On boundary conditions within the solution of macroeconomic dynamic models with rational expectations *

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Abstract

In this paper a solution method is developed, which integrates a transversality condition representing the terminal condition of a dynamic model with an infinite horizon into the solution of a macroeconomic rational expectation model. Thus the transversality condition can be used to decrease the potential degree of indeterminacy within the model by reducing the degrees of freedom. Conditions which assess the relevance of the method, are derived and discussed.

Keywords: multivariate rational expectation models, transversality condition, reduction of indeterminacy

*This is a preliminary version, please do not cite!**

1 Introduction

Within macroeconomic DCGE models one has frequently to deal with the problem of multiple solutions in the sense that the convergence path back to steady state is not defined uniquely by the solution. This problem has been reflected by many contributions to the literature. McCallum (1983, 1999) resolves the indeterminacy problem by picking one specific solution, the socalled MSV-solution (minimal state variable solution). This solution is characterized by the fact that all endogenous variables can be explained by a set of predetermined variables, whereby this solution form holds for all possible parameter values, i.e. changing the parameter values of the model does not add variables to the minimal explaining set. Specifically this include parameter values which eliminate the influence of the predetermined variables on the solution. But to identify this constellation one needs to find very specific orderings of the eigenvalues of the underlying system. In terms of numerical decomposition methods this turns out to be a very demanding task and thus the method is not really practicable for big systems. Onatski (2004) presents the winding number as a criterion for indeterminacy of rational expectation models. According to this criterion there is indeterminacy, if the winding number is smaller than 0. This winding number is reinterpreted as the sum of the partial indices of the Wiener-Hopf decomposition,[1] for the case that those indices are all of the same sign. But Onatski (2006) offers neither a way to limit the degree of indeterminacy nor a criterion for picking out a subset of solutions. Evans e.a. (1998), among others, present the criterion of expectational stability around multiple rational expectations solutions. If the individuals follow non-rational expectations, which are given by e.g. recursive least square learning, for the unknown parameters of the real economic model, they show, assuming that the initial difference of their expectations is not too far away from the real parameters, that there is a positive probability for the convergence of expectations to the real parameters. Thus the economy converges to the rational expectation equilibrium path. In general, this behavior can be used to discriminate between more stable rational expectations equilibria and less stable or even unstable ones, whereby the first one have a higher probability of realization. This claim is questioned by Honkapohja e.a. (2004), who show that most rational expectation solutions for purely forward-looking, linear models are not stable in terms of expectational convergence. An exception are solutions based on expectations driven by Marcov sunspot processes, which enable expectational stability of the associated rational expectation equilibria. With respect to models including in the endogenous variables, the results are more ambiguous, because for plausible parameters both stability and instability may occur in models with sunspot equilibria as well as for as well as for fundamental equilibria.

However, the proposed schemes for picking one or several paths suffer to vary-

[1]Here the Wiener-Hopf decomposition decomposes the Fourier summation of the model's coefficient matrices into a factor within the positive half of the complex plane and one within the negative half of the complex plane, for both of which the same characteristic holds also for their inverses. The partial indices of this decomposition are those values for the argument of the Fourier transform, for which the values of both mentioned functions coincide. Unfortunately the explicit formula given for the computation of the winding number does not work properly. Instead of the integral over the logarithm of the Fourier expansion of the model's coefficients the Cauchy integral formula of this expansion should be used.

ing degrees from a lack of microfoundation in economics. The restriction of the solution to be determined by a minimal set of state variables, while forcing the parameters of the system to be consistent with unrealistic values, does not compensate for a missing rigorous derivation from the economic principles. And the choice of the form of the non-rational expectations imposes an ad-hoc assumption on the model, because again there are no specific reasons identified by economic objectives or constraints of the agents.

A related literature tries to integrate boundary conditions, like e.g. transversality conditions, into the solution of the same type of models consistently. In economic terms these conditions integrate long-run restrictions on the economic system into the solution and secure the efficient usage of wealth across time in the sense that no wealth is wasted. Implicitly this literature uses a slightly different concept of stability as the famous contribution of Blanchard et al. (1980), because the criterion for stable eigenvalues is adjusted for the growth rate of specific variables, which are derived from the transversality condition exogenously given by the economic model itself. Thus the stability concept does not only consist in the idea to restrict the potentially unstable variables in such a way, that one balance each other, but also includes the idea that the transversality conditions of the model might be able to neutralize the explosive potential of unstable variables by imposing specific long run relations between the latter. This concept is especially useful, when the terminal value of the state is not pinned uniquely down by any constraints. In this case the transversality condition can not only restrict the number of paths, but even when there is no converging steady state for the undiscounted system, the transversality condition can force stability on the discounted system. These ideas have been already recognized in Sims (2002), but are not used within the solution concept. Taking a further step Kowal (2005) integrated transversality conditions into a specific linear solution form of the problem, which is also applicable to singular matrix pencils. This paper deals with the more general case that there is not any prespecified solution form and discovers therefore the whole range of possible solutions, but restricts the analysis to regular matrix pencils.

