coordinated expansion of public debt is feasible can-

not find confirmation in our two-good, two-country OLG model with internationally diverging saving rates. Second, if national debt levels are sustainable a unilateral expansion of public debt crowds out private capital in both (groups of) countries, but there is no steady state effect on the real exchange rate if only total factor productivities differ across both groups of countries. This result might help explain why more public debt in a large net foreign debtor country (like the USA) does not have negative impacts on the real exchange rate, and also why, for a net foreign creditor country (like Japan) despite of a large public debt level it does not have positive real exchange rate effects. Third, even if capital income shares are different among both groups of countries the real exchange rate effect of a unilateral expansion of sustainable public debt is nonetheless independent of whether the debt-expanding country is a net foreign creditor or a net foreign debtor, and of how large the net foreign credit or net foreign debt of the country is.

The analysis of a two-country OLG model with internationally differing production elasticities represents the most natural extension of the paper. Moreover, in view of dramatically rising debt to GDP ratios throughout the world a dynamic computable general equilibrium (CGE) model with many countries and multiple commodities and production factors and with estimated or calibrated parameters would be highly relevant.

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Appendix

Optimal consumption and savings of households in Home (*Foreign*)

In order to show how the equations of motion in world market equilibrium are derived, the optimal consumption and savings levels of households are needed for $t = 1, 2, \ldots$. We indicate roughly how optimal consumption and savings for households in Home are obtained. First, insert s_t into the second budget constraint, while taking the international interest parity condition (8) into account. This implies:

$$x_t^1 + e_t y_t^1 + x_{t+1}^2 \big/ \left(1 + i_{t+1}\right) + \left(e_{t+1} y_{t+1}^2\right) \big/ \left(1 + i_{t+1}\right) = w_t - \tau_t \,.$$

Second, maximize (7) subject to this intertemporal budget constraint and solve for optimal consumption quantities and optimal savings. An analogous procedure gives the optimal consumption quantities and optimal savings in Foreign.

(A.1) $$x_t^1 = \left[\zeta \big/ \left(1 + \beta\right)\right]\left[w_t - \tau_t\right]$$

(A.1*) $$x_t^{*,1} = \left[\zeta \big/ \left(1 + \beta^*\right)\right]\left[w_t^* - \tau_t^*\right] e_t$$

(A.2) $$y_t^1 = \left[\left(1 - \zeta\right) \big/ \left(1 + \beta\right)\right]\left(e_t\right)^{-1}\left[w_t - \tau_t\right]$$

(A.2*) $$y_t^{*,1} = \left(1 - \zeta\right) \big/ \left(1 + \beta^*\right)\left(w_t^* - \tau_t^*\right)$$

(A.3) $$x_0^2 = \zeta\left(1 + i_0\right) a_0 G^L \left[k_0 + b - \Phi_0\right], \; x_{t+1}^2 = \beta\zeta \big/ \left(1 + \beta\right)\left(1 + i_{t+1}\right)\left(w_t - \tau_t\right)$$

(A.3*) $$\begin{aligned} x_0^{*,2} &= \zeta G^L a_0 \left[\left(1 + i_0^*\right)\left(k_0^* + b^*\right) + \left(1 + i_0\right)\left(e_0\right)^{-1} \Phi_0\right] e_0, \\ x_{t+1}^{*,2} &= \left(\beta^* \zeta\right) \big/ \left(1 + \beta^*\right)\left(1 + i_{t+1}^*\right)\left(w_t^* - \tau_t^*\right) e_{t+1} \end{aligned}$$

(A.4) $$\begin{aligned} y_0^2 &= \left(1 - \zeta\right) G^L a_0 \left(1 + i_0\right)\left(e_0\right)^{-1}\left[k_0 + b - \Phi_0\right], \\ y_{t+1}^2 &= \beta\left(1 - \zeta\right) \big/ \left(1 + \beta\right)\left(1 + i_{t+1}\right)\left(w_t - \tau_t\right)\left(e_{t+1}\right)^{-1} \end{aligned}$$

(A.4*) $$\begin{aligned} y_0^{*,2} &= \left(1 - \zeta\right) G^L a_0 \left[\left(1 + i_0^*\right)\left(k_0^* + b^*\right) + \left(e_0\right)^{-1}\left(1 + i_0\right)\Phi_0\right], \\ y_{t+1}^{*,2} &= \beta^*\left(1 - \zeta\right) \big/ \left(1 + \beta^*\right)\left(1 + i_{t+1}^*\right)\left(w_t^* - \tau_t^*\right) \end{aligned}$$

