Consumption Externalities, Endogenous Discounting, Heterogeneity and Cycles

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Abstract

This paper studies the role of consumption externalities on equilibrium dynamics and long-run capital distribution of a neoclassical growth model with heterogeneous agents. For simplicity and without loss of generality, we reduce agents' heterogeneity to two types of agents who differ in their initial wealth and discount factor. In contrast to the usual specification of macroeconomic literature, we assume that consumption externalities influence the "intertemporal facet" of agents' preferences; i.e., the discount rate. Our major contribution consists of the following two results. First, we show that our specification establishes a non-degenerate distribution of capital in the steady state. That is, even if households discount their future utility differently, all of them own positive amount of capital at equilibrium. Second, we show that this model can produce Hopf cycles at some range of the parameter values

Key words: Consumption externalities; Discount rate; Agent heterogeneity; Inequality; Indeterminacy.

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1 Introduction

This paper studies the role of consumption externalities on equilibrium dynamics of a neoclassical growth model with heterogeneous agents. For simplicity, we reduce agents' heterogeneity to two types of agents who differ in their initial wealth and discount factor. We depart from the usual specification of dynamic macroeconomic literature in which the external effect of consumption is formulated in a way that gives rise to "Jealousy" or "keeping-up with the Joneses" feature. In particular, we assume that consumption externalities influence the "intertemporal facet" of agents' preferences; i.e., the discount rate.

Our main objective is to study how the interaction between endogenous discount rate and consumption externalities plays an important role in affecting the local dynamics.

This paper is related to two strand of literature. From one side, the way consumption level is related to discounting is studied in previous literature. Drugeon (1998) argues that high economy-wide consumption could be observed as an indication to a high living standards of individuals in the society. Therefore, a rise in aggregate consumption makes agents to discount less their future utility and so they become more patient.[1] Meng (2006) assumes that discount rate depends on aggregate consumption and average income. The author shows that local indeterminacy requires a positive effect of average consumption on the rate of time preference; that is, as the society consumes more, the agent himself becomes even more impatient and is less willing to defer consumption. Finally, Chen and Hsu (2007) consider an economy where agent's discount rate depends on his own consumption, while the aggregate consumption level enters in the instantaneous utility function. They show that indeterminacy requires a decreasing discount rate in individual consumption.

From another side, several papers showed that, in the model of Becker and Foias (1987, 1994),[2] it is possible to establish a non-degenerate capital distribution at the steady state in specific cases. For example, Sarte (1997) assumes that agents who differ in their constant rates of impatience face a progressive tax structure. Epstein and Hynes (1983); Lucas and Stokey (1984; and Boyd (1986) assume that preferences are described by recursive utility functions. Sorger (2002) shows that non-degenerate equilibrium could be also obtained whenever households exercise market power on the capital market.

In this model, the discount rate differs across agent, for one agent, it depends on the other agent's consumption level. We have two main results. First, we show that all agents hold positive capital, at an equilibrium near a steady state. In other terms, contrary to the existing literature including consumption externalities such as Garcia-Penalosa and Turnovsky (2007) who obtain an

[1] As well, Drugeon (1998) argues that living standards of the society determines individuals productivity efficiency. To capture this idea, he introduces the aggregate consumption in the production function.

[2] In Becker and Foias (1987, 1994), whenever agents are heterogeneous with respect to their discount rate, the most agent holds the entire capital stock of the economy at an equilibrium near a steady state.

equilibrium which is equivalent to the representative-agent model,[3] our model captures the heterogeneity across agents in the long-run.

Moreover, the interaction between endogenous discount rate and consumption externalities plays an important role in affecting the dynamics. That is, we show that, for plausible values of elasticities of external effects, the steady state changes its stability through Hopf cycles. This is an added value to previous literature, endogenous cycles have been known to be possible in Ramsey model with heterogeneous agents since Becker and Foias (1994) work. They show that the occurrence of Flip bifurcation requires that the income of patient agent is decreasing function of capital stock (sufficiently low input substitution) and the intertemporal substitutability is weak. Subsequently, Bosi and Seegmuler (2007) showed that two-period cycles can also be observed in the model of Becker and Foias (1994) augmented to include an endogenous labor supply, under the restriction of very small elasticity of capital-labor substitution.

This paper is organised as follows. In the next section, we present the model. Section 3 defines the intertemporal equilibrium and shows the existence of a steady state. In section 4, we analyze the local dynamics. The last section concludes.