2 The Problem and the basic idea for its solution

The basic problem of the paper is to incorporate a boundary condition of the form

$$\lim_{t\to\infty} \mathbf{E}^t \mathbf{W} \begin{pmatrix} v_t \\ w_t \end{pmatrix} = \mathbf{0} \tag{1}$$

into the solution of the condensed form of the linearized equilibrium conditions of an arbitrary macroeconomic dynamic model given by

$$\mathbf{G} \begin{pmatrix} v_{t+1} \\ w_{t+1} \end{pmatrix} = \mathbf{H} \begin{pmatrix} v_t \\ w_t \end{pmatrix} + \mathbf{C} z_t + \tilde{\mathbf{G}} e_{t+1}. \tag{2}$$

Herein the vector of endogenous variables $(v_t^T w_t^T)^T$ does include all leads and lags of the underlying economic model and the coefficient matrices are constructed in order to represent the set of equilibrium conditions of the model.[2] Note that the vector e_{t+1}

[2]Ways to derive this equation are presented in Sims (2002), Hespeler (2008) or Binder et al. (1995)

contains all expectational errors between the periods t and t+1, i.e. all changes in the expectation values for future periods due to revelation of the information available in period t+1.

In the literature there is a whole bunch of solution methods for this problem, whereas each method imposes its own restrictions on the admissible coefficient matrices. This paper deals with those methods which allow for singularity of $\mathbf{G}$, but restrict the matrix pair ($\mathbf{G}$, $\mathbf{H}$) to form a regular matrix pencil. This implies that the model's eigenvalues are well behaved in the sense that they are all elements of $]-\infty,\infty[$. Solution methods to solve this kind of model of varying degrees of generality can be found in Binder (1996), Klein (2000), Sims (2002) and King e.a. (2002). All of these methods derive a solution of the following updating form

$$\begin{pmatrix} v_{t+1} \\ w_{t+1} \end{pmatrix} = \mathbf{\Upsilon}_1 \begin{pmatrix} v_t \\ w_t \end{pmatrix} + \mathbf{\Upsilon}_2 z_t + \sum_{q=1}^{\infty} \mathbf{\Upsilon}_{2+q} E_t z_{t+q}, \tag{3}$$

where $(\mathbf{\Upsilon}_1, ..., \mathbf{\Upsilon}_\infty)$ are defined depending on the characteristics of the used solution method as functions of the original coefficient matrices. Exploiting the decomposition technique, which all methods have in common, we define a transformation $\begin{pmatrix} \tilde{v}_t \\ \tilde{w}_t \end{pmatrix} = \mathbf{P}^H \begin{pmatrix} v_t \\ w_t \end{pmatrix}$, where $\mathbf{P}^H$ is the conjugate transpose of the most left unitary matrix of the Generalized Schur Decomposition of the original matrix pair after reordering according to the ascending order of the generalized eigenvalues. Formally this decomposition looks like $(\mathbf{G}, \mathbf{H}) = (\mathbf{R}\tilde{\mathbf{S}}\mathbf{P}, \mathbf{R}\tilde{\mathbf{T}}\mathbf{P})$, where the most right and most left factors of the products on the equation's left side are unitary and the factors in the middle are upper triangular. Using this transformation within (1) and (3) yields

$$\begin{pmatrix} \tilde{v}_{t+1} \\ \tilde{w}_{t+1} \end{pmatrix} = \mathbf{P}^H \mathbf{\Upsilon}_1 \mathbf{P} \begin{pmatrix} \tilde{v}_t \\ \tilde{w}_t \end{pmatrix} + \mathbf{P}^H \mathbf{\Upsilon}_2 z_t + \sum_{q=1}^{\infty} \mathbf{P}^H \mathbf{\Upsilon}_{2+q} E_t z_{t+q} \tag{4}$$

$$\lim_{t\to\infty} \mathbf{E}^t \mathbf{W}\mathbf{P} \begin{pmatrix} \tilde{v}_t \\ \tilde{w}_t \end{pmatrix} = \mathbf{0} \tag{5}$$

In order to guarantee (5) the limit of the transformed endogenous variables, when time goes to infinity, must be an element of the null space of the coefficient matrix, i.e. $\lim_{t\to\infty} \begin{pmatrix} \tilde{v}_t \\ \tilde{w}_t \end{pmatrix} \in \ker(\mathbf{WP})$. Thus from (4), Theorem 1 in the appendix and the observation that all expected shocks at the infinite horizon of the model are zero it follows that the conditions

$$\mathbf{P}^H \mathbf{\Upsilon}_1 \mathbf{P} \subseteq \ker(\mathbf{WP}) \quad \Leftrightarrow \quad \mathbf{P}^H \mathbf{\Upsilon}_1 \mathbf{P} = \ker(\mathbf{WP})\mathbf{\Sigma}_1 \tag{6}$$

$$\mathbf{P}^H \mathbf{\Upsilon}_2 \subseteq \ker(\mathbf{WP}) \quad \Leftrightarrow \quad \mathbf{P}^H \mathbf{\Upsilon}_2 = \ker(\mathbf{WP})\mathbf{\Sigma}_2 \tag{7}$$

are necessary as well as sufficient in order to integrate (1) consistently into the solution. Equation (6) states that the influence of last period's values of the potentially unstable variables can not violate the transversality condition, while (7) states the same for the remaining influence of any past shock of arbitrary persistence.

