Sustainable government debt in a two-good, two-country overlapping generations model

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Abstract. In recent years maximum sustainable government debt in closed economies once again became a matter of theoretical concern**.** At the present stage of international integration dramatically rising government deficits in large open OECD countries make it imperative to explore limits for national government debt levels which if slightly exceeded would lead to a sudden collapse of the world economy. This paper explores these limits in a two-good, two-country OLG model and analyzes existence and dynamic stability of steady states as well as the transitional dynamics of private capital when government debt levels remain below these limits (debt is sustainable). We find that maximum government debt levels for both countries exist and are negatively related. Moreover, if sustainable government debt is unilaterally expanded, private capital is crowded out in both countries while the terms of trade of the debt-expanding country are unaffected if capital income shares are internationally equal.

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1. Introduction

At the beginning of this decade the sustainability of government deficits and government debt in advanced economies once again became a matter of theoretical concern (Chalk 2000, Rankin and Roffia 2003). While in the older (empirical) literature fiscal policy was regarded as unsustainable if government's intertemporal budget constraint is violated (e.g. Blanchard et al. 1990), the theoretical possibility is now being considered "that, even with a constant stock of government debt, fiscal policy may be unsustainable because a steady state of the economy with non-degenerate values of the variables *may not exist*" (Rankin and Roffia 2003, 218; italics in original).[1] In other words: Government debt is unsustainable if a slightly higher debt stock sets off a process of unstable capital decumulation. The private capital labor ratio (aggregate capital intensity) associated with this unsustainable government debt level is called an 'interior maximum' in contrast to a 'degeneracy' in which the capital intensity approaches zero as a consequence of an excessively high government debt level. Rankin and Roffia (2003) find in their log-linear, Cobb-Douglas version of Diamond's (1965) overlapping generations (OLG) model that "maximum sustainable debt is generally reached at an interior maximum rather than at a degeneracy" (Rankin and Roffia 2003, 220).

Rankin and Roffia's (2003) contribution, although invaluable, is, however, restricted to a closed (or small open) economy setting which precludes the analysis of sustainable government debt in large, open, interdependent economies. At the present stage of international integration dramatically rising government deficits in large open OECD countries make it imperative to explore both limits for national government debt levels which if slightly exceeded in one country would cause a sudden collapse of the world economy and the effects of government debt expansion below those limits on private capital accumulation (economic growth) and international competitiveness as measured by the (external) terms of trade.

To the best of these authors' knowledge limits for government debt in open, interdependent have so far not been investigated at all. Although government debt effects on capital accumulation and terms of trade have been dealt with in the established literature (Feldstein

1986, Frenkel and Razin 1986, Zee 1987, Lin 1994), neither the role of maximum sustainable debt for the existence, dynamic stability and comparative dynamics of steady states has been treated nor the ambiguity of the terms of trade effects of government debt has been resolved. Zee (1987) finds that the terms of trade of the more indebted country deteriorate (improve) if this country is a net foreign debtor (creditor), whereas according to Lin (1994) the terms of trade effect of government debt is independent of the net foreign asset position, and depends only on international differences with respect to production technologies.

To close both of these research gaps, we extend Rankin and Roffia's (2003) closed economy OLG model into a two-good, two-country setting with exogenous growth.[2] To lend empirical support to the two-country setting, we let one country in the model depict the collection of net foreign creditor countries in reality while the other model country represents the net foreign debtor countries as a whole.[3] Since both collections of countries comprise less and developed countries respectively, we assume for simplicity that each model country represents an average of more and less developed countries in reality which can be characterized by identical preferences and similar production technologies.[4]

Given this model setting, we first ask the following questions regarding maximum sustainable debt:[5] Do maximum sustainable government debt levels in both countries always exist? What happens when these limits for sustainable government debt are reached? Are the national limits interdependent and which common factors determine the national debt limits? One main finding is that maximum sustainable government debt levels in both countries always exist.

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1 Thus, a constant (= time-stationary) stock of government debt is not sufficient for sustainability.

2 In particular, we investigate the most obvious type of debt instrument referred to by Rankin and Roffia as 'interest-exclusive' debt, where government debt is treated like a savings account. The remaining two types considered by Rankin and Roffia (2003) are 'interest-inclusive' debt and interest payments on government debt alone.

3 A clear characteristic of the world economy during the past fifteen years has been the development of rising external imbalances as measured by the diverging net foreign asset positions: i.e. while the USA and less-developed countries have become net foreign debtors, Japan, emerging Asia and oil-exporting countries are the net foreign creditors (see IMF 2006, 74; IMF 2008, 35).

4 Cobb-Douglas production functions are defined as 'similar' if production elasticities (or, respectively, capital income shares) are internationally equal, while the scale parameter reflecting the technological level might differ across countries.

5 The first two questions closely follow Rankin and Roffia (2003, 219).

Second, given these limits it is still not clear which role maximum government debt levels play with respect to the existence and characteristics of non-trivial steady state solutions for private capital intensities and with respect to the terms of trade between both countries. Extending Ono's (2002) existence analysis in a closed economy OLG model into our two-country setting, we will see that the magnitude of an appropriately weighted average of domestic and foreign government debt levels relative to the corresponding average of maximal debt levels is decisive for the existence of a unique or multiple non-trivial steady states for capital intensities.

Third, multiple steady state solutions necessitate dynamic stability analysis. From the two approaches to dynamic stability of steady states found in the literature (Gandolfo 1997, 334) we adopt that approach which investigates sufficiency conditions regarding preferences, technologies and policy parameters for dynamic stability.[6] To highlight the differences regarding dynamic stability between closed and large open economies, we compare (existence and) dynamic stability of steady states[7] in autarky to a world market equilibrium in our two-country model, and find only saddle path stability of the steady state with the higher capital intensities.[8]

Utilizing the conditions for saddle path stability of the larger steady state, we reexamine the steady state effects of a unilateral expansion of government debt on terms of trade and domestic and foreign capital intensities. While confirming established OLG wisdom that a unilateral expansion of government debt crowds out private capital in both countries, we do not find that the terms of trade effect depends on the net foreign asset position of the more indebted country.

Finally, it is also of interest to know whether the terms of trade along the transition path towards the new steady state also are independent of the net foreign asset position. Here Zee

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6 The other approach adopted by Zee (1987, 615) assumes asymptotic dynamic stability as a necessary condition for comparative steady state analysis. However, perfect foresight of asset holders in deterministic OLG models makes the assumption of asymptotic stability of the terms of trade dynamics questionable.

7 Note that in log-linear, Cobb-Douglas OLG models of closed economies existence conditions often imply dynamic stability conditions (Ono 2002, Farmer and Wendner 2003). We will see that this is also true in our two-good, two-country OLG model.

8 Saddle-path stability implies that at least one equilibrium variable represents a jump variable. It is natural to suggest that the terms of trade act as a jump variable which responds immediately to parameter shocks. Brecher et al. (2005) also find saddle-path stability in an infinitely lived agent (ILA) two-country model.

(1987, 611) claims that an unexpected expansion of government debt does not impact on the terms of trade in the shock period while their adaptation thereafter depends on the net foreign asset position of the more indebted country.[9] By thoroughly analyzing the transitional dynamics of terms of trade and capital intensities for similar technologies and calculating the transition path numerically for dissimilar technologies, we are essentially able to qualitatively confirm the steady state results.

The paper is organized as follows. In the next section, the existence of non-trivial steady states and the asymptotic stability of one steady state solution in the autarky equilibrium of our two-good, two-country OLG economy are shown. Section 3 presents the world market equilibrium. Section 4 is devoted to the analysis of maximum sustainable government debt and to the investigation of the existence of steady state solutions for private capital intensities under government debt levels below a maximum. In Section 5 we are concerned with the dynamic stability of the steady state solutions and the comparative steady state effects of shocks in sustainable government debt levels. Section 6 is devoted to closed-form solutions of the transitional dynamics towards the steady state under similar technologies and to a numerical calculation of economic transition under dissimilar technologies. Section 7 summarizes and concludes.

2. The autarky equilibrium

The autarky equilibrium of our two-country OLG model is provided by a log-linear, Cobb-Douglas version of the original Diamond (1965) OLG closed economy (as in Rankin and Roffia 2003, 221-222). In this framework, time is discrete and indexed by $t = 0, 1, 2, \ldots$.

In each period, a large number of identical firms operate under perfect competition. Their constant returns-to-scale technology is specified according to a Cobb-Douglas production function.[10] To produce the quantity of output X_t, firms employ two factors of production,

9 However, Zee's (1987) conclusions are questionable if one takes saddle-path stability of the terms of trade dynamics into account.

10 Clearly, to analyze existence, stability and transitional dynamics in autarky, we could have operated with general neoclassical utility and production functions as used by Buiter (1981). However, to highlight the common ground concerning existence of steady states as well as the differences with respect to dynamic stability and transitional dynamics, we assume log-linear preferences and Cobb-Douglas technologies also in autarky.

capital services K_t and labor services N_t, scaled by $M > 0$ to account for a productivity parameter reflecting the technological level:

(1) $$X_t = M\left(A_t\right)^{1-\alpha}\left(K_t\right)^{\alpha},\ 0 < \alpha < 1,$$

whereby $A_t = a_t N_t$ denotes the efficiency-weighted labor input and $0 < a_t$ is labor-efficiency per employee. The corresponding growth factor of efficiency-weighted labor is equal to the product of the time-stationary growth factor of labor productivity G^a, and the population growth factor G^L: $G^A \equiv G^a G^L$. Since firms operate in a fully competitive environment, the production elasticity of capital services α with $0 < \alpha < 1$ represents the capital income share. Analogously, $1-\alpha$ is the labor income share.

Profit maximization implies:

(2) $$q_t = \alpha M\left(k_t\right)^{\alpha-1},\ k_t \equiv K_t / A_t$$

(3) $$w_t = \left(1-\alpha\right) M\, a_t \left(k_t\right)^{\alpha},$$

where k_t is the capital efficiency-labor ratio (= aggregate capital intensity), q_t denotes the real price of capital services and w_t is the real wage rate.

Denoting real investment in capital by I_t, and assuming that the capital stock depreciates completely within one period, capital accumulates over time as follows:

(4) $$K_{t+1} = I_t.$$

As usual in the Diamond-type OLG framework, two generations of homogeneous individuals overlap in each period t. At date t, a new generation of size L_t enters the economy. In each period *t*, the population grows according to an exogenously fixed factor G^L. Each generation lives for two periods, working during the first when young and retiring in the second when old. In the following, the young generation is indexed by superscript 1 (indicating the first period of life) and the old generation is indexed by superscript 2 (indicating the second period of life). Each member of the young generation supplies one unit of labor inelastically to firms and receives the wage rate w_t in return. There is no labor-leisure choice. The young generation allocates labor income to per-capita consumption of the *X*-

commodity, x_t^1 and to real per-capita savings s_t. When old, it spends all revenues for per-capita consumption x_{t+1}^2.

Households are identical within as well as across generations. Life-time utility of generation *t*, U_t^1, depends on per-capita consumption in the working and retirement period in the following log-linear way:

$$U_t^1 = \ln x_t^1 + \beta \ln x_{t+1}^2 , \tag{5}$$

where β ($0 < \beta < 1$) denotes the future discount factor of the young generation. The decision problem (in real and per-capita terms) for a representative member of generation *t* is thus to maximize (5) subject to the working and retired period budget constraints:

$$\begin{aligned} x_t^1 + s_t &= w_t - \tau_t , \\ x_{t+1}^2 &= \left(1 + i_{t+1}\right) s_t , \end{aligned} \tag{6}$$

with $s_t \equiv K_{t+1}/L_t + B_{t+1}/L_t$.