2 The model

The economy we consider is populated by infinitely-lived heterogeneous agents. The heterogeneity stems from different initial wealth and different discount factor. For the sake of simplicity and without loss of generality, we consider two types of agents, and denote the size of the *ith* class of households as n_i which is constant over time. Agents are assumed to be identical within each group and so we consider a representative agent for each type.

In addition, agents are assumed to be *status seekers.* From one side, the utility function of a representative agent of group i depends on own consumption as well as on consumption level of a representative agent of the other group. From the other side, consumption externalities influence the "intertemporal facet" of agents' preferences; i.e., the discount rate.

Time is continuous and the enviroment is deterministic. A representative agent of type i is endowed with $k_{i,0} > 0$, supplies inelastically one-unit of labor at each period. Further, given a sequence of real interest rates on capital $\{r_t\}$ and wages rate $\{w_t\}$, agent i chooses a sequence of consumption and capital $\{c_{i,t}, k_{i,t}\}_{t=0}^{+\infty}$ which maximizes his life-time utility function (1) under his budget constraint (2):

$$\max_{c_{i,t},k_{i,t}} \int_0^\infty u_i\left(c_{i,t}, c_{j,t}\right) \exp\left\{-\int_0^t \delta_i\left(c_{j,v}\right) dv\right\} dt \tag{1}$$

[3] Garcia-Penalosa and Turnovsky (2007) study a neoclassical growth model with heterogeneous agents and consumption externalities. They assume that agents' preferences are quasi-homothetic so that the aggregate behavior of the economy is independent of wealth distribution. Thus this framework generates an equilibrium which is equivalent to the representative-agent model.

subject to

$$\dot{k}_{i,t} = r_t k_{i,t} + w_t l_{i,t} - c_{i,t} \tag{2}$$

The utility function (1) satisfies the following assumption:

Assumption 1 *For $i \neq j$, the instantaneous utility function $u_i(c_i, c_j)$, is twice continuously differentiable and satisfies $u_{i,1}(c_i, c_j) > 0 > u_{i,11}(c_i, c_j)$. Further, the time-preference rate $\delta_i(c_j) > 0$ is twice continuously differentiable and $\delta_i'(c_j) \lesseqgtr 0$.*

Setting up the current-value Hamiltonian of agent i's maximization problem, $H = u_i(c_{i,t}, c_{j,t}) + \lambda_{i,t}[r_t k_{i,t} + w_t l_{i,t} - c_{i,t}]$, where $\lambda_{i,t} > 0$ is the co-state variable, first-order conditions with respect to consumption and to capital ($H_c = 0$ and $H_k = \lambda_{i,t}\delta_i(c_{j,t}) - \dot{\lambda}$) imply the following four dynamic equations:

$$\delta_i(c_{j,t}) - r_t = \frac{u_{i,11}c_{i,t}}{u_{i,1}}\frac{\dot{c}_{i,t}}{c_{i,t}} + \frac{u_{i,12}c_{j,t}}{u_{i,1}}\frac{\dot{c}_{j,t}}{c_{j,t}} \tag{3}$$

$$\dot{k}_{i,t} = r_t k_{i,t} + w_t l_{i,t} - c_{i,t} \tag{4}$$

where (3) is the intertemporal Euler equation and (4) is the resources constraint. Further, a rational agent takes account of transversality condition in choosing his optimal consumption and capital:

$$\lim_{t\to+\infty} \exp\left\{-\int_0^t \delta_i(c_{j,v})\, dv\right\} u_{i,1}(c_{i,t}, c_{j,t})\, k_{i,t} = 0 \tag{5}$$

For the felicity function $u_i(c_i, c_j)$, we define the following elasticities. First, let $\varepsilon_{i,11} \equiv u_{i,11}c_i/u_{i,1}$ be the elasticity of marginal utility of own consumption, which has a negative sign. In addition, the elasticity of intertemporal substitution in own consumption equals $-1/\varepsilon_{i,11}$. Second, let $\varepsilon_{i,12} \equiv u_{i,12}c_j/u_{i,1}$ be the elasticity of marginal utility with respect to the other agent's consumption. The sign of this elasticity depends on how agent i responds to the consumption level of agent j. If agent i is a *conformist* who wants to be similar to agent j (*keeping-up with the Joneses*) then $\varepsilon_{i,12} > 0$. However, if agent i is an *anti-conformist* who wants to be different from agent j (*running-away from the Joneses*) then $\varepsilon_{i,12} < 0$.