3 Integration into explicit solution algorithms

In this section equation (6) is incorporated into two different solution algorithms, which thus both fulfill also the transversality condition directly derived from the underlying economic model. The both examples chosen are the solution algorithm presented in Sims(2002) and the more general algorithm presented in Hespeler (2008). Of course equation (6) can be incorporated into more solution algorithms, either by simply checking, if the coefficient matrix of the endogenous variables fulfills (6), for the case that there are no degrees of freedom or by reducing the available degrees of freedom by imposing this equation.

Starting with the algorithm provided in Sims (2002) it suffices to point out that this algorithm actually coincides with the first case of the more general algorithm presented in Hespeler (2008) for the case that there is a unique solution. As Sims (2002) himself emphasizes "[...] if the solution is [...] not unique, [...], the system generates one of the multiple solutions to [...]. If one is interested in the full set of non-unique solutions, one has to add back, as additional 'disturbances' the component [...] left undetermined."[3] But this is done in the algorithm of Hespeler (2008) automatically by using the concept of the pseudoinverse and the resulting degrees of freedom in any underdetermined solution. Hence we do not need to present the integration of transversality conditions into Sims's algorithm explicitly, but we can refer to the case of an unique solution within the algorithm presented in Hespeler (2008)

The solution proposed in Hespeler (2008) can be summarized by the following expression

$$
\begin{aligned}
\begin{pmatrix} v_{t+1} \\ w_{t+1} \end{pmatrix} = & \begin{pmatrix} \tilde{\mathbf{P}}_{11}\tilde{\mathbf{S}}_{11}^{-1} & \mathbf{0} \\ \tilde{\mathbf{P}}_{21}\tilde{\mathbf{S}}_{11}^{-1} & \mathbf{0} \end{pmatrix} \cdot \\
& \begin{pmatrix} \tilde{\mathbf{T}}_{11}\tilde{\mathbf{P}}_{11}^{H} + \left(\tilde{\mathbf{T}}_{12} - \mathbf{\Phi}\tilde{\mathbf{T}}_{22}\right)\tilde{\mathbf{P}}_{21}^{H} & \tilde{\mathbf{T}}_{11}\tilde{\mathbf{P}}_{12}^{H} + \left(\tilde{\mathbf{T}}_{12} - \mathbf{\Phi}\tilde{\mathbf{T}}_{22}\right)\tilde{\mathbf{P}}_{22}^{H} \\ \tilde{\mathbf{T}}_{11}\tilde{\mathbf{P}}_{11}^{H} + \left(\tilde{\mathbf{T}}_{12} - \mathbf{\Phi}\tilde{\mathbf{T}}_{22}\right)\tilde{\mathbf{P}}_{21}^{H} & \tilde{\mathbf{T}}_{11}\tilde{\mathbf{P}}_{12}^{H} + \left(\tilde{\mathbf{T}}_{12} - \mathbf{\Phi}\tilde{\mathbf{T}}_{22}\right)\tilde{\mathbf{P}}_{22}^{H} \end{pmatrix} \begin{pmatrix} v_t \\ w_t \end{pmatrix} \\
& + \mathbf{\Xi} z_t \\
& + \begin{pmatrix} \tilde{\mathbf{P}}_{11}\tilde{\mathbf{S}}_{11}^{-1}\left(\tilde{\mathbf{S}}_{12} - \mathbf{\Phi}\tilde{\mathbf{S}}_{22}\right) - \tilde{\mathbf{P}}_{12} \\ \tilde{\mathbf{P}}_{21}\tilde{\mathbf{S}}_{11}^{-1}\left(\tilde{\mathbf{S}}_{12} - \mathbf{\Phi}\tilde{\mathbf{S}}_{22}\right) - \tilde{\mathbf{P}}_{22} \end{pmatrix} \sum_{q=1}^{\infty} (\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{S}}_{22})^{s-1}\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}}' E_{t+1}(z_{t+q}).
\end{aligned}
\tag{8}
$$

Herein two cases can be distinguished. In the first case a consistent solution for the equation $\tilde{\mathbf{R}}_1^H\tilde{\mathbf{G}}e_{t+1} = \mathbf{\Phi}\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}e_{t+1}$ exists, i.e. the influence of the expectational error on the system's stable variables can be explained as a function of its influence on the system's unstable variables. Hence a solution for $\mathbf{\Phi}$ is derived as the following function of the arbitrary matrix $\mathbf{Z}$ of appropriate:

$$
\mathbf{\Phi} = \tilde{\mathbf{R}}_1^H\tilde{\mathbf{G}}(\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+ + \mathbf{Z}(\mathbf{I} - \tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}(\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+). \tag{9}
$$

[3] Sims (2002), p.12.