In the first budget constraint, τ_t is a lump-sum tax. In the second constraint, i_{t+1} is the real interest rate in period $t+1$. Savings are allocated to government bonds B_{t+1}/L_t and ownership claims to physical capital K_{t+1}/L_t. Since the assets are perfect substitutes in households' portfolios, the no-arbitrage condition $i_{t+1} = q_{t+1} - 1$ holds.

The government collects lump sum tax τ_t to finance the costs of government debt per-efficiency capita, $b_t \equiv B_t/A_t$:

$$G^A b_{t+1} + \tau_t/a_t = \left(1 + i_t\right) b_t ,\ G^A \equiv G^a G^L . \tag{7}$$

As in Diamond (1965, 1137), it is assumed that the government runs a 'constant-stock' fiscal policy (for more details see Azariadis 1993, 319 or De la Croix and Michel 2002, 216-226): $b_{t+1} = b_t = b,\ \forall t$. The lump sum tax becomes endogenous and is determined by

$$\tau_t = b\left[\alpha M\left(k_t\right)^{\alpha-1} - G^A\right] a_t . \tag{8}$$

Because of the competitive nature of the economy, markets clear in each period. In equilibrium, the demand for labor is equal to the total number of agents born at time *t*:

$$(9) \qquad N_t = L_t .$$

Since the rates of return on both assets equalize, asset market equilibrium (in real, per-efficiency-capita terms) implies:

$$(10) \qquad s_t = G^A \left(k_{t+1} + b \right) a_t .$$

The composite commodity produced is used either for consumption purposes by both generations or as an investment good. The product market equilibrium is expressed by the following condition:

$$(11) \qquad x_t = \left(1/a_t \right) x_t^1 + \left(1/G^L \right)\left(1/a_t \right) x_t^2 + G^A k_{t+1}.$$

In accordance with Walras' Law, one market clearing equation is redundant.

An intertemporal equilibrium is fully described by the following first-order difference equation (see the derivation of equation (12) in the appendix):

$$(12) \qquad k_{t+1} = \sigma_0 k_t^{\alpha} - \sigma \left\{ b \left[\left(1 + i_t \right) / G^A - 1 \right] \right\} - b ,$$

where $\sigma_0 \equiv \left(1 - \alpha \right) \sigma \left(M / G^A \right)$ with $\sigma \equiv \beta / \left(1 + \beta \right)$, $1 + i_t = \alpha M \left(k_t \right)^{\alpha - 1}$, and $k_0 > 0$ given.

As usual, a steady state is a fixed point $k_{t+1} = k_t \equiv k$ of equation (12). We need information about the number and the quality of fixed-point or steady state solutions. As Ono (2002, 82-83) demonstrates in a related model setting, a restriction on the policy parameter b is needed which ensures that for all admissible structural parameters $\alpha, \beta, G^L, G^A, M$ a non-trivial solution $0 < k < \bar{k}$ of equation (12) exists, whereby $\bar{k}$ is the solution to the equation $H\left(k \right) \equiv \left(M / G^A \right) \left(k \right)^{\alpha} - k = 0$. Clearly, $\bar{k} = \left(M / G^A \right)^{1/(1-\alpha)}$.

To prove rigorously the existence of non-trivial steady state solutions for the capital intensity, let the parameter vector $\omega \equiv \left(\alpha, \beta, G^A, M, b \right)$ be an element of the parameter space $\Omega = \left[0,1 \right]^2 \times \mathbb{R}_3^+$. Finally, rewrite equation (12) as in the following time one map:

$$(12.1) \qquad k_{t+1} = \Gamma\left(k_t \right) \equiv \begin{cases} \mathrm{T}\left(k_t \right) & \text{if } \mathrm{T}\left(k_t \right) > 0 \\ 0 & \text{if } \mathrm{T}\left(k_t \right) \leq 0, \end{cases}$$

where $\mathrm{T}\left(k_t \right) \equiv \sigma_0 \left(k_t \right)^{\alpha} - b \sigma_1 \left(k_t \right)^{\alpha - 1} - b \left(1 - \sigma \right)$, $\sigma_1 \equiv \alpha \sigma \left(M / G^A \right)$.

The following Proposition 1 characterizes the whole spectrum of steady state solutions in our log-linear, Cobb-Douglas OLG model and provides sufficient conditions for the existence of non-trivial steady states.[11] In particular, a maximum level of government debt, denoted by $\bar{b}$, is analytically shown to exist. The policy parameter $\bar{b}$ corresponds exactly to Rankin and Roffia's (2003, 224) maximum sustainable government debt level.

PROPOSITION 1 (Existence of steady state solutions).

For any $\omega \in \Omega$ *there exists* $\bar{b} \in \mathbb{R}_{++}$ *such that*

(i) *for* $b < \bar{b}$ *there are one trivial* ($k = 0$) *and two non-trivial steady states* k^L *and* k^H *with* $0 < k^L < k^H < \bar{k}$,

(ii) *for* $b = \bar{b}$ *there are one trivial and one non-trivial steady state,*

(iii) *for* $b > \bar{b}$ *there is only the trivial steady state.*

PROOF. See the appendix.

<<< Figure 1 about here >>>

Figure 1 illustrates the first case (i) of Proposition 1. Since function $\Gamma(k_t)$ cuts the 45° line twice, there are two non-trivial steady states, a lower one at $k = k^L$, and a higher one at $k = k^H$. When $b = \bar{b}$, the graph of $\Gamma(k_t)$ shifts downwards such that the 45° line is tangential to $\Gamma(k_t)$ and only one non-trivial steady state exists.

In order to highlight the differences regarding the dynamic stability notions among the autarkic (closed economy) and the two-country equilibrium we also report Proposition 2 which states that for the economically most interesting case $b < \bar{b}$, the higher steady state solution is asymptotically stable while the lower one is asymptotically unstable.

PROPOSITION 2 (Dynamic stability of steady states).

Suppose $b < \bar{b}$ *holds. The steady state solution* $k = k^H$ *is asymptotically stable, while the steady state* $k = k^L$ *is asymptotically unstable. Moreover, the transition path is non-oscillatory and monotone.*

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[11] De la Croix and Michel (2002, 219-226) prove the existence of non-trivial steady state solutions for general utility and neoclassical production functions, but do not deal with maximum

PROOF. Non-oscillatory and monotone dynamics of capital follow from the fact that, for positive values of k_t, function $\mathrm{T}(k_t)$ is a strictly increasing and strictly concave function. Since in the case of $b < \bar{b}$ there are two non-trivial steady state solutions, $\mathrm{T}(k_t)$ crosses the identity line first from below and then from above. This implies asymptotic instability of k^L and asymptotic stability of k^H. ■

REMARK 1. For $b > \bar{b}$ the trivial steady state is globally stable. If $b = \bar{b}$, the dynamic system undergoes a 'fold' bifurcation (Azariadis 1993, Appendix A5) which means that the two steady states k^H and k^L collapse to a single steady state which is asymptotically unstable for any initial value k_0 below the unique steady state value. As mentioned above, $\bar{b}$ corresponds to Rankin and Roffia's (2003, 219-220) maximum sustainable government debt which if slightly exceeded "sets off a process of unstable capital decumulation" such that "a steady state might suddenly cease to exist."

3. The world market equilibrium

Let us now extend the autarky model towards the intertemporal market equilibrium OLG model of the world economy, which consists of two interdependent countries, Home and Foreign. As in Zee (1987), there are two tradable goods, X and Y^*, and each country specializes in the production of a unique composite commodity, which can be used for consumption as well as for investment purposes. The commodity produced in Home is designated by X and the one produced in Foreign by Y^*.[12] Both countries are identical with respect to intertemporal consumer preferences. In accordance with Lin (1994) we assume Cobb-Douglas production functions in both countries with different technological levels but equal production elasticities of capital (capital income shares) across countries, i.e. $M \neq M^*, \alpha = \alpha^*$ (for short, we will refer to this assumption as internationally 'similar' production technologies).

The production sector in Foreign is described by the following equations:

(1*) $$Y_t^* = M^*\left(A_t^*\right)^{1-\alpha}\left(K_t^*\right)^{\alpha}, \; A_t^* = a_t N_t^*,$$

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sustainable debt.

12 Henceforth all variables referring to Foreign are denoted by an asterisk.

(2*) $$q_t^* = \alpha M^* \left(k_t^*\right)^{\alpha-1}, k_t^* \equiv K_t^* \big/ A_t^* ,$$

(3*) $$w_t^* = \left(1-\alpha\right) M^* \, a_t \left(k_t^*\right)^{\alpha} ,$$

(4*) $$K_{t+1}^* = I_t^* .$$

In the world economy, both countries are open to international trade in goods and assets (government bonds). As in Zee (1987, 605), only "the domestically produced commodity can be purchased and stored by domestic residents as capital to be used in home-country production in the following period." Physical capital is therefore internationally immobile. The population does not migrate between countries.

In this framework, domestic as well as foreign households choose between consumption of domestic, x_t^1 ($y_t^{*,1}$) and of foreign commodities, y_t^1 ($x_t^{*,1}$).

The budget constraint (in real and per-capita terms) of the household living in Home, when young is

(13) $$x_t^1 + \left(1/p_t\right) y_t^1 + s_t = w_t - \tau_t ,$$

whereby $$s_t \equiv K_{t+1}/L_t + B_{t+1}^H/L_t + \left(1/p_t\right) B_{t+1}^{*,H}/L_t ,$$

and when old is

(14) $$x_{t+1}^2 + \left(1/p_{t+1}\right) y_{t+1}^2 = \left(1+i_{t+1}\right)\left(K_{t+1}/L_t + B_{t+1}^H/L_t\right) + \left(1+i_{t+1}^*\right)\left(1/p_{t+1}\right)\left(B_{t+1}^{*,H}/L_t\right),$$

where p_t denotes the (external) terms of trade (units of the foreign good per unit of the domestic good), while B_{t+1}^H/L_t and $B_{t+1}^{*,H}/L_t$ denote the stocks of domestic and of foreign government bonds which the household in Home plans to hold at the beginning of period $t+1$. Clearly, domestic real capital, domestic bonds and also foreign bonds are perfectly substitutes from the perspective of Home's younger household.

Home households preferences are represented by the following intertemporal log-linear utility function:

(15) $$U_t = \zeta \ln x_t^1 + \left(1-\zeta\right) \ln y_t^1 + \beta\left[\zeta \ln x_{t+1}^2 + \left(1-\zeta\right) \ln y_{t+1}^2\right],$$

whereby $0<\zeta<1$ ($1-\zeta$) is the expenditure share for domestic (foreign) commodities. Each household maximizes the utility function (15) subject to the budget constraints defined by equations (13) and (14). The optimal consumption and savings quantities of the household in Home are given in the appendix (see equations (A.1)-(A.5)).