For the time-preference function $\delta_i(c_j)$, we define the elasticity $\zeta_i = \delta_i' c_j/\delta_i$, which measures the sensitivity of discounting of agent i to the consumption of agent j. In this framework, we leave open the possibility of increase in j's consumption producing a positive or a negative effet over i's dicount rate.[4]

[4] Drugeon (1998) assumes that higher consumption level in the economy indicates to higher standard of living which in turn leads individuals to discount less their future utility and so they become more patient. However, Meng (2006) shows that local indeterminacy requires a positive effect of economy-wide consumption on time preference rate.

On the production side, assume that firms are identical. We consider an exogenous production function $F(K_t, L_t)$ which is homogeneous of degree one and satisfies the following assumption:

Assumption 2 *Let $k_t \equiv K_t/L_t$ be the capital per capita. The technology $f(k)$ is a continuous function of the capital per capita $k \geq 0$, positive-valued and differentiable. Furthermore, $f''(k) < 0 < f'(k)$, for $k > 0$, and $f(0) = 0$, $\lim_{k \to 0} f'(k) = +\infty$ and $\lim_{k \to +\infty} f'(k) = 0$.*

A representative firm is assumed to take the factor prices r_t and w_t and technology $F(K_t, L_t)$ as given and maximizes its profit. We then get

$$\begin{aligned} r_t &= f'(k_t) \\ w_t &= f(k_t) - k_t f'(k_t) \end{aligned} \tag{6}$$

For the production function, $\sigma \equiv [kf'(k)/f - 1] f'(k)/kf''(k)$ is the elasticity of capital-labor substitution, $s \equiv f'(k)k/f(k) \in (0,1]$ is the capital share of the total income and finally we have $f''k/f' = -(1-s)/\sigma$.

3 Intertemporal equilibrium

We start by giving a definition of an intertemporal equilibrium:

Definition 1 *An intertemporal equilibrium is a sequence $\left(r_t, w_t, K_t, L_t, (k_{i,t}, l_{i,t}, c_{i,t})_{i=1}^{2}\right)$ which satisfies the following conditions:*

1. *given the strictly positive sequence $(r_t, w_t)_{t=0}^{\infty}$, $(K_t, L_t)_{t=0}^{\infty}$ solves firm's program for $t = 0, 1, \ldots, \infty$;*
2. *given $(r_t, w_t)_{t=0}^{\infty}$, $(k_{i,t}, l_{i,t}, c_{i,t})_{t=1}^{\infty}$ solves the program of agent i, for $i = 1, 2$;*
3. *the capital market clears: $K_t = n_1 k_{1,t} + n_2 k_{2,t}$, for $t = 0, 1, \ldots, \infty$;*
4. *the labor market clears: $L_t = n_1 + n_2$, for $t = 0, 1, \ldots, \infty$;*
5. *the product market clears: $\dot{K}_t = F(K_t, L_t) - C_t$, where $C_t = \sum_{i=1}^{2} n_i c_{i,t}$ is the aggregate consumption.*

Let $N_i \equiv n_i/(n_i + n_j) \in [0,1]$ be the relative size of the *ith* class of households, which is constant over time, with $N_i + N_j = 1$. The capital per capita at time t is given by $k_t = N_i k_{i,t} + N_j k_{j,t}$. Further, we denote the share of capital per capita supplied by type i at time t by $\theta_i \equiv N_i k_{i,t}/k_t$, with $\theta_i \in [0,1]$ and $\theta_1 + \theta_2 = 1$.

We can now characterize the four-dimensional system as follows:

Proposition 1 *Let assumptions* (1) *and* (2) *hold. An intertemporal equilibrium with perfect forsight is a sequence of* $\{c_{i,t}, k_{i,t}\}_{t=0}^{+\infty}$ *that solves the four-dimensional dynamic system that consists of Euler equations*

$$\begin{bmatrix} \dot{c}_{1,t} \\ \dot{c}_{2,t} \end{bmatrix} = \frac{c_{1,t}c_{2,t}}{\varepsilon_{1,11}\varepsilon_{2,11} - \varepsilon_{1,12}\varepsilon_{2,12}} \begin{bmatrix} \varepsilon_{2,11}/c_{2,t} & -\varepsilon_{1,12}/c_{2,t} \\ -\varepsilon_{2,12}/c_{1,t} & \varepsilon_{1,11}/c_{1,t} \end{bmatrix} \begin{bmatrix} \delta_1(c_{2,t}) - f'(k_t) \\ \delta_2(c_{1,t}) - f'(k_t) \end{bmatrix} \tag{7}$$

and the resources constraints, for $i = 1, 2$*,*

$$\dot{k}_{i,t} = f'(k_t)\, k_{i,t} + f(k_t) - k_t f'(k_t) - c_{i,t} \tag{8}$$

subject to the initial aggregate endowment $k_{i,0} > 0$ *and the transversality condition (5).*