The coefficient $\boldsymbol{\Xi}$ takes on the value

$$\boldsymbol{\Xi} \equiv \begin{pmatrix} \tilde{\mathbf{P}}_{11}\tilde{\mathbf{S}}_{11}^{-1}\left(\tilde{\mathbf{R}}_1^H - \boldsymbol{\Phi}\tilde{\mathbf{R}}_2^H\right) \\ \tilde{\mathbf{P}}_{21}\tilde{\mathbf{S}}_{11}^{-1}\left(\tilde{\mathbf{R}}_1^H - \boldsymbol{\Phi}\tilde{\mathbf{R}}_2^H\right) \end{pmatrix} \tilde{\mathbf{C}}. \tag{10}$$

In the second case there is no consistent solution for the mentioned equation available. In this case the matrix $\boldsymbol{\Phi}$ is arbitrary,[4] while the coefficient $\boldsymbol{\Xi}$ is given as

$$\boldsymbol{\Xi} \equiv \begin{pmatrix} \tilde{\mathbf{P}}_{11}\tilde{\mathbf{R}}_1^H(\tilde{\mathbf{C}} + \tilde{\mathbf{G}}\boldsymbol{\Lambda}) \\ \tilde{\mathbf{P}}_{21}\tilde{\mathbf{R}}_1^H(\tilde{\mathbf{C}} + \tilde{\mathbf{G}}\boldsymbol{\Lambda}) \end{pmatrix}. \tag{11}$$

Herein the new factor $\boldsymbol{\Lambda}$ is defined by

$$\boldsymbol{\Lambda} = -(\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}} - (\mathbf{I} - (\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})\mathbf{Z}, \tag{12}$$

where $\mathbf{Z}$ is again an arbitrary matrix of appropriate dimension. In this solution the factor $\boldsymbol{\Lambda}$ explains the expectational error of a given period as a linear function of the preceding exogenous shock term. Therefore $\boldsymbol{\Lambda}$ is the solution to the equation $\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}\boldsymbol{\Lambda} = -\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}}$.

3.1 First Case

Applying (6) and (7) for the first case leaves one with the condition

$$\begin{pmatrix} \tilde{\mathbf{S}}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{T}}_{11} & \left(\tilde{\mathbf{T}}_{12} - \boldsymbol{\Phi}\tilde{\mathbf{T}}_{22}\right) \\ \tilde{\mathbf{T}}_{11} & \left(\tilde{\mathbf{T}}_{12} - \boldsymbol{\Phi}\tilde{\mathbf{T}}_{22}\right) \end{pmatrix} = \ker(\mathbf{WP})\boldsymbol{\Sigma}_1 \tag{13}$$

$$\begin{pmatrix} \tilde{\mathbf{S}}_{11}^{-1}\left(\tilde{\mathbf{R}}_1^H - \boldsymbol{\Phi}\tilde{\mathbf{R}}_2^H\right) \\ \mathbf{0} \end{pmatrix} \tilde{\mathbf{C}} = \ker(\mathbf{WP})\boldsymbol{\Sigma}_2. \tag{14}$$

Using (13) the first off-diagonal blockmatrix of the left side can be solved for $\boldsymbol{\Phi}$. Setting the solution equal to (9) the resulting equation can be solved for $\mathbf{Z}(\mathbf{I} - \tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}(\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+)$. Inserting this solution into (9) and substituting the obtained experssion for $\boldsymbol{\Phi}$ in (14) yields an equation in the two undetermined submatrices $(\boldsymbol{\Sigma}_1)_1$ and $(\boldsymbol{\Sigma}_2)_1$ of the form

$$\tilde{\mathbf{S}}_{11}^{-1}\tilde{\mathbf{R}}_1^H\tilde{\mathbf{C}} - \tilde{\mathbf{S}}_{11}^{-1}\tilde{\mathbf{T}}_{12}\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}} = \begin{pmatrix}(\ker(\mathbf{WP}))_1 & -(\ker(\mathbf{WP}))_1\end{pmatrix} \begin{pmatrix} \boldsymbol{\Sigma}_2 \\ (\boldsymbol{\Sigma}_1)_2\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}} \end{pmatrix}. \tag{15}$$