The corresponding budget constraints for the household in Foreign are:

(13*) $$p_t x_t^{*,1} + y_t^{*,1} + s_t^* = w_t^* - \tau_t^*,$$

whereby $s_t^* \equiv K_{t+1}^*/L_t^* + B_{t+1}^{*,F}/L_t^* + p_t\left(B_{t+1}^F/L_t^*\right),$

(14*) $$p_{t+1}x_{t+1}^{*,2} + y_{t+1}^{*,2} = \left(1+i_{t+1}^*\right)\left(K_{t+1}^*/L_t^* + B_{t+1}^{*,F}/L_t^*\right) + p_{t+1}\left(1+i_{t+1}\right)\left(B_{t+1}^F/L_t^*\right).$$

As before, B_{t+1}^F/L_t^* ($B_{t+1}^{*,F}/L_t^*$) denotes the stock of domestic (foreign) government bonds which the household of Foreign plans to hold at the beginning of period $t+1$.

The utility function of the household in Foreign is:

(15*) $$U_t^* = \zeta \ln x_t^{*,1} + \left(1-\zeta\right)\ln y_t^{*,1} + \beta\left[\zeta \ln x_{t+1}^{*,2} + \left(1-\zeta\right)\ln y_{t+1}^{*,2}\right].$$

Since government bonds are assumed to be perfectly mobile across Home and Foreign, a real international interest parity condition holds between the two countries:

(16) $$\left(1+i_{t+1}\right)\left(p_{t+1}/p_t\right) = \left(1+i_{t+1}^*\right).$$

The lump-sum tax rate in Foreign is determined as follows:

(8*) $$\tau_t^* = b^*\left[\alpha M^*\left(k_t^*\right)^{\alpha-1} - G^A\right]a_t.$$

Clearing of the labor market in Foreign requires:

(10*) $$N_t^* = L_t^*.$$

Furthermore, without loss of generality, we assume that the supply of labor is equal across countries, i.e. $L_t = L_t^*$.

The product market clearing condition of Home reads as follows:

(17) $$x_t = \left(1/a_t\right)x_t^1 + \left(1/G^L\right)\left(1/a_t\right)x_t^2 + G^A k_{t+1} + \left(1/a_t\right)x_t^{*,1} + \left(1/G^L\right)\left(1/a_t\right)x_t^{*,2},$$

whereas foreign product market clearing demands:

(17*) $$y_t^* = \left(1/a_t\right) y_t^{*,1} + \left(1/G^L\right)\left(1/a_t\right) y_t^{*,2} + G^A k_{t+1}^* + \left(1/a_t\right) y_t^1 + \left(1/G^L\right)\left(1/a_t\right) y_t^2 .$$

The world market for Home bonds clears according to:

(18) $$B_t = B_t^H + B_t^F ,$$

and symmetrically for Foreign bonds we have:

(18*) $$B_t^* = B_t^{*,H} + B_t^{*,F} .$$

The world asset market clearing condition requires that the total amount of savings in the world equals the total world demand for assets from Home and Foreign:

(19) $$s_t / a_t + \left(1/p_t\right)\left(s_t^* / a_t\right) = G^A \left[k_{t+1} + b + \left(1/p_t\right)\left(k_{t+1}^* + b^*\right) \right].$$

4. Intertemporal equilibrium dynamics and existence of steady states

This section derives the intertemporal equilibrium dynamics of the two-country model and investigates the existence and multiplicity of steady states. Thereafter, the existence of maximum sustainable government debt levels in Home and in Foreign is analyzed.

From the international interest parity condition (16), the equation of motion of the terms of trade follows, and we have:

(20) $$p_{t+1} = p_t \left(\frac{M^*}{M} \right) \frac{\left(k_{t+1}\right)^{1-\alpha}}{\left(k_{t+1}^*\right)^{1-\alpha}} .$$

By inserting the optimal saving function for Home (A.5) and the analogous function for Foreign into the world asset market clearing condition (19) and considering the profit maximizing conditions (3) and (3*) as well as the equations for lump taxes (8) and (8*), we obtain the following difference equation describing the law of motion of the international asset market:

(21) $$\begin{aligned} p_t k_{t+1} + k_{t+1}^* &= p_t \sigma_0 \left(k_t\right)^\alpha - p_t b \left\{ \sigma \left[\left(1+i_t\right)/G^A - 1 \right] + 1 \right\} + \sigma_0^* \left(k_t^*\right)^\alpha \\ &\quad - b^* \left\{ \sigma \left[\left(1+i_t^*\right)/G^A - 1 \right] + 1 \right\},\ \sigma_0^* \equiv \left(1-\alpha\right)\sigma\left(M^*/G^A\right),\ 1+i_t^* \equiv \alpha M^* \left(k_t^*\right)^{\alpha-1} . \end{aligned}$$

From the two national product market clearing conditions (17) and (17*), the third dynamic equation is obtained:

(22) $$p_t k_{t+1} - \left[\zeta/(1-\zeta)\right]k_{t+1}^* = \left(M/G^A\right)p_t\left(k_t\right)^\alpha - \left(M^*/G^A\right)\left[\zeta/(1-\zeta)\right]\left(k_t^*\right)^\alpha .$$

Equations (20)-(22) represent the three-dimensional dynamic system of the two-good, two-country OLG model with similar technologies across countries.

As in autarky, the first step needed when analyzing the system dynamics of world market equilibrium is to investigate the existence of steady state solutions, i.e. $k_{t+1} = k_t = k$, $k_{t+1}^* = k_t^* = k^*$, $p_{t+1} = p_t = p$. Lemma 1 provides a characterization of non-trivial steady state solutions for the case of similar production technologies in both countries, i.e. $\alpha = \alpha^*$ but $M \neq M^*$.

LEMMA 1. *At a fixed point of the dynamic system* (20)-(22)*, Foreign and Home capital intensities are related by* $k^* = \mu k$, $\mu \equiv \left(M^*/M\right)^{1/(1-\alpha)}$*; Home's terms of trade are* $p = \mu\left[\zeta/(1-\zeta)\right]$*; and Home's capital intensity is found by solving the following equation* $k = \sigma_0 k^\alpha - \vartheta\left\{\sigma\left[(1+i)/G^A - 1\right]+1\right\}$ *with* $\vartheta \equiv \zeta b + (1-\zeta)\mu^{-1}b^*$.

The new (exogenous) parameter ϑ can be regarded as the weighted average of domestic and foreign government debt levels, and its magnitude plays an important role for the existence and uniqueness of steady state solutions, as becomes apparent in Proposition 3 below.

COROLLARY 1. *If the expenditure share for the domestic commodity* ζ *equals* $1/2$ *and* $M = M^*$*, the terms of trade* p *are equal to* 1.[13]

To investigate thoroughly the existence of non-trivial steady state solutions and the existence of maximum sustainable government debt levels in both countries, let $\omega^* \equiv \left(\alpha, \beta, G^A, M, M^*, \vartheta\right)$ be the parameter vector and $\Omega^* = [0,1]^2 \times \mathbb{R}_+^4$ be the parameter space in the world market equilibrium.

PROPOSITION 3 (Existence of steady state solutions).

For any $\omega^* \in \Omega^*$ *there exists* $\bar{\vartheta} \in \mathbb{R}_{++}$ *such that*

(i) *for* $\vartheta < \bar{\vartheta}$ *there are one trivial* ($k = 0$) *and two non-trivial steady states* k^L *and* k^H *with* $0 < k^L < k^H < \bar{k}$,

[13] Bianconi (2003, 29) considers a similar 'symmetric' steady state in an ILA framework.

(ii) *for* $\vartheta = \bar{\vartheta}$ *there are one trivial and one non-trivial steady state,*

(iii) *for* $\vartheta > \bar{\vartheta}$ *there is only the trivial steady state.*

PROOF. In the proof of Proposition 1, substitute the parameter b for the parameter ϑ.■

Several comments with respect to Proposition 3 are in order. First, due to the assumptions of internationally identical preferences and similar technologies, the sufficient conditions for the existence of non-trivial steady states in closed and large open economies are closely related. The international differences are depicted by the exogenous parameter ϑ, and hence its magnitude is decisive for the existence of non-trivial steady states in the two-country model. Second, two non-trivial steady state solutions exist if $\vartheta < \bar{\vartheta}$ where $\bar{\vartheta} \equiv \zeta \bar{b} + (1-\zeta)\mu^{-1}\bar{b}^*$ is defined as the weighted average of maximum sustainable government debt levels in Home ($\bar{b}$) and in Foreign ($\bar{b}^*$). Third, as in the closed economy case there always exists a finite maximum of the weighted average of domestic and foreign government debt levels, $\bar{\vartheta}$, and it too occurs at an interior maximum rather than at a degeneracy. This claim is illustrated by Figure 2 in which ϑ is plotted as function of k using the determining equation for k in Lemma 1.[14] Mathematically, if $\vartheta = \bar{\vartheta}$ holds, the dynamic system undergoes a 'saddle-node bifurcation' (Azariadis 1993, 152) which represents the two-dimensional analogue to the fold bifurcation in the one-dimensional dynamic system of the autarky equilibrium. Again, a slight variation of the government debt levels in Home or in Foreign sets off a process of unstable decumulation of private capital intensities.

<< Figure 2 about here >>

Fourth, the determinants of maximum sustainable government debt in each country also need to be identified. Focusing on the home country, we obtain immediately from Lemma 1: $\bar{b} = \left(\sigma_0 k^\alpha - k\right) \Big/ \left\{\zeta\left[\sigma\left((1+i)/G^A - 1\right)+1\right]\right\} - (1-\zeta)(\zeta\mu)^{-1}\bar{b}^*$. Differentiating with respect to k and setting the result equal to zero yields $k^{\max}$. In particular, $k^{\max}$ can be determined by the solution of the following quadratic equation:

14 The explanation for the shape of the curve plotted in Figure 2 is similar to that used by Rankin and Roffia (2003, 224) for their closed economy.

$\alpha(1-\alpha)\left(M/G^{A}\right)^{2}\left(\sigma\left(k^{\max}\right)^{\alpha-1}\right)^{2}-\alpha\sigma\left[1+(1-\alpha)\sigma\right]\left(M/G^{A}\right)\left(k^{\max}\right)^{\alpha-1}-1+\sigma=0$.[15] Utilizing the signs of the coefficients it is not difficult to show that there is only one positive real root for $\left(k^{\max}\right)^{\alpha-1}$. Insertion of this unique solution for $k^{\max}$ into the equation determining $\bar{b}$ yields

$$\bar{b}\left(k^{\max}\right)\equiv\left(\sigma_{0}\left(k^{\max}\right)^{\alpha}-k^{\max}\right)\Big/\left\{\zeta\left[\sigma\left(\alpha M\left(k^{\max}\right)^{\alpha-1}\big/G^{A}-1\right)+1\right]\right\}-(1-\zeta)(\zeta\mu)^{-1}\bar{b}^{*}.$$

Thus, for given structural parameters α, β, G^{A}, M, M^{*}, and ζ, this equation reveals a negative relationship between maximum sustainable government debt in Home and in Foreign. In other words, the higher the maximum sustainable government debt in Home, the lower it has to be in Foreign and vice versa.

It is also interesting to look at the effects of changes in the structural parameters α, β, G^{A} and M on the weighted average of maximal sustainable government debt in Home and in Foreign. As regards the effects of α and β on $\bar{\vartheta}$, we refer to Table 1 which shows that an increase in α and a decrease in β reduce $\bar{\vartheta}$ and the associated levels of capital intensities.[16] Moreover, a higher α and a lower β reduce the maximum sustainable debt as a ratio to Home's maximal capital intensity. "The reason is that [an increase in] α and [a decrease in] β both lower the incentive to save: α because, by increasing the profit share and reducing the wage share in income, it shifts income from the first to the second period of life; and β because it raises the degree of 'impatience' in the consumers' intertemporal preferences" (Rankin and Roffia 2003, 227).