At the steady state: $\dot{c}_{i,t} = 0$ and $\dot{k}_{i,t} = 0$. We get (for $i \neq j$)

$$\delta_i(c_j) = f' \tag{9}$$

$$c_i = (k_i - k) f' + f \tag{10}$$

where $k = N_i k_i + N_j k_j$ and $N_i \equiv n_i / (n_i + n_j) \in [0, 1]$.

From the dynamic system (7)-(8), consumption externalities appear only in Euler equations and have two effects. From one hand, external effects have *intertemporal effect*, that is, agent i's marginal rate of substitution between consumption at different dates is affected by agent j's consumption. This leads agent i to substitute inefficiently consumption across periods. We observe that such an effect disappears at the steady state. Consequently, whenever consumption externalities are introduced in utility function, they result in an inefficient equilibrium path while do not affect long-run equilibrium.[5]

From another hand, external effects in consumption have an *intratemporal effect* as they not only influence the transition path but also the steady state equilibrium.

4 Local dynamics

We linearize the four-dimensional system (7) and (8) around the symmetric steady state (9) and (10). We get the four-dimensional linear system

$$\varepsilon_{1,12}\,\zeta_2 \frac{\dot{c}_{1,t}}{c_1} - \varepsilon_{2,11}\zeta_1 \frac{\dot{c}_{2,t}}{c_2} + \frac{(1-s)\,\theta_1}{\sigma}(\varepsilon_{1,12} - \varepsilon_{2,11})\frac{\dot{k}_{1,t}}{k_1} + \frac{(1-s)\,\theta_2}{\sigma}(\varepsilon_{1,12} - \varepsilon_{2,11})\frac{\dot{k}_{2,t}}{k_2} = 0 \tag{11}$$

[5] See Fisher and Hof (2000a, 2000b), Liu and Turnovsky (2005) and Alonso-Carrera et al. (2004, 2005, 2006).

$$\varepsilon_{1,11}\,\zeta_2\frac{\dot{c}_{1,t}}{c_1}-\varepsilon_{2,12}\zeta_1\frac{\dot{c}_{2,t}}{c_2}-\frac{(1-s)\,\theta_1}{\sigma}\left(\varepsilon_{2,12}-\varepsilon_{1,11}\right)\frac{\dot{k}_{1,t}}{k_1}-\frac{(1-s)\,\theta_2}{\sigma}\left(\varepsilon_{2,12}-\varepsilon_{1,11}\right)\frac{\dot{k}_{2,t}}{k_2}=0 \tag{12}$$

$$\left(N_1\left(1-s\right)+s\theta_1\right)\frac{\dot{c}_{1,t}}{c_1}+\left(\frac{s\theta_1\left(1-s\right)}{\sigma}\left(\theta_1-N_1\right)-s\theta_1\right)\frac{\dot{k}_{1,t}}{k_1}+\frac{s\left(1-s\right)}{\sigma}\theta_2\left(\theta_1-N_1\right)\frac{\dot{k}_{2,t}}{k_2}=0 \tag{13}$$

$$\left(\left(1-s\right)N_2+s\theta_2\right)\frac{\dot{c}_{2,t}}{c_2}+\frac{s\theta_1\left(1-s\right)}{\sigma}\left(\theta_2-N_2\right)\frac{\dot{k}_{1,t}}{k_1}+\left(\frac{s\theta_2\left(1-s\right)}{\sigma}\left(\theta_2-N_2\right)-s\theta_2\right)\frac{\dot{k}_{2,t}}{k_2}=0 \tag{14}$$

In the following, we will focus on the case with no consumption external effects in the felicity function.[6] Further, we leave open the possibility of increase in j's consumption producing a positive or a negative effet over i's dicount rate.[7]

Assumption 3 *Let* $\varepsilon_{1,12}=\varepsilon_{2,12}=0$ *and* $\zeta_i \lesseqgtr 0$.

The linear system (11)-(14) can be written in matrix form as follows:

$$\begin{bmatrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{k}_1 \\ \dot{k}_2 \end{bmatrix} = J \begin{bmatrix} c_{1,t}-c_1 \\ c_{2,t}-c_2 \\ k_{1,t}-k_1 \\ k_{2,t}-k_2 \end{bmatrix} \tag{15}$$

where J is the jacobian matrix.