(A.5) $s_t = \sigma\left[w_t - \tau_t\right], \sigma \equiv \beta/(1+\beta)$

(A.5*) $s_t^* = \sigma^*\left[w_t^* - \tau_t^*\right]$

Proof of Proposition 1

To proof Proposition 1, let's start with $F_k(k) = \alpha(1-\alpha)\xi\left(M/G^A\right)k^{\alpha-1}$ $+\alpha(1-\alpha)\vartheta\left(M/G^A\right)k^{\alpha-2} - 1$. It is immediate that (i) $F_k(k)$ is a continuous and strictly decreasing function, that (ii) $\lim_{k\to 0} F_k(k) = \infty$ and (iii) that $\lim_{k\to\infty} F_k(k) = 0$. Hence, an Intermediate Value Theorem guaranties for each ϑ the existence of a κ which solves $F_k(\kappa,\vartheta) = 1$. Moreover, the solution is unique since $F_{kk}(k,\vartheta) < 0$. Hence, $\kappa = \mathrm{K}(\vartheta)$. Note also that $\mathrm{K}(\vartheta)$ is a strictly increasing function because $F_{kk}(k,\vartheta) < 0$ and $F_{k\vartheta}(k,\vartheta) > 0$. To obtain ER2, combine $F_k(\kappa,\vartheta) = 1$ and $F(\kappa,\vartheta,b^*) = \kappa$. $F_k(\kappa,\vartheta) = 1 \Leftrightarrow \alpha(1-\alpha)\left(M/G^A\right)\times$ $\times(\xi\kappa + \vartheta)\kappa^{\alpha-2} = 1$. $F(\kappa,\vartheta,b^*) = \kappa \Leftrightarrow \kappa = \kappa^{\alpha-2}\left(M/G^A\right)\left[\xi(1-\alpha)\kappa^2 - \alpha\vartheta\kappa\right] -$ $-\left[(1-\sigma)/\sigma\right]\vartheta - (1-\zeta)\mu^{-1}b^*\left(1-\sigma^*/\sigma\right)$. Eliminating $\kappa^{\alpha-2}$ from both equations, gives the following result:

$\left[\alpha(1-\alpha)\left(M/G^A\right)(\xi\kappa+\vartheta)\right]^{-1} = \left\{\kappa + \left[(1-\sigma)/\sigma\right]\vartheta + (1-\zeta)\mu^{-1}b^*\left(1-\sigma^*/\sigma\right)\right\} /$ $\left(M/G^A\right)\left[\xi(1-\alpha)\kappa^2 - \alpha\vartheta\kappa\right]$. Rearrange this equation and solve it for κ, and you will get: $\kappa^2 - 2\eta\kappa - \nu = 0$ with η and ν as defined in text after Lemma 1. From this quadratic polynomial immediately follows: $\kappa_+ \equiv \eta + \sqrt{\eta^2 + \nu}$. Reinserting κ_+ into $\alpha(1-\alpha)\left(M/G^A\right)(\xi\kappa+\vartheta)\kappa^{\alpha-2} = 1$, you will obtain ER2. ER1 follows from inspection of Figure 1.■

Derivation of the Jacobian matrix for the dynamic system

To show roughly how we obtained the elements of the Jacobian matrix (23), we now describe the main steps taken in the derivation of $\partial e_{t+1}/\partial e_t = 1 + (1-\alpha)(H/k)$. First, take the total differential of (15) with respect to all variables:

$de_{t+1} = \left(M^*/M\right)^{-1}\left(k\right)^{\alpha-1}\left(k^*\right)^{1-\alpha} de_t - e\left(M/M^*\right)\left(1-\alpha\right)\left(k\right)^{\alpha-2}\left(k^*\right)^{1-\alpha} dk_{t+1}$
$+e\left(M/M^*\right)\left(1-\alpha\right)\left(k\right)^{\alpha-1}\left(k^*\right)^{-\alpha} dk^*_{t+1}$. Second, solve the left-hand sides of (16) and (17) simultaneously with respect to the total differentials of k_{t+1} and k^*_{t+1}. Third, form the partial differentials $\partial k_{t+1}/\partial e_t$ and $\partial k^*_{t+1}/\partial e_t$ while taking the results of the second step into account. Fourth, evaluate the total differential of the first step at a steady state solution and consider the infinitesimal changes of k_{t+1} and k^*_{t+1} only with respect to e_t: $\partial e_{t+1}/\partial e_t = 1 + e\left(1-\alpha\right)\left[\left(k^*\right)^{-1}\partial k^*_{t+1}/\partial e_t - k^{-1}\partial k_{t+1}/\partial e_t\right]$. The last step is to insert the partial differentials evaluated at a steady state solution from step three into the above equation.