<<< Table 1 about here >>>

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[15] This formula represents a generalization of equation (15) in Rankin and Roffia (2003, 227).

[16] The results reported in Table 1 are, qualitatively speaking, largely similar to those which Rankin and Roffia's (2003, 228) report in their Table 1. The only difference is that maximum debt as a ratio to maximum capital intensity in our model changes as β rises, which is due to our slightly different specification of the intertemporal utility function.

Table 2 shows that for given numerical values of α and β ($\alpha = 0.3, \beta = 0.8$) a higher G^A reduces $\overline{\vartheta}$ and maximal capital intensity while a higher M raises $\overline{\vartheta}$ and maximal capital intensity, but that maximum sustainable debt as a ratio to Home's capital intensity is unaffected by these parameter changes. The intuition is that a larger natural growth factor reduces savings per efficiency capita and hence the supply on the international asset market leaving less room for government debt, while a larger factor productivity in Home raises Home production and Home wage income and hence increases the supply on asset markets enabling more government debt.

<<< Table 2 about here >>>

Finally, in Table 3 we investigate in numerical terms the trade-off between maximum government debt in Home and in Foreign for different values of ζ $(1-\zeta)$ as well as for M and M^*, whereby $\alpha = 0.3, \beta = 0.8, G^A = 1$.

<<< Table 3 about here >>>

5. Stability of steady states and comparative steady state analysis of an increase in Home's sustainable government debt

Given that in both countries government debt levels are sustainable, i.e. they remain below the maximum sustainable level, this section is devoted to the investigation of dynamic stability of the steady state solutions and to comparative steady state analysis based on the stability analysis.

To analyze the dynamic stability properties in the neighborhood of non-trivial steady state solutions, the equilibrium dynamics is linearly approximated in a small neighborhood of each of the steady states The Jacobian matrix of the dynamic system (20)-(22) can be written as follows:[17]

[17] For hints on how to derive the elements of the Jacobian matrix J see the appendix.

$$(23) \qquad J=\begin{bmatrix} \partial p_{t+1}/\partial p_t & \partial p_{t+1}/\partial k_t & \partial p_{t+1}/\partial k_t^* \\ \partial k_{t+1}/\partial p_t & \partial k_{t+1}/\partial k_t & \partial k_{t+1}/\partial k_t^* \\ \partial k_{t+1}^*/\partial p_t & \partial k_{t+1}^*/\partial k_t & \partial k_{t+1}^*/\partial k_t^* \end{bmatrix} \equiv \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{bmatrix} \text{ with}$$

$j_{11}=1+(1-\alpha)(H/k)$, $j_{12}=\left[(1-\alpha)(1+i)p\right]/\left(G^A k\right)$, $j_{13}=-\left[(1-\alpha)(1+i)p\right]/\left(G^A k^*\right)$,

$j_{21}=p^{-1}\left[(1-\zeta)H-\zeta\Phi\right]$, $j_{22}=\left[(1+i)/G^A\right]\left[1-\zeta+\zeta\sigma(1-\alpha)(1+b/k)\right]$,

$j_{23}=-(1-\zeta)\mu^{-1}\left[(1+i)/G^A\right]\left[1-\sigma(1-\alpha)\left(1+b^*/k^*\right)\right]$,

$j_{31}=-\zeta\mu(H+\Phi)/p$, $j_{32}=-\zeta\mu\left[(1+i)/G^A\right]\left[1-\sigma(1-\alpha)(1+b/k)\right]$,

$j_{33}=\left[(1+i)/G^A\right]\left[\zeta+(1-\zeta)\sigma(1-\alpha)\left(1+b^*/k^*\right)\right]$, whereby

$H\equiv\left(M/G^A\right)k^\alpha-k=\left[(1+i)/\left(G^A\alpha\right)-1\right]k$ and $\Phi\equiv k+b\left\{\sigma\left[(1+i)/G^A-1\right]+1\right\}-\sigma_0 k^\alpha$.

To correctly evaluate local dynamic stability of non-trivial steady states in the world market equilibrium, information on the eigenvalues of the Jacobian matrix (23) denoted by $\lambda_i, i=1,2,3$ is needed. Lemma 2 provides this information.

LEMMA 2. *The three eigenvalues of the Jacobian evaluated at non-trivial steady states read as follows:* $\lambda_1=(1+i)/\left(G^A\alpha\right), \lambda_2=\alpha, \lambda_3=(1+i)(1-\alpha)\sigma(1+\vartheta/k)/G^A$.

PROOF. See the proof of Lemma 2 in the appendix.

Knowledge of the three eigenvalues of the Jacobian enables us to state Proposition 4, which deals with the dynamic stability of both steady states. In contrast to the asymptotic stability of non-trivial steady states in autarky, only saddle path stability of steady states can be obtained.

PROPOSITION 4 (Dynamic stability of steady states).

Suppose that $\vartheta<\bar{\vartheta}$ *holds.*[18] *Then, the steady state solution* $\left(p,k^H,k^{*,H}\right)$ *is saddle-path stable, i.e.* $\lambda_1>1,\ \lambda_2<1, \lambda_3<1$, *while the steady state solution* $\left(p,k^L,k^{*,L}\right)$ *is saddle-path unstable* ($\lambda_1>1,\ \lambda_2<1,\lambda_3>1$).

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18 The eigenvalues of the Jacobian evaluated at the single steady state associated with $\vartheta=\bar{\vartheta}$ are as follows: the first is larger than one, the second eigenvalue equals α and the third is equal to unity. As in the case of $\vartheta<\bar{\vartheta}$ not all eigenvalues of the Jacobian are less than one, as Zee (1987) would have it!

PROOF. From Lemma 2 we know that $\lambda_2 = \alpha$ and by assumption $\alpha < 1$ holds. Second, $\lambda_1 = (1+i)/(G^A \alpha) \quad > 1$ follows from $k^L < k^H < \bar{k}$, since $k < \bar{k} \Leftrightarrow \quad 1 < (M/G^A) k^{\alpha-1}$ $= (1+i)/(G^A \alpha)$. $\lambda_3 < 1 \Leftrightarrow \; (1+i)(k^H)(1-\alpha)\sigma\left[1+\left(\vartheta/k^H\right)\right]/(G^A) < 1$. This follows from the following facts: (i) $(1+i)(k^H)(1-\alpha)\sigma\left[1+\left(\vartheta/k^H\right)\right]/(G^A)$ is equal to the derivative of the function $\mathrm{T}(k) = \sigma_0 k^\alpha - \vartheta\sigma_1 k^{\alpha-1} - \vartheta(1-\sigma)$ with respect to *k* evaluated at the higher steady state, (ii) function $\mathrm{T}(k)$ is strictly concave and (iii) its graph cuts the 45° line at the higher steady state from above (substitute ϑ for *b* in Figure 1 above). On the other hand, for $\lambda_3 > 1$ we have $(1+i)(k^L)(1-\alpha)\sigma\left[1+\left(\vartheta/k^L\right)\right]/(G^A) > 1$ which is implied by the fact that the graph of $\mathrm{T}(k)$ cuts the 45° line at the lower steady state from below. ■[19]

Now that we know that in the case of $\vartheta < \bar{\vartheta}$ only the higher steady state solution k^H is saddle-path stable, let us turn to comparative steady state analysis in a small neighborhood of k^H. We first investigate how the capital intensities of Home and Foreign and the terms of trade respond to an infinitesimal shock in Home's sustainable government debt.

To clarify the role of the net foreign asset position of Home for the steady state effects of sustainable government debt in our two-country OLG model, we follow Zee (1987, 609) and write the condition for the international asset market equilibrium in the following equivalent form:

(24) $$p = -\Phi^*/\Phi,\ \Phi \equiv k + b\left\{\sigma\left[(1+i)/G^A - 1\right] + 1\right\} - \sigma_0 k^\alpha \;,$$

$$\Phi^* \equiv k^* + b^*\left\{\sigma\left[(1+i)/G^A - 1\right] + 1\right\} - \sigma_0^* \left(k^*\right)^\alpha,$$

whereby Φ (Φ^*) denotes the net foreign asset position of Home (Foreign).

PROPOSITION 5 (Steady state effects of unilateral sustainable budget policy).
Suppose there is an infinitesimal change of b while b^* remains unchanged (*unilateral sustainable budget policy*). *There is then a negative relationship between the change of capital*

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[19] The proof of Proposition 4 confirms the conjecture mentioned in footnote 8 above.

intensities in Home and Foreign and per-capita debt, i.e. $dk/db<0$ and $dk^*/db=\mu\, dk/db<0$, *while the terms of trade are not affected at all, i.e.* $dp/db=0$.

PROOF. $dp/db=0$ is obvious from Lemma 1. To prove $dk/db<0$, differentiate totally (24) with respect to k,k^*,p and b. Since $dp=0$ and $dk^*=\mu dk$ hold, we obtain: $\left\{p\left[1-\sigma/G^A(1-\alpha)(1+i)\left(1+b/k^H\right)\right]+\left[1-\sigma/G^A(1-\alpha)(1+i)\left(1+b^*\mu^{-1}/k^H\right)\right]\mu\right\}dk=$ $-p\left(1+\sigma\left[\left(1+i-G^A\right)/G^A\right]\right)db$. After inserting $p=\mu\zeta/(1-\zeta)$ and collecting terms, we get the following: $\left[1-\sigma/G^A(1-\alpha)(1+i)\left(1+\vartheta/k^H\right)\right]dk=-\zeta\left(1+\sigma\left[\left(1+i-G^A\right)/G^A\right]\right)db$ or: $dk/db=-\zeta\left(1+\sigma\left[\left(1+i-G^A\right)/G^A\right]\right)\Big/\left[1-\lambda_3\right]$. Since $\lambda_3<1$ from Proposition 4, $dk/db<0$.■

Proposition 5 claims that a larger government debt in Home always crowds out private capital in Home *and* in Foreign, while the terms of trade do not respond at all. The intuition behind this result is as follows. In the steady state an increase of sustainable government debt causes lump-sum taxes to rise in order to pay for the additional interest. As a consequence, net wage and savings of the young household decline. To restore equilibrium, capital intensities in both countries decrease according to a fixed proportion, and hence real interest in Home and Foreign increases equivalently thus leaving, in accordance with international interest parity condition (20), the terms of trade unaffected.

An important implication of Proposition 5 is that the net foreign asset position of the more indebted country is not decisive at all for the steady state response of the terms of trade to a larger (or lower) sustainable government debt. This implication plainly contradicts Zee's (1987, 617) claim that a "higher level of domestic government debt leads to a fall (rise) in the terms of trade if, at the initial steady state, the home country is a net debtor (creditor)."