The matrix $(\boldsymbol{\Sigma}_1)_2$ comprises the l last columns up of the matrices $\boldsymbol{\Sigma}_1$, where l is equal to the row dimension of $\tilde{\mathbf{S}}_{11}$. The matrix $(\ker(\mathbf{WP}))_1$ contains the according first rows of $\ker(\mathbf{WP})$. The second row of (14) formed by the lower blockmatrices of the coefficients and the diagonal part of the lower row of (13) impose additional consistency requirements in the form of $\boldsymbol{\Sigma}_2 = \ker((\ker(\mathbf{WP}))_2)\mathbf{K}$ and $(\boldsymbol{\Sigma}_1)_2 = \ker((\ker(\mathbf{WP}))_2)(\mathbf{V})_2$, where both

[4] This is actually again a generalization of Hespeler (2008), where $\boldsymbol{\Phi}$ was pinned down as $\boldsymbol{\Phi} = \tilde{\mathbf{S}}_{12}$. But it turns out that this is not necessary, but that any arbitrary matrix $\boldsymbol{\Phi}$ with appropriate dimensions solves the problem.

matrices $(\mathbf{V})_2$ and $\mathbf{K}$ are arbitrary but of appropriate dimensions. After substituting those expressions into (15) this system should be still consistent. Hence it can be solved for the most rightward unknown factor on the right side.

$$\begin{aligned} \begin{pmatrix} \mathbf{K} \\ (\mathbf{V})_2\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}} \end{pmatrix} &= \mathbf{\Psi}^+\left(\tilde{\mathbf{S}}_{11}^{-1}\tilde{\mathbf{R}}_1^H - \tilde{\mathbf{S}}_{11}^{-1}\tilde{\mathbf{T}}_{12}\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\right)\tilde{\mathbf{C}} + \left(\mathbf{I} - \mathbf{\Psi}^+\mathbf{\Psi}\right)\mathbf{D} \qquad (16) \\ \mathbf{\Psi} &\equiv \big((\ker(\mathbf{WP}))_1 \ker((\ker(\mathbf{WP}))_2) \quad -(\ker(\mathbf{WP}))_1 \ker((\ker(\mathbf{WP}))_2)\big) \end{aligned}$$

Herein $\mathbf{D}$ is an arbitrary matrix of appropriate dimensions. Recalling the solution for $\mathbf{Z}(\mathbf{I} - \tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}(\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+)$, which depends actually on the submatrix $(\mathbf{\Sigma}_1)_2$, and solving for the latter yields

$$\begin{aligned} (\mathbf{V})_2 &= \left(\mathbf{\Psi}^+\left(\tilde{\mathbf{S}}_{11}^{-1}\tilde{\mathbf{R}}_1^H - \tilde{\mathbf{S}}_{11}^{-1}\tilde{\mathbf{T}}_{12}\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\right)\tilde{\mathbf{C}} + \left(\mathbf{I} - \mathbf{\Psi}^+\mathbf{\Psi}\right)\mathbf{D}\right)_2 \cdot \qquad (17) \\ &\quad (\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}})^+ + \mathbf{E}(\mathbf{I} - \tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}}(\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}})^+), \end{aligned}$$

where $\mathbf{E}$ is an arbitrary matrix of appropriate dimensions and the expression $(.)_2$ refers to the rows of the right side in (16) defining the lower block of the left side. This expression is substituted into the solution for $\mathbf{Z}(\mathbf{I} - \tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}(\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+)$ and the resulting equations is solved for $\mathbf{Z}$ in order to obtain

$$\begin{aligned} \mathbf{Z} &= \Bigg(-\tilde{\mathbf{S}}_{11}\mathbf{\Psi}_1\bigg(\left(\mathbf{\Psi}^+\left(\tilde{\mathbf{S}}_{11}^{-1}\tilde{\mathbf{R}}_1^H - \tilde{\mathbf{S}}_{11}^{-1}\tilde{\mathbf{T}}_{12}\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\right)\tilde{\mathbf{C}} + \left(\mathbf{I} - \mathbf{\Psi}^+\mathbf{\Psi}\right)\mathbf{D}\right)_2 \cdot \qquad (18) \\ &\quad (\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}})^+ + \mathbf{E}(\mathbf{I} - \tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}}(\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}})^+)\bigg) + \tilde{\mathbf{T}}_{12}\tilde{\mathbf{T}}_{22}^{-1} - \tilde{\mathbf{R}}_1^H\tilde{\mathbf{G}}(\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+\Bigg) \cdot \\ &\quad \left(\mathbf{I} - \tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}(\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+\right)^+ + \mathbf{B}\left(\mathbf{I} - \left(\mathbf{I} - \tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}(\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+\right)\left(\mathbf{I} - \tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}(\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+\right)^+\right). \end{aligned}$$

Like $\mathbf{D}$ and $\mathbf{E}$, $\mathbf{B}$ is an arbitrary matrix of appropriate dimensions. Finally, also the off-diagonal part of the second row as well as the diagonal part of the first row of (13) must hold, which together yield the condition that the equation

$$\tilde{\mathbf{S}}_{11}^{-1}\tilde{\mathbf{T}}_{11} = (\ker(\mathbf{WP}))_1 \ker((\ker(\mathbf{WP}))_2)(\mathbf{V})_1 \qquad (19)$$

must be solvable for $(\mathbf{V})_1$. Hence, the conditions under which (6) and (7) hold, have been established.