The characteristic equation of J is

$$P\left(\lambda\right)=\lambda^4+b_1\lambda^3+b_2\lambda^2+b_3\lambda+b_4=0 \tag{16}$$

where

$$\begin{aligned} b_1 &= -\left(\lambda_1+\lambda_2+\lambda_3+\lambda_4\right) \\ b_2 &= \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4 \\ b_3 &= -\left(\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_2\lambda_3\lambda_4+\lambda_1\lambda_3\lambda_4\right) \\ b_4 &= \lambda_1\lambda_2\lambda_3\lambda_4 \end{aligned}$$

[6] Barbar and Barinci (2009) show that the introduction of consumption externalities in a model based on Becker and Foias (1994) only modifies the range of parameters values giving rise to cycles of period two. In other terms, consumption externalities in preferences do not affect local dynamics.

[7] Meng (2006) shows that the introduction of consumption externalities in time-preference rate plays a crucial role in the appearence of local indeterminacy.

which are given by

$$b_1 = s\left(1 - \frac{1}{\sigma}(1-s)(2\theta_1 - 1)(\theta_1 - N_1)\right) \tag{17}$$

$$b_2 = \theta_1\theta_2 s^2 + \frac{\theta_1 R_1 \varepsilon_{2,11} - \theta_2 R_2 \varepsilon_{1,11}}{\sigma} - S\zeta_1\zeta_2 \tag{18}$$

$$b_3 = \frac{s\theta_1\theta_2}{\sigma}(\varepsilon_{2,11}R_1 - \varepsilon_{1,11}R_2) - \frac{\theta_2 R_2}{\sigma}S\zeta_2 - \left(\zeta_2 + \frac{\theta_1 R_1}{\sigma b_1}\right) b_1 S\zeta_1 \tag{19}$$

$$b_4 = -S\theta_1\theta_2 s^2 \left(\frac{R_2}{s\sigma}\zeta_2 + \left(\zeta_2 + \frac{R_1}{s\sigma}\right)\zeta_1\right) \tag{20}$$

where

$$R_1 \equiv (1-s)\left[s\theta_1 + (1-s)N_1\right] \in (0,1)$$

$$R_2 \equiv (1-s)\left[s\theta_2 + (1-s)N_2\right] \in (0,1)$$

$$S \equiv -\varepsilon_{1,11}\varepsilon_{2,11} < 0$$

In order to analyze the local stability of the steady state (9) and (10), we study how the coefficients of the characteristic polynomial (16) vary with some parameters of interest. In particular, we want to examine the existence of Hopf cycles, using the method provided by several papers such as Liu (1994); Asada and Yoshida (2003) and Manfredi and Fanti (2004).[8]

Hopf bifurcation occurs whenever a pair of complex conjugated eigenvalues crosses the imaginary axis while the other eigenvalues have non-zero real parts.[9] According to Liu (1994), the best and simplest indicator to examine the existence of a pair of purely imaginary eigenvalues is the higher-order Routh-Hurwicz (RH) determinant which is given by

$$\Phi = \begin{vmatrix} b_1 & 1 & 0 \\ b_3 & b_2 & b_1 \\ 0 & b_4 & b_3 \end{vmatrix} = b_1 b_2 b_3 - b_3^2 - b_1^2 b_4 \tag{21}$$

In other terms, the characteristic polynomial (16) has (at least) one purely imaginary pair eigenvalues if $\Phi = 0$[10] is satisfied at some critical value of the parameter of interest.

[8] We focus our attention on the appearence of Hopf cycles since the analytical detection of local indeterminacy is very hard in four-dimensional model.

[9] Liu (1994) refers to this definition as "General Hopf Bifurcation, GHB". However, if Hopf cycles appear due to a pair of complex conjugate eigenvalues which crosses the imaginary axis while the other 'non-bifurcating' eigenvalues have *negative* real parts, Liu (1994) refers to this type as "Simple Hopf Bifurcation, SHB". As the former is reducible to the later, we apply the what is called "General Hopf Bifurcation, GHB" for our model.

[10] In literature, such a condition is referred to as the "Hopf bifurcation boundary".

For our four-dimensional system, we apply the method provided by Asada and Yoshida (2003). Accordingly, Hopf cycles appear if and only if either of the following two conditions is satisfied:

(A) $b_1 b_3 > 0$, $b_4 \neq 0$ and $\Phi = 0$.