To emphasize, in contrast, the independence of steady state terms of trade from the foreign net asset position of the more indebted country, we present in Figure 3 and Figure 4, respectively, the phase lines (steady state lines) of the dynamic system (20)-(22) when Home is a net foreign creditor or a net foreign debtor.[20]

<<< Figure 3 about here >>>

<<< Figure 4 about here >>>

There are two steady state lines in a (k_t, p_t) diagram, termed the AA- and the CC-curve. The AA-curve (= equation (24)) can be interpreted as geometrical locus of all pairs (k_t, p_t) which assures international asset market clearing. The CC-curve ($p = \mu\left[\zeta/(1-\zeta)\right]$) represents all (k_t, p_t) combinations which induce equilibrium in the combined commodity markets of Home and Foreign. Clearly, the CC-curve is horizontal in the (k_t, p_t) diagram. Upon differentiating (24) with respect to k while taking into account $k^* = \mu k$, it is not difficult to show that the slope of the AA-curve is determined as follows: $dp/dk_{|KK} = -\left(p\,\partial\Phi/\partial k + \mu\left(\partial\Phi^*/\partial k^*\right)\right)/\Phi$. Since the numerator on the right hand side of this expression is always larger than zero,[21] the slope of the AA-curve depends on the sign of Φ: if Home is a net foreign creditor ($\Phi < 0$) (Figure 3), the AA-curve is positively sloped, and its slope is negative, if Home is a net foreign debtor ($\Phi > 0$) (Figure 4).

Let us now consider a marginal change of *b*. It is clear that a *b* shock has no impact on the CC-curve. It does, however shift the AA-curve. Analytically, this shift is determined as follows: $dp/db_{|KK} = -p\left(\partial\Phi/\partial b\right)\Phi^{-1}$. Since $\partial\Phi/\partial b = \left[\sigma(1+i) + (1-\sigma)G^A\right]/G^A$ >0, the sign of the foreign asset position of Home governs the shift: if Home is a net foreign creditor,

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20 The steady state lines depicted in Figure 3 and Figure 4 are derived from the following parameter set using MATHEMATICA 6.0: $G^A = 1.0$, $\beta = 0.8$, $\zeta = 0.5$, $M = M^* = 4.5$, $\alpha = \alpha^*$ $= 0.3$, and for $\Phi < 0$ we set $b = 0.2$, $b^* = 0.4$ (with the order reversed for $\Phi > 0$).

21 To show that the numerator on the right hand side is larger than zero notice that $\partial\Phi/\partial k = 1-(1-\alpha)(1+i)\sigma(1+b/k)$ and a similar expression for Foreign hold. Averaging over both expressions and taking into account the stability condition $\lambda_3 < 1$ (at $k = k^H$) we see that the claim is true.

$dp/db_{|KK} > 0$, hence the AA-curve shifts upwards. The opposite is true when Home is a net foreign debtor. But whatever the shift of the AA-curve, the terms of trade do not change whether Home is a net foreign creditor or a net foreign debtor.[22]

6. Transitional impacts of shocks in sustainable government debt

Knowing that the steady state terms of trade effect of government debt is independent of the net foreign asset position, it is natural to ask whether the transition path of the terms of trade towards the new steady state is also independent of the net foreign asset position or not. Here again Zee (1987, 611) claimed that in the period after the expansion of government debt the terms of trade would fall (rise) if Home is a net debtor (creditor), and that in the shock period itself there is no impact on the terms of trade at all.

In investigating the transitional effects of marginal changes in Home's government debt on Home and Foreign capital intensities and on the terms of trade, we have to take into account that the initial steady state is only saddle-path and not asymptotically stable (as assumed by Zee). This implies that one of the endogenous dynamic variables is a 'jump' variable not exogenously determined by initial conditions. A natural conjecture is that the terms of trade represent the jump variable which immediately responds to a policy shock, while the 'sluggish' capital stocks (per efficiency capita) in Home and Foreign do not adapt because their values are historically fixed.

To get more information about the analytical structure of the three-dimensional equilibrium dynamics around the stable steady state solution, we approximate (20)-(22) in a small neighborhood of $\left(p,k^H,k^{*,H}\right)$:

$$\begin{bmatrix} p_{t+1} \\ k_{t+1} \\ k^*_{t+1} \end{bmatrix} = \left[J\left(p,k^H,k^{*,H}\right)\right]\begin{bmatrix} p_t \\ k_t \\ k^*_t \end{bmatrix} + \left[I - J\left(p,k^H,k^{*,H}\right)\right]\begin{bmatrix} p \\ k^H \\ k^{*,H} \end{bmatrix}. \tag{25}$$

The general solution of the first-order linear difference equation system (25) takes the following form:

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[22] One might object that the irrelevance of the net foreign asset position for the terms of trade effect of sustainable government debt depends on our assumption of similar technologies. However, as Lin (1994, 102) showed in a related framework, and we will show in the next section,

$$p_t = p + \kappa_2 \upsilon_2^p \left(\lambda_2\right)^t + \kappa_3 \upsilon_3^p \left(\lambda_3\right)^t,$$

(26) $$k_t = k + \kappa_2 \upsilon_2^k \left(\lambda_2\right)^t + \kappa_3 \upsilon_3^k \left(\lambda_3\right)^t,$$

$$k_t^* = k^* + \kappa_2 \upsilon_2^* \left(\lambda_2\right)^t + \kappa_3 \upsilon_3^* \left(\lambda_3\right)^t.$$

Here $\kappa_i,\, i = 2,3$ denote constants determined by initial conditions for capital intensities in Home and Foreign, while $\upsilon_i = \left(\upsilon_i^p, \upsilon_i^k, \upsilon_i^*\right)^T,\, i = 2,3$ is the eigenvector associated with the eigenvalues within the unit circle $\lambda_i,\, i = 2,3.$ Note that the eigenvector associated with the eigenvalue larger than unity is excluded from (26) by setting $\kappa_1 = 0$. However, this exclusion implies that the equilibrium dynamics must not start from any feasible combination $\left(p_0, k_0, k_0^*\right)$ in the neighborhood of $\left(p, k^H, k^{*,H}\right)$, and that the initial combination of dynamic variables has to be located on the stable submanifold in the $\left(p_t, k_t, k_t^*\right)$-space. If $\left(p_0, k_0, k_0^*\right)$ belongs to the stable submanifold, the economy converges on $\left(p, k^H, k^{*,H}\right)$, otherwise the system dynamics strays in finite time. Before expanding these informal claims thoroughly in Proposition 6, Lemma 3 describes the eigenvectors associated with the less than unity eigenvalues.

LEMMA 3. *The eigenvectors associated with the eigenvalues of the Jacobian* (23) *within the unit circle* $\upsilon_i,\, i = 2,3$ *read as follows:*

$$\upsilon_2 = \left(-p/k, 1/\alpha + \gamma, \gamma\mu\right)^T, \upsilon_3 = \left(0,1,\mu\right)^T, \gamma \equiv \left[\zeta b\left(1 - \sigma + \sigma\lambda_1\right)\right] / \left[k\left(\alpha - \lambda_3\right)\right].$$

PROOF. See the Appendix.

The eigenvalues of Lemma 2 and the eigenvectors in Lemma 3 enable us to present in Proposition 6 a closed form solution of the transitional dynamics around the higher (saddle path stable) steady state $\left(p, k^H, k^{*,H}\right)$.

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the independence of the terms of trade effect of sustainable government debt from the net foreign asset position remains true even under dissimilar production technologies.

PROPOSITION 6 (Transitional dynamics around the higher steady state). *A linear approximation of the two-country, two-good equilibrium dynamics* (20)-(22) *evaluated at* $\left(p, k^H, k^{*,H}\right)$ *takes the following form:*

$$p_t = p\left\{1+\left(\alpha/k^H\right)\left[\mu^{-1}\left(k_t^* - k^{*,H}\right)-\left(k_t - k^H\right)\right]\right\}, \tag{27}$$

$$k_{t+1} = k_t + \alpha\left\{\left[\left(1-\alpha\right)/\alpha - \left(\lambda_3 - \alpha\right)\gamma\right]\left(k^H - k_t\right) + \left(1/\alpha + \gamma\right)\left(\lambda_3 - \alpha\right)\mu^{-1}\left(k_t^* - k^{*,H}\right)\right\}, \tag{28}$$

$$k_{t+1}^* = k_t^* + \alpha\left[\left(1-\lambda_3\right)/\alpha - \gamma\left(\lambda_3 - \alpha\right)\right]\left(k^{*,H} - k_t^*\right) + \alpha\gamma\left(\alpha - \lambda_3\right)\mu\left(k_t - k^H\right), \tag{29}$$

whereby k_0 and k_0^* are exogenously given.

PROOF. See the appendix.

The equilibrium dynamics depicted by equations (27)-(29) enable us to evaluate the immediate effects of a shock in the per-capita government debt of Home. Suppose that the shock occurs in period $t = 0$, it is unannounced and permanent. In view of Proposition 6, Corollary 2 below describes the immediate impacts of a small government-debt shock in Home if the economy starts on the stable submanifold in the neighborhood of $\left(p, k^H, k^{*,H}\right)$.

COROLLARY 2. *Suppose there is a finite but small change of* b *(while* b^* *remains unchanged) in period* $t = 0$*, such that* $\vartheta < \overline{\vartheta}$ *holds even after the* b*-shock. Then, the terms of trade of the shock period* p_0 *remain unchanged* ($dp_0/db = 0$), *while Home and Foreign capital intensities one period later exhibit a negative response to the policy shock as follows*: $dk_1^*/db = \mu\left(dk_1/db\right) < 0$.

PROOF. We know from Lemma 1 that steady-state capital intensities in Home and Foreign are related as follows: $k^{*,H} = \mu k^H$. Since the economy starts in a steady state and initial capital intensities do not respond to the policy shock, it must be true that $k_0^* = \mu k_0$. Moreover, $dp/db = 0$ holds. Hence, in view of (27) for $t = 0$, $dp_0/db = 0$ follows immediately. In order to show that $dk_1^*/db = \mu\left(dk_1/db\right) < 0$ is true, we consider (28) and (29) for $t = 0$, and after slight manipulations the following equations are obtained:

$$k_1 = k_0 + \alpha\left\{\gamma\left(\lambda_3 - \alpha\right)\left[k^H - k_0 + \mu^{-1}\left(k_0^* - k^{*,H}\right)\right]\right\} + \left(1-\alpha\right)\left(k^H - k_0\right) + \left(\lambda_3 - \alpha\right)\mu^{-1}\left(k_0^* - k^{*,H}\right),$$

$k_1^* = k_0^* + \alpha\gamma\left(\lambda_3 - \alpha\right)\left[\mu\left(k_0 - k^H\right) + \left(k^{*,H} - k_0^*\right)\right] + \left(1-\lambda_3\right)\left(k^{*,H} - k_0^*\right)$. On account of $k_0^* = \mu k_0$ and $k^{*,H} = \mu k^H$, the equations for k_1 and k_1^* collapse to the following equations for both variables: $k_1^* = \mu k_1 = \mu\left[k_0 + \left(1-\lambda_3\right)\left(k^H - k_0\right)\right]$. Since $dk^H/db < 0$ and $\lambda_3 < 1$, it follows that $dk_1^*/db = \mu dk_1/db < 0$.■

Thus, an increase in Home's sustainable government debt in period $t = 0$ unambiguously reduces the capital intensities of Home and Foreign in period 1, but has no immediate impact on the terms of trade in the shock period.

The insensitivity of the initial terms of trade with respect to permanent changes in Home's government debt is most easily explained if we focus in the shock period on the Golden Rule case (i.e. $1+i_0 = G^A$) in which lump sum taxes do not respond to $b-$variations (see equation (8)). As a consequence, per-capita savings in Home do not respond to the policy shock because the net wage of Home's young household is unchanged. The reason is that k_0 and therefore the gross wage income and lump sum taxes remain unaffected. In Foreign there is no policy change, hence per-capita savings in Foreign do not adapt to the policy shock in Home. Moreover, assume for simplicity that $\zeta = 1/2$, $G^A = 1$ and $M = M^*$ hold. Under these assumptions Corollary 2 implies that $\Delta k_1 = \Delta k_1^*$. Evaluating equation (22) for $t = 0$, we obtain an equation determining the initial terms of trade: $p_0 = \left[\zeta/\left(1-\zeta\right)\right]\left[k_1^* - M\left(k_0^*\right)^\alpha\right]\Big/\left[k_1 - M\left(k_0\right)^\alpha\right]$. Hence, since $\Delta k_1 = \Delta k_1^*$, $\Delta p_0 = 0$.