Proof of Lemma 2

To calculate the eigenvalues and the eigenvectors of the Jacobian, we use the characteristic equation of the Jacobian (21): $\left(J-\lambda_i I\right)v_i = 0$, whereby I denotes the identity matrix, and the characteristic equation in expanded form reads as follows:

$$\left(J-\lambda_i I\right)v_i = 0 \Leftrightarrow \begin{pmatrix} j_{11}-\lambda_i & j_{12} & j_{13} \\ j_{21} & j_{22}-\lambda_i & j_{23} \\ j_{31} & j_{32} & j_{33}-\lambda_i \end{pmatrix}\begin{pmatrix} \upsilon_i^e \\ \upsilon_i^k \\ \upsilon_i^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Multiplying the second row by $-\mu$, adding it to the third row, and multiplying the first row by $k/\left(1-\alpha\right)$ yields the following equivalent equation

$$\begin{pmatrix} \frac{k}{\left(1-\alpha\right)}\left(1-\lambda_i\right)+H & -\frac{\left(1+i\right)e}{G^A} & \frac{\left(1+i\right)e\mu^{-1}}{G^A} \\ j_{21} & j_{22}-\lambda_i & j_{23} \\ \frac{\mu H}{e} & \mu\left[\lambda_i - \frac{\left(1+i\right)}{G^A}\right] & \frac{\left(1+i\right)}{G^A}-\lambda_i \end{pmatrix}\begin{pmatrix} \upsilon_i^e \\ \upsilon_i^k \\ \upsilon_i^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Multiplying the first row by $-\mu/e$ times and adding it to the last row we get

$$\begin{pmatrix} \frac{k}{(1-\alpha)}(1-\lambda_i)+H & -\frac{(1+i)e}{G^A} & \frac{(1+i)e\mu^{-1}}{G^A} \\ j_{21} & j_{22}-\lambda_i & j_{23} \\ -\frac{k\mu(1-\lambda_i)}{e(1-\alpha)} & \mu\lambda_i & -\lambda_i \end{pmatrix} \begin{pmatrix} \upsilon_i^e \\ \upsilon_i^k \\ \upsilon_i^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Finally subtracting the third row $\mu^{-1}(1+i)e/(\lambda_i G^A)$ times from the first row leads to

(A.6) $$\begin{pmatrix} \frac{k}{(1-\alpha)}(1-\lambda_i)\left(1-\frac{1+i}{\lambda_i}\right)+H & 0 & 0 \\ j_{21} & j_{22}-\lambda_i & j_{23} \\ -\frac{k\mu(1-\lambda_i)}{e(1-\alpha)} & \mu\lambda_i & -\lambda_i \end{pmatrix} \begin{pmatrix} \upsilon_i^e \\ \upsilon_i^k \\ \upsilon_i^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Equation (A.6) can be solved if and only if the determinant of the matrix in (A.6) vanishes, i.e. if either

(A.7) $$\begin{vmatrix} j_{22}-\lambda_i & j_{23} \\ \mu & -1 \end{vmatrix} = 0, \text{ or}$$

(A.8) $$\frac{k}{(1-\alpha)}(1-\lambda_i)\left(1-\frac{1+i}{\lambda_i}\right)+H=0.$$

There are thus two cases to be distinguished: Case 1 in which (A.7) holds and case 2 for which (A.8) is true. Let us consider both cases in turn.