(B) $b_1 = 0$, $b_3 = 0$ and $b_4 < 0$.

The paramets of interest are the elasticities of the external effects (ζ_1 and ζ_2) and the elasticity of capital-labor substitution σ.

We choose ζ_1 as the bifurcation parameter. Here we observe that b_1 depends only on the elasticity of input-substitution σ. This makes rise the following cases according to the sign of b_1:

Case 1 *If* $(\theta_1 - N_1)(2\theta_1 - 1) > 0$, *then define*

$$\sigma^* \equiv (1 - s)(\theta_1 - N_1)(2\theta_1 - 1)$$

1. *Whenever* $\sigma < \sigma^*$, *then* $b_1 < 0$, *the appearence of Hopf cycles requires* $b_3 < 0$, $b_4 \neq 0$ *and* $\Phi = 0$.
2. *Whenever* $\sigma > \sigma^*$, *then* $b_1 > 0$, *Hopf bifurcation requires* $b_3 > 0$, $b_4 \neq 0$ *and* $\Phi = 0$.
3. *Whenever* $\sigma = \sigma^*$, *then* $b_1 = 0$, *Hopf bifurcation requires that* $b_3 = 0$ *and* $b_4 < 0$.

Case 2 *If* $(\theta_1 - N_1)(2\theta_1 - 1) < 0$, *then* $b_1 > 0$. *Hopf cycles require that* $b_3 > 0$, $b_4 \neq 0$ *and* $\Phi = 0$.

The following proposition characterizes the local dynamics, based on the cases above.

Proposition 2 *Consider the critical values* $\zeta_{1,3}^0$, $\zeta_{1,H1}$ and $\zeta_{1,H2}$ *that are respectively given in the Appendix by (23), (28) and (29), we have the following:*

1. *Whenever* $\sigma < \sigma^*$: *For* $\zeta_2 < -\theta_1 R_1/\sigma b_1$, *the system changes its stability through Hopf cycles at* $\zeta_1 = \zeta_{1,H1}$, *while for all* $\zeta_2 > -\theta_1 R_1/\sigma b_1$, *Hopf cycles appear at* $\zeta_{1,H2}$.
2. *Whenever* $\sigma > \sigma^*$ *or* $(\theta_1 - N_1)(2\theta_1 - 1) < 0$: *For* $\zeta_2 < -\theta_1 R_1/\sigma b_1$, *the system changes its stability through Hopf bifurcation at* $\zeta_1 = \zeta_{1,H1}$, *while for* $\zeta_2 > -\theta_1 R_1/\sigma b_1$, *Hopf bifurcation occurs at* $\zeta_1 = \zeta_{1,H2}$.
3. *Whenever* $\sigma = \sigma^*$: *The system changes its stability through Hopf cycles at* $\zeta_1 = \zeta_{1,3}^0$.

Proof :

(1) Whenever $\sigma < \sigma^*$, $b_1 < 0$ and so the appearence of Hopf cycles requires $b_3 < 0$, $b_4 \neq 0$ and $\Phi = 0$. This leads to two cases:

(1.1) For $\zeta_2 \in (-\infty, -\theta_1 R_1/\sigma b_1)$.

In this case, b_3 increases with ζ_1 and $b_3 = 0$ at $\zeta_1 = \zeta_{1,3}^0$. Thus, $b_3 < 0$ for all $\zeta_1 < \zeta_{1,3}^0$. The appearence of Hopf bifurcation requires the existence of a critical value for ζ_1 which belongs to the interval $\left(-\infty, \zeta_{1,3}^0\right)$ and differs from $\zeta_{1,4}^0$ [since $b_4\left(\zeta_{1,4}^0\right) = 0$] and at which $\Phi = 0$. To check this, we study the properties of the RH determinant: $\Phi(\zeta_1)$ has a maximum at ζ_1^*, and $\Phi\left(\zeta_{1,3}^0\right) < 0$ and $\Phi \to -\infty$, as $\zeta_1 \to -\infty$. As a result, there exists one solution for $\Phi = 0$ that is denoted by $\zeta_{1,H1} \in \left(-\infty, \zeta_{1,3}^0\right)$ and given by (28) if and only if $Q > 0$.

It could be directly observed that such a solution exists in two cases:

- Whenever $\zeta_2 \in (-\infty, -R_1/s\sigma)$ and $\zeta_{1,4}^0 < \zeta_{1,3}^0$, Hopf bifurcation arises at $\zeta_1 = \zeta_{1,H1}$, with $\Phi\left(\zeta_{1,H1}\right) = 0$ and $\zeta_{1,H1} \neq \zeta_{1,4}^0$.