If initial terms of trade remain unchanged and the initial steady state is Golden Rule, the question remains open of what causes the crowding out of private capital in period 1 in Home and in Foreign. To answer this question we have to investigate which domestic and foreign endogenous variables in the shock period are adapting to the expansion of government debt in Home, and which endogenous variables remain unchanged. Starting with the latter, it is clear that the production of Home's and Foreign's commodity as well as the consumption demand for both goods by the younger households in Home and Foreign do not adapt to the policy shock. What is true for the consumption demand of Home and Foreign younger households, is, however, not true with respect to the consumption demand of Home's old household for the domestic and the foreign commodity.

To see this, let's look at the consumption demand of Home's old household for the domestic good in the shock period, $x_0^2 = \zeta G^L a_0 \left(1+i_0\right)\left[k_0 + b - \Phi_0\right]$ (equation A.3 in the appendix). On the right hand side of this equation a_0, i_0, k_0 and Φ_0 are historically fixed while b is the policy parameter. When sustainable government debt per capita b in Home increases, then x_0^2 rises because a larger sustainable government debt raises the wealth of the old household in Home, and her larger consumption crowds out k_1 on account of the market clearing condition (17) in the shock period. The larger wealth induces the old household in Home also to increase the consumption of the Foreign commodity $y_0^2 = \left(1-\zeta\right) G^L a_0 \left(1+i_0\right) p_0 \left[k_0 + b - \phi_0\right]$ (equation A.4 in the appendix) which crowds out k_1^* in view of the market clearing condition (17*). In this way the old household in Home, enriched by the interest-inclusive repayment of the larger debt of Home's government, transmits the domestic debt policy shock from Home to Foreign.

Knowing that there is no immediate impact on the terms of trade in the shock period, while in the after shock period both capital intensities decline, it remains to be analyzed what happens thereafter. Corollary 3 claims that the terms of trade still remain unaffected while capital intensities continue to fall towards their lower steady state values.

COROLLARY 3. *Suppose there is in period* $t = 0$ *a finite but small change of* b *(while* b^* *remains unchanged) such that* $\vartheta < \overline{\vartheta}$ *holds even after the* b*-shock. Then, the terms of trade of the periods following the shock,* $p_t, t = 1, 2, ..$ *remain unchanged* ($dp_t/db = 0$, $t = 1, 2, ...$), *while Home and Foreign capital intensities in all periods later exhibit a proportional negative response to the policy shock* ($dk_{t+1}^*/db = \mu dk_{t+1}/db < 0$, $t = 1, 2, ..$).

PROOF. We know that $dp/db = 0$ and from the proof of Corollary 2 we know that $k_1^* = \mu k_1$ holds. Evaluating (27) at $t = 1$ we see immediately that $dp_1/db = 0$ since again $k^{*,H} = \mu k^H$ holds. Similarly as in the proof of Corollary 2, we obtain the following equations: $k_2^* = \mu k_2 = \mu\left[k_1 + \left(1-\lambda_3\right)\left(k^H - k_1\right)\right]$. Reiterating this procedure for $t = 2, 3...$, and calculating the derivatives of the capital intensities in Home and Foreign of each period with respect to b the proof is completed. ■

As a final step, we extend our analysis to unequal capital income shares (dissimilar technologies) across countries, i.e. $\alpha \neq \alpha^*$. Due to analytical complexity, we resort to numerical illustrations of four typical cases. They follow from possible combinations of net foreign asset positions in Home and in Foreign ($\Phi < 0 \wedge \Phi^* > 0$ or $\Phi > 0 \wedge \Phi^* < 0$) and larger-smaller relationships between the magnitudes of domestic and foreign capital income shares ($\alpha < \alpha^*$ or $\alpha > \alpha^*$). In case 1, Home is a net foreign creditor ($\Phi < 0$) and her capital income share is less than in Foreign ($\alpha < \alpha^*$), while in case 2 the same holds for capital income shares and Home is a net foreign debtor ($\Phi > 0$). Analogously, in case 3 Home is a net foreign creditor ($\Phi < 0$) with a larger capital income share than in Foreign while in case 4 the same holds for capital income shares with Home being a net foreign debtor.

<<< Table 4 about here >>>

Table 4 reports on the results of an increase in *b* by 20% for the impact in the shock period and the new steady state. [23] For the numerical analysis we assume the following common parameter set $G^A = 1.0$, $\beta = 0.8$, $\zeta = 0.5$, $M = M^* = 4.5$. We set for $\alpha < \alpha^*$ $\alpha = 0.25, \alpha^* = 0.3$ (with values being reversed for $\alpha > \alpha^*$) and for $\Phi < 0$ we set $b = 0.2, b^* = 0.6$ (again reversing values for $\Phi > 0$).

As the results reported in Table 4 show, the most obvious difference between dissimilar and similar technologies concerns the terms of trade. First, the terms of trade do adapt to a policy shock *both* in the shock period *and* thereafter. Second, the new steady state values are higher (lower) compared to the pre-shock steady state when $\alpha > \alpha^*$ ($\alpha < \alpha^*$), regardless of whether Home is a net foreign debtor or net foreign creditor. Third, in cases 1 and 2 the terms of trade in the shock period, p_0, are larger than their old steady state value, and they converge from above towards the lower, new steady state value, while in cases 3 and 4 the opposite holds.

In spite of these differences one basic characteristic of the transition path of the terms of trade under similar technologies generalizes to the case of dissimilar technologies: the transition path is independent of the sign and magnitude of the net foreign asset position of the more indebted country. This is in clear contrast to Zee's (1987, 611) claim mentioned above.[24] Given that the two countries in our model truly represent the collection of net foreign creditor and net foreign debtor countries, the result implies that the transitional terms of trade effect (if any) is independent of whether sustainable government debt is expanded in a net foreign creditor country (like Japan) or in a net foreign creditor country (like the USA).

7. Summary and conclusions

This paper is concerned with sustainable government debt in a two-good, two-country OLG model under exogenous growth. In contrast to earlier notions of government debt sustainability, constant stocks of government debt per (efficiency) capita are defined as sustainable if a slight expansion of government debt in one country does not lead to a sudden collapse of the world economy. We first explore the limits for a weighted average of national government debt levels at which the collapse, i.e. a saddle-node bifurcation occurs (maximum sustainable government debt). Second, we investigate the existence and dynamic stability of steady states for private capital intensities and terms of trade as well as their transitional dynamics when government debt levels remain below these limits (are sustainable).

Regarding maximum sustainable government debt, we find that an upper limit for the weighted average of domestic and foreign levels of government debt analogous to Rankin and Roffia's (2003) maximum sustainable government debt always exists, and it occurs at a non-trivial steady state (interior maximum) rather than at a trivial steady state solution for capital intensities. The determinants of this maximum in the two-country world economy are similar to those found in the closed economy: a higher capital income share, more impatience, a higher natural growth factor and less factor productivity limit the range for maximum sustainable debt. Moreover, national maximum debt levels corresponding to the overall maximum are negatively related. Besides the factors which determine the global maxi-

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[23] The transitional dynamics were calculated numerically using the NLP solver of GAMS 2.5, version 21.5.

[24] Zee's (1987) conclusion appears to be based on his presumption of asymptotic stability of the terms of trade dynamics. Both the irresponsiveness of the initial terms of trade and the dependence of the terms of trade dynamics thereafter on the sign of the net foreign asset position (Zee 1987, 611) can be traced back to this presumption.

mum, this relationship also depends on the expenditure share for the domestic (foreign) good and on internationally differences in technological levels.

As regards the second main topic, we find that if the sum of domestic and foreign government debt levels weighted by the expenditure share for the domestic and foreign good and the ratio of foreign to domestic technological levels remains below the corresponding maximum (where government debt is sustainable), two non-trivial (non-degenerate) steady states for private capital intensities exist in both countries. The existence condition for non-trivial steady state solutions also implies that the steady state with the higher capital intensity is saddle path stable while the other is saddle path unstable - a result which contrasts with the presumption of Zee (1987) that the terms of trade dynamics in his two-good, two-country OLG model are asymptotically stable.

The proof of the saddle-path stability of the steady state with the higher level of domestic and foreign capital intensities provides us with a methodological justification reexamining the effects of a unilateral expansion of sustainable government debt on steady state terms of trade as well as on steady state domestic and foreign capital intensities. We are able to confirm Lin's (1994) result that the steady state terms of trade are unaffected by a unilateral government debt shock if the capital income shares are equal across both countries. This accords exactly with our assumption of internationally similar production technologies. We have to reject Zee's (1987) conclusion that the steady state effect of government debt shocks on the terms of trade depends on the net foreign asset position of the more indebted country. As regards the steady state effects of unilateral debt policy on private capital intensities in Home and Foreign, we affirm the negative relationship as already stated by Zee (1987) and Lin (1994). Here, both approaches to dynamic stability in two-country OLG models apparently lead to the same results.

The knowledge that the steady state with the larger capital intensity is saddle-path stable also enables a thorough investigation of the transitional effects of small variations in sustainable government debt. Although our transitional analysis takes the jump character of the terms of trade fully into account, we find for the case of internationally similar technologies that the initial terms of trade are not affected by an unexpected and unannounced shock in Home's sustainable government debt, while Home and Foreign capital intensities in the after-shock period decline in a fixed proportion, absorbing in this way alone the full policy shock. We also show that under similar technologies the terms of trade do not respond to government debt shocks along the transition path towards the new steady state too. Again, the international borrower-lender status of a country does not matter for this result.

Finally, a numerical analysis of four typical parameter constellations under dissimilar production technologies shows that unilateral budget policy also affects the terms of trade along the transition path towards the steady state. In accordance with the notion of saddle-path stability the terms of trade in the shock period not only immediately adapt to the policy shock but they usually overshoot their new steady state value. Moreover, while the transition path of the terms of trade depends on the relative magnitude of domestic, in comparison to foreign, capital income shares, it is independent of the sign of the net foreign asset position in Home. The numerical results, obtained from typical parameter sets, contradict two main conclusions which Zee (1987) derived analytically: Namely that (1) the terms of trade in the shock period are unimpaired by a government debt shock, and (2) the rise or fall of the terms of trade along the transition path towards the new steady state depends on the sign of Home's net foreign asset position.

What can we conclude about the effects of an expansion of the stock of government debt on terms of trade and capital accumulation in a world economy consisting of two groups of countries characterized by opposite net foreign asset positions but equal capital income shares? First, if government debt levels are maximal even a slight expansion of government debt in one country sets off an implosion of private capital throughout the world economy irrespective of whether the more indebted country is a net foreign creditor or a net foreign debtor country. Hence, the often heard claim that an internationally coordinated expansion of government debt does not impact negatively on private capital accumulation and growth cannot be verified in our two-good, two-country OLG model if government debt levels are maximal. Second, if government debt levels are sustainable a unilateral expansion of government debt crowds out private capital in both (groups of) countries, but there is no effect on the terms of trade even if technological levels differ significantly across both groups of countries. Despite the simplicity of our model context, this result might still help explain why more government debt in a large net foreign debtor country (like the USA) does not have negative terms of trade effects, and also why, for a net foreign creditor country (like Japan) it does not have positive terms of trade effects. Third, even if capital income shares are different among both groups of countries the terms of trade effect of a unilateral expansion of sustainable government debt is nonetheless independent of whether the debt-expanding country is a net foreign creditor or a net foreign debtor, and of how large the net foreign credit or net foreign debt of the country is.