3.2 Second Case

Applying (6) and (7) for the second case yields the conditions

$$\begin{aligned} \begin{pmatrix} \tilde{\mathbf{S}}_{11}^{-1}\tilde{\mathbf{T}}_{11} & \tilde{\mathbf{S}}_{11}^{-1}\left(\tilde{\mathbf{T}}_{12} - \mathbf{\Phi}\tilde{\mathbf{T}}_{22}\right) \\ \mathbf{0} & \mathbf{0} \end{pmatrix} &= \ker(\mathbf{WP})\mathbf{\Sigma}_1 \qquad (20) \\ \begin{pmatrix} \tilde{\mathbf{R}}_1^H\left(\tilde{\mathbf{C}} - \tilde{\mathbf{G}}\mathbf{\Lambda}\right) \\ \mathbf{0} \end{pmatrix} &= \ker(\mathbf{WP})\mathbf{\Sigma}_2. \qquad (21) \end{aligned}$$

Thus the first row of (21) can be solved for $\mathbf{\Lambda}$, the solution can be substituted into (12) and the resulting expression can be solved for $\mathbf{Z}$ yielding

$$\begin{aligned}\mathbf{Z} &= \left(\tilde{\mathbf{R}}_1^H\tilde{\mathbf{G}}(\mathbf{I}-(\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})\right)^+\left(\tilde{\mathbf{R}}_1^H\tilde{\mathbf{C}}-(\ker((\mathbf{W}\mathbf{P}_2))_1\mathbf{\Sigma}_2-\tilde{\mathbf{R}}_1^H\tilde{\mathbf{G}}(\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}}\right) \\ &\quad -\left(\mathbf{I}-(\tilde{\mathbf{R}}_1^H\tilde{\mathbf{G}}(\mathbf{I}-(\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}))^+(\mathbf{I}-\tilde{\mathbf{R}}_1^H\tilde{\mathbf{G}}((\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}))\right)\mathbf{B}\end{aligned} \tag{22}$$

Herein $\mathbf{B}$ is an arbitrary matrix of appropriate dimensions. The second rows of (20) and (21) give conditions for the matrices $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$

$$\mathbf{\Sigma}_2 = \ker((\ker(\mathbf{W}\mathbf{P}))_2)\mathbf{F} \qquad \mathbf{\Sigma}_1 = \ker((\ker(\mathbf{W}\mathbf{P}))_2)\mathbf{J}, \tag{23}$$

because these ones must be linearly generated by the null space of the second row of the transversality conditions's nullspace. Like $\mathbf{B}$, the matrix $\mathbf{F}$ is arbitrary, but it has the same number of columns as the left side in (21). Consistency of (20) pins down the solutions for the matrices $(\mathbf{J})_1$ and $\mathbf{\Phi}$, the latter as a function of the arbitrary, but appropriately dimensioned matrix $(\mathbf{J})_2$, by the equations

$$\tilde{\mathbf{S}}_{11}^{-1}\tilde{\mathbf{T}}_{11} = (\ker(\mathbf{W}\mathbf{P}))_1\ker((\ker(\mathbf{W}\mathbf{P}))_2)(\mathbf{J})_1 \tag{24}$$
$$\mathbf{\Phi} = \tilde{\mathbf{T}}_{12}\tilde{\mathbf{T}}_{22}^{-1}-\tilde{\mathbf{S}}_{11}(\ker(\mathbf{W}\mathbf{P}))_1\ker((\ker(\mathbf{W}\mathbf{P}))_2)(\mathbf{J})_2\tilde{\mathbf{T}}_{22}^{-1}. \tag{25}$$

We are now able to interpret the presence and influence of the transversality condition (1) on the model's solution. In both cases it can be seen that the transversality condition restricts the available degrees of freedom. In the first case it imposes an additional restriction on the matrix $\mathbf{\Phi}$, which explains the influence of the expectational error on the system's stable variables as a function of its influence on the system's unstable variables, while in the more general second case the matrix $\mathbf{\Lambda}$, which explains the model's expectational error in the next period as a linear function of the currently expected shock term, is constrained.