- Whenever $\zeta_2 \in (-R_1/s\sigma, -\theta_1 R_1/\sigma b_1)$ and $\zeta_{1,4}^0 > \zeta_{1,3}^0$, the system changes its stability at $\zeta_{1,H1}$ by the appearence of a purely imaginary pair generating Hopf cycles.

(1.2) For $\zeta_2 \in (-\theta_1 R_1/\sigma b_1, +\infty)$.

First, b_3 decreases with ζ_1 and $b_3 = 0$ at $\zeta_1 = \zeta_{1,3}^0$, thus $b_3 < 0$ for all $\zeta_1 > \zeta_{1,3}^0$. Second, consider the RH determinant $\Phi(\zeta_1)$, for all $\zeta_1 \in \left(\zeta_{1,3}^0, +\infty\right)$, we note that $\Phi\left(\zeta_{1,3}^0\right) < 0$ and $\Phi \to +\infty$ as $\zeta_1 \to +\infty$, and Φ has a minimum at ζ_1^*. As a result, we deduce that there exists one solution, denoted by $\zeta_{1,H2}$, such that $\Phi\left(\zeta_{1,H2}\right) = 0$ and $\zeta_{1,H2} \neq \zeta_{1,4}^0$. Therefore, the system changes its stability through Hopf bifurcation at $\zeta_1 = \zeta_{1,H2}$.

(2) Whenever $\sigma > \sigma^*$ or $(\theta_1 - N_1)(2\theta_1 - 1) < 0$, we have $b_1 > 0$. Thus there exists a purely imaginary pair eigenvalues and so the system changes its stability through Hopf cycles if and only if $b_3 > 0$, $b_4 \neq 0$ and $\Phi = 0$. Since $b_1 > 0$, there is at least one of the other eigenvalues with negative real part.

(2.1) For $\zeta_2 \in (-\infty, -\theta_1 R_1/\sigma b_1)$.

First, b_3 is decreasing in ζ_1 and $b_3 = 0$ at $\zeta_1 = \zeta_{1,3}^0$. Therefore, $b_3 > 0$ for all $\zeta_1 \in \left(-\infty, \zeta_{1,3}^0\right)$. Furthermore, the function $\Phi(\zeta_1)$ has a minimum at ζ_1^*, $\Phi\left(\zeta_{1,3}^0\right) < 0$ and $\Phi \to +\infty$, as $\zeta_1 \to -\infty$. As a result, there exists one critical value of ζ_1 at which $\Phi = 0$, denote it by $\zeta_{1,H1}$. Then Hopf cycles appear at $\zeta_1 = \zeta_{1,H1}$, provided that $\zeta_{1,H1} \neq \zeta_{1,4}^0$.

(2.2) For $\zeta_2 \in (-\theta_1 R_1/\sigma b_1, +\infty)$.

First, b_3 increases with ζ_1 and $b_3 = 0$ at $\zeta_1 = \zeta_{1,3}^0$. Therefore, $b_3 > 0$ for all $\zeta_1 \in \left(\zeta_{1,3}^0, +\infty\right)$. Moreover, $b_4 \neq 0$ for all $\zeta_1 \neq \zeta_{1,4}^0$. The appearance of Hopf cycles requires the existence of at least one (at most two) critical value of $\zeta_1 \in \left(\zeta_{1,3}^0, +\infty\right)$ at which $\Phi = 0$ and that differs from $\zeta_{1,4}^0$.

We observe that the function $\Phi(\zeta_1)$ has a maximum at ζ_1^*, $\Phi\left(\zeta_{1,3}^0\right) < 0$ and $\Phi \to -\infty$, as $\zeta_1 \to +\infty$. Hopf cycles arise if and only if $\Phi(\zeta_1^*) > 0$, that is, $Q > 0$. Given that the later condition holds, Hopf bifurcation appears at $\zeta_{1,H2}$ which belongs to $\left(\zeta_{1,3}^0, +\infty\right)$ and at which $\Phi = 0$.

(3) For $\sigma = \sigma^*$ then $b_1 = 0$. The appearence of Hopf cycles require $b_3 = 0$ and $b_4 < 0$.