A dynamic computable general equilibrium (CGE) model with many countries and multiple commodities and production factors, and with estimated or calibrated parameters would be a natural extension of the present work. Another area for extension might entail an analysis of the welfare consequences of unilateral debt policy, in both a steady state as well as in a transitional dynamics setting.

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Appendix

Optimal consumption and savings of households in Home (*Foreign*)

In order to show how the equations of motion in world market equilibrium are derived, the optimal consumption and savings levels of households are needed for $t = 1, 2, \ldots$. We indicate roughly how optimal consumption and savings for households in Home are obtained. First, insert s_t into the second budget constraint, while taking the international interest parity condition (16) into account. This implies:

$$x_t^1 + (1/p_t)\, y_t^1 + x_{t+1}^2 \big/ (1 + i_{t+1}) + y_{t+1}^2 \big/ \left[p_{t+1} (1 + i_{t+1}) \right] = w_t - \tau_t .$$

Second, maximize (15) subject to this intertemporal budget constraint and solve for optimal consumption quantities and optimal savings. An analogous procedure gives the optimal consumption quantities and optimal savings in Foreign.

(A.1) $$x_t^1 = \left[\zeta \big/ (1+\beta) \right] \left[w_t - \tau_t \right]$$

(A.1*) $$x_t^{*,1} = \left[\zeta \big/ (1+\beta) \right] \left[w_t^* - \tau_t^* \right] \big/ p_t$$

(A.2) $$y_t^1 = \left[(1-\zeta) \big/ (1+\beta) \right] p_t \left[w_t - \tau_t \right]$$

(A.2*) $$y_t^{*,1} = (1-\zeta) \big/ (1+\beta) \left(w_t^* - \tau_t^* \right)$$

(A.3) $$x_0^2 = \zeta (1 + i_0)\, a_0 G^L \left[k_0 + b - \Phi_0 \right], \; x_{t+1}^2 = \beta\zeta \big/ (1+\beta)(1 + i_{t+1})(w_t - \tau_t)$$

(A.3*) $$x_0^{*,2} = \zeta G^L a_0 \left[\left(1 + i_0^*\right)\left(k_0^* + b^*\right) + (1 + i_0)\, p_0 \Phi_0 \right] \big/ p_0 ,$$
$$x_{t+1}^{*,2} = \beta\zeta \big/ (1+\beta)\left(1 + i_{t+1}^*\right)\left(w_t^* - \tau_t^*\right) \big/ p_{t+1}$$

(A.4) $$y_0^2 = (1-\zeta)\, G^L a_0 (1 + i_0)\, p_0 \left[k_0 + b - \phi_0 \right],$$
$$y_{t+1}^2 = \beta (1-\zeta) \big/ (1+\beta)(1 + i_{t+1})(w_t - \tau_t)\, p_{t+1}$$

(A.4*) $$y_0^{*,2} = (1-\zeta)\, G^L a_0 \left[\left(1 + i_0^*\right)\left(k_0^* + b^*\right) + p_0 (1 + i_0)\, \phi_0 \right],$$
$$y_{t+1}^{*,2} = \beta (1-\zeta) \big/ (1+\beta)\left(1 + i_{t+1}^*\right)\left(w_t^* - \tau_t^*\right)$$

(A.5) $$s_t = \sigma \left[w_t - \tau_t \right], \; \sigma \equiv \beta \big/ (1+\beta)$$

(A.5*) $s_t^* = \sigma\left[w_t^* - \tau_t^*\right]$

Derivation of the equation of motion in autarky

First, insert (A.5) into (10) and you will obtain: $G^A\left(k_{t+1}+b\right) = \sigma\left[\left(w_t/a_t\right) - \left(\tau_t/a_t\right)\right]$. Second, substitute for w_t/a_t and τ_t/a_t the right-hand sides of (3) and (8), respectively. Third, collecting variables gives (12).

Proof of Proposition 1

Let $\Psi(b) \equiv F\left(\mathrm{K}(b), b\right)$, where $F(k,b) \equiv \mathrm{T}(k) - k = \sigma_0 k^{\alpha} - b\sigma_1 k^{\alpha-1} - b(1-\sigma) - k$ and $\mathrm{K}(b)$ is the solution of equation $F_k(k,b) = 0$ for a given value of b. Since (i) $F_k(k,b)$ is a continuous and strictly decreasing function, (ii) $\lim_{k\to 0} F_k(k,b) = \infty$, and (iii) $\lim_{k\to\infty} F_k(k,b) = -1$, an Intermediate Value Theorem guaranties for each b the existence of a κ which solves $F_k(\kappa,b) = 0$. Moreover, the solution is unique since $F_{kk}(k,b) < 0$. Hence, $\kappa = \mathrm{K}(b)$. Note also that $\mathrm{K}(b)$ is a strictly increasing function because $F_{kk}(k,b) < 0$ and $F_{kb}(k,b) > 0$. Since $F_b(k,b) < 0$, an envelope theorem implies that $\Psi(b)$ is a strictly decreasing function with $\Psi(0) > 0$, and $\lim_{b\to\infty} \Psi(b) < 0$. Continuity of $\Psi(b)$ implies the existence of $\bar{b}$ such that $\Psi(\bar{b}) = 0$. For $b \in \left[0, \bar{b}\right)$, $\Psi(b) > 0$, while $\Psi(b) < 0$ for $b \in \left(\bar{b}, \infty\right)$.■

Derivation of the Jacobian matrix for the dynamic system

To show roughly how we obtained the elements of the Jacobian matrix (23), we now describe the main steps taken in the derivation of $\partial p_{t+1}/\partial p_t = 1 + (1-\alpha)(H/k)$. First, take the total differential of (34) with respect to all variables:

$$dp_{t+1} = \left(M^*/M\right)\left(k_{t+1}\right)^{1-\alpha}\left(k_{t+1}^*\right)^{\alpha-1} dp_t + p_t\left(M^*/M\right)(1-\alpha)\left(k_{t+1}\right)^{-\alpha}\left(k_{t+1}^*\right)^{\alpha-1} dk_{t+1}$$
$$-p_t\left(M^*/M\right)(1-\alpha)\left(k_{t+1}\right)^{1-\alpha}\left(k_{t+1}^*\right)^{\alpha-2} dk_{t+1}^*$$

Second, solve the left-hand sides of (21) and (22) simultaneously with respect to the total differentials of k_{t+1} and k_{t+1}^*. Third, form the partial differentials $\partial k_{t+1}/\partial p_t$ and $\partial k_{t+1}^*/\partial p_t$ while taking the results of the second step into account. Fourth, evaluate the total differential of the first step at a steady state solution and consider the infinitesimal changes of k_{t+1} and k_{t+1}^* only with respect to p_t:

$\partial p_{t+1}/\partial p_t = 1 + p(1-\alpha)\left[k^{-1}\partial k_{t+1}\big/\partial p_t - \left(k^*\right)^{-1}\partial k^*_{t+1}/\partial p_t\right]$. The last step is to insert the partial differentials evaluated at a steady state solution from step three into the above equation.

Proof of Lemma 3[25]

To calculate the eigenvalues and the eigenvectors of the Jacobian, we use the characteristic equation of the Jacobian (23): $(J-\lambda_i I)v_i = 0$, whereby I denotes the identity matrix, and the characteristic equation in expanded form reads as follows:

$$\left(J-\lambda_i I\right)v_i = 0 \Leftrightarrow \begin{pmatrix} j_{11}-\lambda_i & j_{12} & j_{13} \\ j_{21} & j_{22}-\lambda_i & j_{23} \\ j_{31} & j_{32} & j_{33}-\lambda_i \end{pmatrix}\begin{pmatrix} \upsilon_i^p \\ \upsilon_i^k \\ \upsilon_i^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Subtracting the second row μ times from the first row and multiplying the first row by $k/(1-\alpha)$ yields the following equivalent equation

$$\begin{pmatrix} \frac{k}{(1-\alpha)}(1-\lambda_i)+H & (1+i)p & -(1+i)p\mu^{-1} \\ j_{21} & j_{22}-\lambda_i & j_{23} \\ \frac{-\mu H}{p} & \mu\left[\lambda_i-(1+i)\right] & (1+i)-\lambda_i \end{pmatrix}\begin{pmatrix} \upsilon_i^p \\ \upsilon_i^k \\ \upsilon_i^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Adding the first row μ/p times to the last row we get

$$\begin{pmatrix} \frac{k}{(1-\alpha)}(1-\lambda_i)+H & (1+i)p & -(1+i)p\mu^{-1} \\ j_{21} & j_{22}-\lambda_i & j_{23} \\ \frac{k\mu(1-\lambda_i)}{p(1-\alpha)} & \mu\lambda_i & -\lambda_i \end{pmatrix}\begin{pmatrix} \upsilon_i^p \\ \upsilon_i^k \\ \upsilon_i^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Finally subtracting the third row $\mu^{-1}(1+i)p/\lambda_i$ times from the first row leads to

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25 For mathematical assistence in proving Lemma 3 we are particularly indebted to Andreas Rainer.

$$\text{(A.6)} \qquad \begin{pmatrix} \frac{k}{(1-\alpha)}(1-\lambda_i)\left(1-\frac{1+i}{\lambda_i}\right)+H & 0 & 0 \\ j_{21} & j_{22}-\lambda_i & j_{23} \\ \frac{k\mu(1-\lambda_i)}{p(1-\alpha)} & \mu\lambda_i & -\lambda_i \end{pmatrix}\begin{pmatrix} \upsilon_i^p \\ \upsilon_i^k \\ \upsilon_i^* \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Equation (A.6) can be solved if and only if the determinant of the matrix in (A.6) vanishes, i.e. if either

$$\text{(A.7)} \qquad \begin{vmatrix} j_{22}-\lambda_i & j_{23} \\ \mu & -1 \end{vmatrix}=0, \text{ or}$$

$$\text{(A.8)} \qquad \frac{k}{(1-\alpha)}(1-\lambda_i)\left(1-\frac{1+i}{\lambda_i}\right)+H=0.$$

There are thus two cases to be distinguished: Case 1 in which (A.7) holds and case 2 for which (A.8) is true. Let us consider both cases in turn.

Case 1. Using the definition of j_{22} and j_{23}, equation (A.7) straightforwardly leads to

$$\lambda_3 = j_{22}+\mu j_{23} = \frac{(1+i)\sigma(1-\alpha)}{G^A}\left(1+\frac{\vartheta}{k}\right).$$

To determine its corresponding eigenvector, we use (A.6). Because $\frac{k}{(1-\alpha)}(1-\lambda_i)\left(1-\frac{1+i}{\lambda_i}\right)+H\neq 0$, it follows that $\upsilon_3^p=0$, and thus (A.6)

$$\begin{pmatrix} j_{22}-\lambda_i & j_{23} \\ \mu\lambda_3 & -\lambda_3 \end{pmatrix}\begin{pmatrix} \upsilon_3^k \\ \upsilon_3^* \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

with the solution claimed in Lemma 3 as can be seen as follows: The first row leads to $\upsilon_3^*=\mu\upsilon_3^k$, and the second row as a result of the value of λ_3 can then be solved identically, i.e. we can chose $\upsilon_3^k=1$.