4 Relevance of the Method

In order to assess the proposed method's relevance the rank restrictions imposed by the solutions can be used. Looking at the second case one recognizes immediately that for a consistent solution the following conditions need to hold

$$\operatorname{rank}((\ker(\mathbf{W}\mathbf{P}))_1) \geq \operatorname{rank}(\tilde{\mathbf{S}}_{11}^{-1}\tilde{\mathbf{T}}_{11}) \tag{26}$$
$$\operatorname{rank}\left((\ker(\mathbf{W}\mathbf{P}))_1\ker((\ker(\mathbf{W}\mathbf{P}))_2)\right) \geq \operatorname{rank}(\tilde{\mathbf{S}}_{11}^{-1}\tilde{\mathbf{T}}_{11}). \tag{27}$$

Defining the ranks on the left sides of this equations as $\xi = \operatorname{rank}((\ker(\mathbf{W}\mathbf{P}))_1)$ and $\psi = \operatorname{rank}(\xi, \dim(P)-\operatorname{rank}(\mathbf{W})-\zeta)$, where ζ is defined as $\zeta = \operatorname{rank}((\ker(\mathbf{W}\mathbf{P}))_2)$, notice that there are upper bounds for these measures given by

$$\xi \leq \min(\dim(\tilde{\mathbf{S}}_{11}^{-1}), \dim(\mathbf{P})-\operatorname{rank}(\mathbf{W}))) \tag{28}$$
$$\zeta \leq \dim(\mathbf{P})-\max(\dim(\tilde{\mathbf{S}}_{11}^{-1}), \operatorname{rank}(\mathbf{W})) \tag{29}$$
$$\psi \leq \min\left(\min(\dim(\tilde{\mathbf{S}}_{11}^{-1}), \dim(\mathbf{P})-\operatorname{rank}(\mathbf{W})), \max(\dim(\tilde{\mathbf{S}}_{11}^{-1}), \operatorname{rank}(\mathbf{W}))-\operatorname{rank}(\mathbf{W})\right) \tag{30}$$

It is now possible to give conditions of the existence of a solution with integrated transversality conditions for the case that all three measures adopt their upper limits. In fact three different constellations allow for the existence of such a solution. In the first constellation the sum of the dimensions of the system and the number of its stable eigenvalues diminished by the rank of the transversality condition, i.e. the rank of the matrix $\mathbf{W}$, needs to exceed the rank of the system's coefficient for the current period. In the second constellation the dimension of the system needs to exceed the number of the stable eigenvalues, which in turn needs to exceed the rank of the transversality condition. Additionally the difference between the twofold of the number of the stable eigenvalues and the rank of the transversality condition needs to exceed the rank of the system's coefficient for the current period. And finally, in the third constellation the dimension of the system needs to exceed the rank of the transversality condition, which needs to exceed the number of stable eigenvalues, which in turn needs to exceed the rank of the system's coefficient for the current period. Hence either

$$\operatorname{rank}(\tilde{\mathbf{T}}) \leq \dim(\mathbf{P}) - \operatorname{rank}(\mathbf{W}) + \dim(\tilde{\mathbf{S}}_{11}^{-1}) < 2\dim(\tilde{\mathbf{S}}_{11}^{-1}), \tag{31}$$

$$\operatorname{rank}(\tilde{\mathbf{T}}) \leq 2\dim(\tilde{\mathbf{S}}_{11}^{-1}) - \operatorname{rank}(\mathbf{W}) \quad \wedge \quad \dim(\mathbf{P}) > \dim(\tilde{\mathbf{S}}_{11}^{-1}) > \operatorname{rank}(\mathbf{W}) \tag{32}$$

or

$$\operatorname{rank}(\tilde{\mathbf{T}}) \leq \dim(\tilde{\mathbf{S}}_{11}^{-1}) < \operatorname{rank}(\mathbf{W}) < \dim(\mathbf{P}) \tag{33}$$

need to hold. While these conditions are relatively difficult to interpret in terms of economics, two direct inferences can be drawn. Firstly, the constraint on the rank of $\tilde{\mathbf{T}}$ limits the relevance of the method to big systems including leads and/or lags of endogenous variables, because those systems tend to be presented by a matrix $\mathbf{H}$ characterized by a certain degree of sparsity and hence a relatively low rank. One should recognize that the rank of $\tilde{\mathbf{T}}$ is higher than the number of unstable eigenvalues. Hence the basic result follows, that for existence there must be a certain upper threshold for the number of unstable eigenvalues. Moreover the rank of $\tilde{\mathbf{T}}$ decreases with the number of generalized eigenvalues equal to zero. Secondly, the rank of $\mathbf{W}$ has an ambivalent influence on the first and the third constellations and an unambiguous one on the second. In the first (third) constellation a lower rank of this matrix increases (decreases) the chances that the left inequalities are fulfilled, while it decreases (increases) the chances that the right inequality holds. In the second constellation a lower rank of $\mathbf{W}$ increases the chances that both of the conditions hold.

Nevertheless these conditions are only necessary and not sufficient conditions for the existence of a solution which integrates the transversality condition. The reason is that the conditions above are derived for the case that all measures, ζ, ξ and ψ, take on their upper bounds. But in general these measures are free to be smaller than their upper bounds and therefore, even if the discussed conditions hold, there is the possibility that the transversality condition can not be integrated into a consistent solution of the dynamic model.