First, $b_3 = 0$ at $\zeta_1 = \zeta_{1,3}^0$. Then $\zeta_{1,3}^0$ is the critical value at which Hopf cycles appear if and only if it belongs to the interval at which $b_4 < 0$. In order to determine this interval, two cases arise:

(3.1) For $\zeta_2 \in (-\infty, -R_1/s\sigma)$, then b_4 is decreasing in ζ_1 and $b_4 = 0$ at $\zeta_1 = \zeta_{1,4}^0$. Thus $b_4 < 0$ for all $\zeta_1 \in \left(\zeta_{1,4}^0, +\infty\right)$. Hopf cycles arise iff $\zeta_{1,3}^0 \in \left(\zeta_{1,4}^0, +\infty\right)$.

(3.2) For $\zeta_2 \in (-R_1/s\sigma, +\infty)$, then b_4 is increasing in ζ_1 and $b_4 = 0$ at $\zeta_1 = \zeta_{1,4}^0$. Thus $b_4 < 0$ for all $\zeta_1 \in \left(-\infty, \zeta_{1,4}^0\right)$. Hopf cycles arise iff $\zeta_{1,3}^0 \in \left(-\infty, \zeta_{1,4}^0\right)$.

In each of the above cases, we have just shown the existence of a pair of complex conjugated eigenvalues which crosses the imaginary axis at $\zeta_{1,3}^0$.

5 Appendix

We compute the following critical values:

Let $b_2(\zeta_1) = 0$, we get $\zeta_1 = \zeta_{1,2}^0$, where

$$\zeta_{1,2}^0 \equiv \frac{1}{S\zeta_2}\left(\theta_1\theta_2 s^2 + \frac{\theta_1 R_1 \varepsilon_{2,11} - \theta_2 R_2 \varepsilon_{1,11}}{\sigma}\right) \tag{22}$$

Let $b_3(\zeta_1) = 0$, we get $\zeta_1 = \zeta_{1,3}^0$, where

$$\zeta_{1,3}^0 \equiv \frac{\frac{s\theta_1\theta_2}{\sigma}\left(R_1\varepsilon_{2,11} - R_2\varepsilon_{1,11}\right) - \frac{\theta_2 R_2}{\sigma} S\zeta_2}{\left(\zeta_2 + \frac{\theta_1 R_1}{\sigma b_1}\right) b_1 S} \tag{23}$$

Let $b_4(\zeta_1) = 0$, we get $\zeta_1 = \zeta_{1,4}^0$, where

$$\zeta_{1,4}^0 = -\frac{\frac{R_2}{s\sigma}\zeta_2}{\zeta_2 + \frac{R_1}{s\sigma}} \tag{24}$$

Further, the function $\Phi(\zeta_1)$ is given by

$$\Phi(\zeta_1) = -b_1^2 S^2 \left(\zeta_2 + \frac{\theta_1 R_1}{\sigma b_1}\right) \frac{\theta_1 R_1}{\sigma b_1} \left[(\zeta_1 - \zeta_1^*)^2 - Q\right] \tag{25}$$

where

$$Q \equiv \left(\zeta_{1,3}^0 - \zeta_1^*\right)^2 + \frac{\theta_1 \theta_2 s^2}{\zeta_2 + \frac{\theta_1 R_1}{\sigma b_1}} \frac{\zeta_2 + \frac{R_1}{s\sigma}}{S} \frac{\zeta_{1,3}^0 - \zeta_{1,4}^0}{\frac{\theta_1 R_1}{\sigma b_1}} \tag{26}$$

and

$$\zeta_1^* = \frac{1}{2\frac{\theta_1 R_1}{\sigma b_1}} \left(\left(\zeta_2 + 2\frac{\theta_1 R_1}{\sigma b_1}\right)\zeta_{1,3}^0 - \zeta_2 \zeta_{1,2}^0 + \frac{\theta_1 \theta_2 s^2}{\zeta_2 + \frac{\theta_1 R_1}{\sigma b_1}} \frac{\zeta_2 + \frac{R_1}{s\sigma}}{S}\right) \tag{27}$$

which solves $\partial\Phi/\partial\zeta_1 = 0$.

In addition,

$$\zeta_{1,H1} \equiv \zeta_1^* - \sqrt{Q} \tag{28}$$

$$\zeta_{1,H2} \equiv \zeta_1^* + \sqrt{Q} \tag{29}$$

are the solutions of $\Phi(\zeta_1) = 0$ and satisfy $\partial\Phi\left(\zeta_{1,H1}\right)/\partial\zeta_1 \neq 0$ and $\partial\Phi\left(\zeta_{1,H2}\right)/\partial\zeta_1 \neq 0$.

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