Case 2. In this case we know that $\frac{k}{(1-\alpha)}(1-\lambda_i)\left(1-\frac{1+i}{\lambda_i}\right)+H=0$. Since $H/k=(1+i)/\alpha-1$, it follows that $(1+i)/\alpha-1>1$ and $\lambda_2=\alpha$ as claimed in Lemma 2.

The eigenvector associated with the second eigenvalue can be found as follows: $\frac{k}{(1-\alpha)}(1-\lambda_i)\left(1-\frac{1+i}{\lambda_i}\right)+H=0$ implies that υ_2^p can be chosen freely, so for instance we can take $\upsilon_2^p = p/k$. Therefore (A.6) reduces to

$$\begin{pmatrix} j_{22}-\lambda_i & j_{23} \\ \mu\alpha & -\alpha \end{pmatrix}\begin{pmatrix} \upsilon_2^k \\ \upsilon_2^* \end{pmatrix} = \begin{pmatrix} -j_{21}\dfrac{p}{k} \\ -\mu \end{pmatrix}.$$

The second row yields $\upsilon_2^* = \mu\left(\upsilon_2^k + \alpha^{-1}\right)$. The first row equals

$$\left(j_{22}-\alpha\right)\upsilon_2^k + j_{23}\mu\left(\upsilon_2^k+\alpha^{-1}\right) = -j_{21}\left(p/k\right).$$

After substituting for the third eigenvalue,

$\upsilon_2^k = \dfrac{1-\lambda_3/k+\zeta\left(b/k\right)\left[1-\sigma+(1+i)\left(\sigma/\left(\alpha G^A\right)\right)\right]}{\lambda_3-\alpha}$ results. ■

Proof of Proposition 6:

Insert the eigenvectors from Lemma 3 into the second and third equation of (26) and solve simultaneously for $\kappa_2\left(\lambda_2\right)^t$ and $\kappa_3\left(\lambda_3\right)^t$. The results are as follows: $\kappa_2\left(\lambda_2\right)^t = \left[k_t - k^H - \left(k_t^* - k^{*,H}\right)\right]/\left(\upsilon_2^k - \upsilon_3^k\right) = \alpha\left[k_t - k^H - \mu^{-1}\left(k_t^* - k^{*,H}\right)\right], \kappa_3\left(\lambda_3\right)^t = \alpha\mu^{-1}\upsilon_2^k \times\left(k_t^* - k^{*,H}\right) - \alpha\mu^{-1}\upsilon_2^*\left(k_t - k^H\right) = \alpha\left[\left(1/\alpha+\gamma\right)\mu^{-1}\left(k_t^* - k^{*,H}\right) - \gamma\left(k_t - k^H\right)\right]$. The next step is to consider the second and the third equation of (26) for $t+1$ and t, and then to subtract the latter from the former. We get the following results: $k_{t+1}-k_t = \left(1/\alpha+\gamma\right)\left(\lambda_2-1\right)\kappa_2\left(\lambda_2\right)^t + \left(\lambda_3-1\right)\kappa_3\left(\lambda_3\right)^t$, $k_{t+1}^* - k_t^* = \gamma\left(\lambda_2-1\right)\kappa_2\left(\lambda_2\right)^t + \left(\lambda_3-1\right)\kappa_3\left(\lambda_3\right)^t$. The last step is to insert into these equations the equations for $\kappa_2\left(\lambda_2\right)^t$ and $\kappa_3\left(\lambda_3\right)^t$ from above, and to collect terms. As a consequence, (28) and (29) are obtained. Finally, inserting the equations for $\kappa_2\left(\lambda_2\right)^t$ and $\kappa_3\left(\lambda_3\right)^t$ into the first equation of (26) and remembering that $\upsilon_2^p = -p/k, \upsilon_3^p = 0$ holds, we obtain (27).■

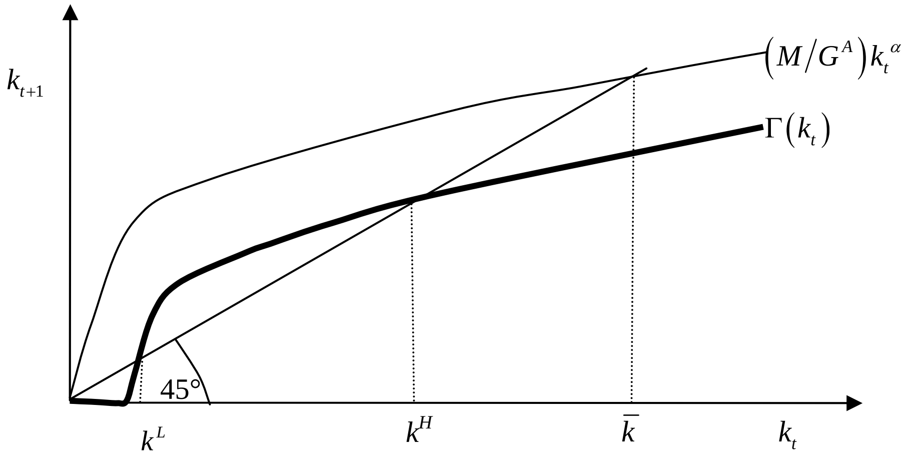

Fig. 1: Existence and asymptotic stability of steady states in autarky

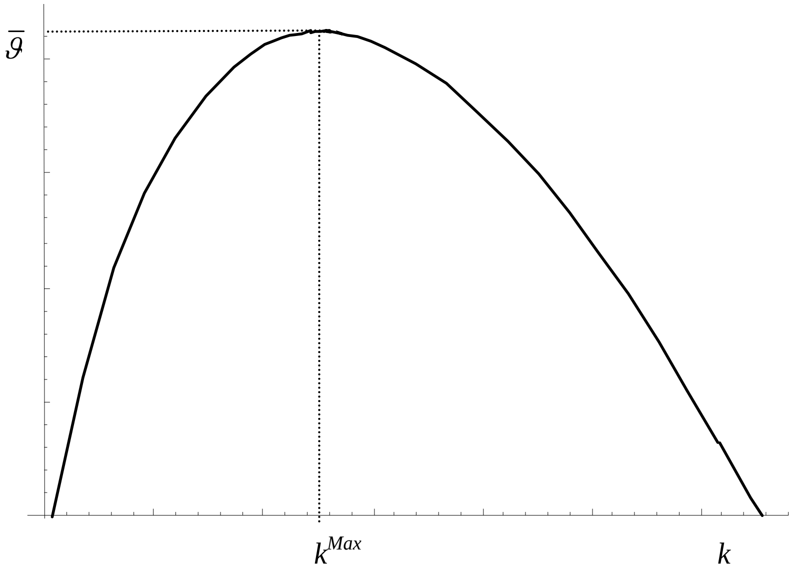

Fig. 2: The relationship between ϑ and k .

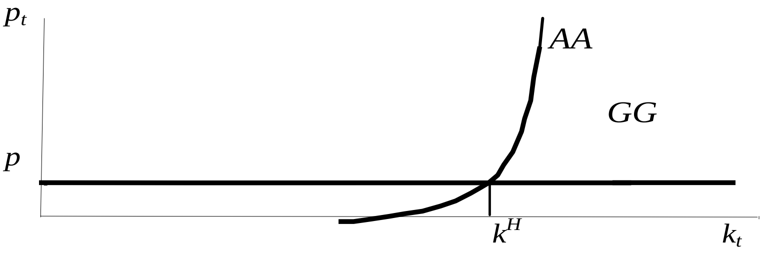

Fig. 3: Home is net foreign creditor

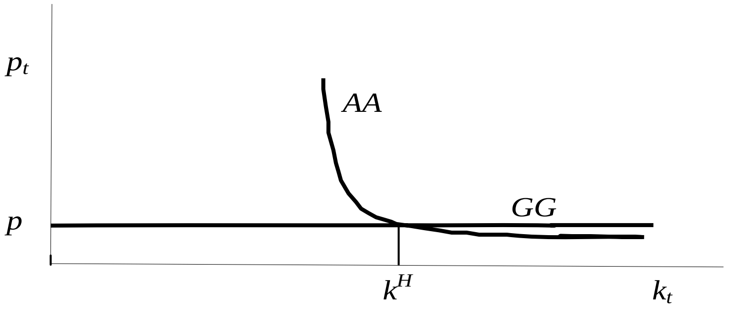

Fig. 4: Home is net foreign debtor

α	$\beta = 0.8$			$\beta = 0.9$		
	$\overline{\vartheta}$	k^{Max}	$\overline{\vartheta}/k^{\max}$	$\overline{\vartheta}$	k^{Max}	$\overline{\vartheta}/k^{\max}$
0.3	0.425384	0.614177	0.692608	0.475879	0.677987	0.7019
0.4	0.232212	0.53608	0.433167	0.262703	0.599591	0.438138

Table 1: The effects of capital income share and future discount factor on maximum sustainable debt and associated maximal capital intensity.

	$M = 4.5$			$M = 5$		
G^A	$\bar{\vartheta}$	k^{Max}	$\bar{\vartheta}/k^{\max}$	$\bar{\vartheta}$	k^{Max}	$\bar{\vartheta}/k^{\max}$
1.0	0.425384	0.614177	0.692608	0.49448	0.71394	0.692608
1.2	0.327842	0.473345	0.692608	0.381095	0.550232	0.692608

Table 2: The effects of the natural growth factor and the level of total factor productivity on maximum sustainable debt and associated maximal capital intensity.

ζ	$M = 4.5,\ M^* = 5.0$	$M = 5,\ M^* = 4.5$
0.5	$\bar{b} = 0.85078 - 0.860625\bar{b}^*$	$\bar{b} = 0.988961 - 1.16243\bar{b}^*$
0.6	$\bar{b} = 0.708974 - 0.57351\bar{b}^*$	$\bar{b} = 0.824134 - 0.774955\bar{b}^*$
0.4	$\bar{b} = 1.06346 - 1.2904\bar{b}^*$	$\bar{b} = 1.2362 - 1.74365\bar{b}^*$

Table 3: The effects of the expenditure share for the domestic commodity and different technological levels on the relationship between maximum sustainable debt in Home and in Foreign.

	Pre shock steady state			Shock period			Post shock steady state		
	p	k	k^*	p_0	k_1	k_1^*	p	k	k^*
Case 1: $\Phi < 0$ $\alpha < \alpha^*$	1.01758	0.89455	1.15150	1.01916	0.87554	1.12661	1.01647	0.85103	1.09158
Case 2: $\Phi > 0$ $\alpha < \alpha^*$	1.01741	0.88742	1.14167	1.02223	0.82926	1.06545	1.01238	0.73358	0.93101
Case 3: $\Phi < 0$ $\alpha > \alpha^*$	0.98289	1.14167	0.88742	0.98136	1.11680	0.86841	0.98380	1.08238	0.84433
Case 4: $\Phi > 0$ $\alpha > \alpha^*$	0.98272	1.15150	0.89455	0.97867	1.07250	0.83465	0.98732	0.94597	0.74458

Table 4: Effects of an increase in debt per capita ($\Delta b/b = 0.2$) on Home and Foreign capital intensities and Home terms of trade.