Case 1. Using the definition of j_{22} and j_{23}, equation (A.7) straightforwardly leads to

$$\lambda_3 = j_{22}+\mu j_{23} = \frac{(1+i)(1-\alpha)}{G^A}\left(\zeta\sigma\left(1+\frac{b}{k}\right)+(1-\zeta)\sigma^*\left(1+\frac{b^*}{k^*}\right)\right).$$

To determine its corresponding eigenvector, we use (A.6). Because $\frac{k}{(1-\alpha)}(1-\lambda_i)\left(1-\frac{1+i}{\lambda_i}\right)+H \neq 0$, it follows that $\upsilon_3^e = 0$, and thus (A.6)

$$\begin{pmatrix} j_{22}-\lambda_3 & j_{23} \\ \mu\lambda_3 & -\lambda_3 \end{pmatrix} \begin{pmatrix} \upsilon_3^k \\ \upsilon_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

with the solution claimed in Lemma 3 as can be seen as follows: The first row leads to $\upsilon_3^* = \mu\upsilon_3^k$, and the second row as a result of the value of λ_3 can then be solved identically, i.e. we can chose $\upsilon_3^k = 1$.

Case 2. In this case we know that $\frac{k}{(1-\alpha)}(1-\lambda_i)\left(1-\frac{1+i}{\lambda_i}\right)+H=0$. Since $H/k=(1+i)/\alpha-1$, it follows that $(1+i)/\alpha-1>1$ and $\lambda_2=\alpha$ as claimed in Lemma 2.

The eigenvector associated with the second eigenvalue can be found as follows: $\frac{k}{(1-\alpha)}(1-\lambda_i)\left(1-\frac{1+i}{\lambda_i}\right)+H=0$ implies that υ_2^e can be chosen freely, so for instance we can take $\upsilon_2^e = e/k$. Therefore (A.6) reduces to

$$\begin{pmatrix} j_{22}-\lambda_i & j_{23} \\ \mu\alpha & -\alpha \end{pmatrix}\begin{pmatrix} \upsilon_2^k \\ \upsilon_2^* \end{pmatrix} = \begin{pmatrix} -j_{21}\dfrac{e}{k} \\ -\mu \end{pmatrix}.$$

The second row yields $\upsilon_2^* = \mu\left(\upsilon_2^k + \alpha^{-1}\right)$. The first row equals

$$(j_{22}-\alpha)\upsilon_2^k + j_{23}\mu\left(\upsilon_2^k+\alpha^{-1}\right) = -j_{21}(e/k).$$

After substituting for the third eigenvalue,

$\upsilon_2^k = \dfrac{1-\lambda_3/k+\zeta(b/k)\left[1-\sigma+(1+i)\left(\sigma/\left(\alpha G^A\right)\right)\right]}{\lambda_3-\alpha}$ results. ■

Proof of Proposition 4:

Insert the eigenvectors from Lemma 3 into the second and third equation of (24) and solve simultaneously for $\kappa_2(\lambda_2)^t$ and $\kappa_3(\lambda_3)^t$. The results are as follows:

$\kappa_2(\lambda_2)^t = \left[k_t - k^H - \left(k_t^* - k^{*,H}\right)\right]\Big/\left(\upsilon_2^k - \upsilon_3^k\right) = \alpha\left[k_t - k^H - \mu^{-1}\left(k_t^* - k^{*,H}\right)\right]$, $\kappa_3(\lambda_3)^t = \alpha\mu^{-1}\upsilon_2^k$ $\times\left(k_t^* - k^{*,H}\right) - \alpha\mu^{-1}\upsilon_2^*\left(k_t - k^H\right) = \alpha\left[(1/\alpha+\gamma)\mu^{-1}\left(k_t^* - k^{*,H}\right) - \gamma\left(k_t - k^H\right)\right]$. The next step is to consider the second and the third equation of (24) for $t+1$ and t, and then to subtract the latter from the former. We get the following results: $k_{t+1} - k_t =$ $(1/\alpha+\gamma)(\lambda_2-1)\kappa_2(\lambda_2)^t + (\lambda_3-1)\kappa_3(\lambda_3)^t$, $k_{t+1}^* - k_t^* = \gamma(\lambda_2-1)\kappa_2(\lambda_2)^t + (\lambda_3-1)\kappa_3(\lambda_3)^t$.

The last step is to insert into these equations the equations for $\kappa_2(\lambda_2)^t$ and $\kappa_3(\lambda_3)^t$ from above, and to collect terms. As a consequence, (25) and (26) are obtained. Finally, inserting the equations for $\kappa_2(\lambda_2)^t$ and $\kappa_3(\lambda_3)^t$ into the first equation of (24) and remembering that $\upsilon_2^e = -e/k, \upsilon_3^e = 0$ holds, we obtain (27).■