5 Example economies

In order to use the transversality condition for limiting the indeterminacy within the model's solution, an economy with at least two assets is essential. The two, or more, assets, generate some degrees of freedom within the transversality condition, and thus allow this condition to spread across a certain domain. It is exactly the intersection of that domain with the manifold of possible solutions to the rational expectation model, which forms the set of solutions fulfilling the transversality condition. Hence reducing the transversality condition on a single variable is in the extreme case, i.e. a model with only one state, equivalent to reducing the terminal condition on this state to a single point. This strongly reduces the probability that this condition holds for the set of the solutions generated by the rational expectation model. But allowing for two or more state variables, i.e. assets, enables the transversality condition to play a role in reducing potential indeterminacies. This reflects the fact that in indeterminacy there are less unstable variables than non-state variables. Hence these non-state variables are determined together with the state variables in stable block. But the initial conditions on the states do not give enough information in order to generate a fully determined solution. Therefore additional information on the states contained in the transversality condition may help to decrease the degree of indeterminacy in the solution, so long as this information is not perpendicular to the set of solutions generated by the rational expectation model.

Following this argument an economy with two production sectors, i.e. two different technologies and thus two different capital stocks, is used to demonstrate the limited relevance of the method for small-scale models. The model is more or less of the New Keynesian type, featuring imperfect competition, risk aversion and endogeneity of the capital stock. For simplifying reasons prices are assumed to be fully flexible. However price rigidity would not change the result. Starting with the representative consumer this one consumes a Dixit-Stiglitz bundle of consumption good varieties, $c_t = (\int_0^1 x_{it}^{\frac{\eta-1}{\eta}} di)^{\frac{\eta}{\eta-1}}$ and invests into a riskless asset b_t, or $\hat{b}_t$ in real terms, supplied by a risk-neutral financial intermediator. The gross return on this asset, where $(1 + r_t)$ is the gross rate of return, together with the household's share in the profits of the firms, i.e. Π_t, is the income of the household. Hence the household's budget constraint is equal to $\int_0^1 p_{it} x_{it} di + b_t = (1+r_{t-1})b_{t-1}+\Pi_t$. Her utility for the entire time horizon is given by $E(U_t) = \sum_{q=0}^{\infty} \beta^q \frac{c_{t+q}^{(1-\sigma)}}{1-\sigma}$ and depends exclusively on current and future consumption, where c_t denotes consumption in t. Solving the two stages of the problem, i.e. static cost minimization and and dynamic utility maximization, yields, after reducing the resulting system, to the following equations describing the household optimum

$$\beta^{-1} c_t^{-\sigma} - \frac{1 + r_{t+1}}{1 + \pi_{t+1}} c_{t+1}^{-\sigma} = 0 \quad (34)$$

$$c_t + \hat{b}_t - \frac{(1 + r_t)}{1 + \pi_t} \hat{b}_{t-1} - \hat{\Pi}_t = 0. \quad (35)$$

Herein, as in all following equations, future variables denote the expected values of these variables for the according periods. Additionally a transversality conditioned needs to be imposed, which for the infinite horizon of the model takes the form

$$\lim_{t\to\infty} \beta^t \hat{b}_t = 0. \tag{36}$$

Turning to the production sector the technologies of the producers of the different varieties are characterized by decreasing marginal returns in capital k_{it}: $y_t^i = k_{it}^{\alpha_i}$. Their capital stocks are governed by the investment decision according to $k_{i(t+1)} = k_{it}(1-\delta_i) + I_{it}\ \forall i$, where I_{it} is the real investment in period t and δ_i is the depreciation rate. The possibility of disinvestment is included by assumption, but the possibility to disinvest so heavily that the capital stock becomes negative, is excluded; thus $k_{it} \geq 0\ \forall\ i \in (j,l)\ \wedge\ \forall\ t$. Since the producers are imperfect monopolists, they set their prices in order to maximize profits. Therefore the problem can again be split into two parts. In the first one the firm's expected costs consisting of debt service and depreciation is minimized. In the second one expected profits are maximized. Solving both problems yields, after reducing the resulting equations,

$$\hat{s}_t^i = \frac{1+r_t}{(1+\pi_t)\alpha_i k_{it}^{\alpha_i-1}} \left(1 - \beta\frac{(1-\delta_i)(1+r_{t+1})(1+\pi_t)}{(1+r_t)(1+\pi_{t+1})}\right) \tag{37}$$

$$\frac{1-\eta}{\eta}\hat{p}_{it} - \hat{s}_{it} = 0. \tag{38}$$

Additionally, for each firm i the boundary condition $\lim_{t\to\infty} E_0\beta^{t-1}k_{it} = 0$ needs to hold, in order to prevent the waste of any resources. Remembering the assumption that there are only two different technologies in the economy and assume that a fraction χ of the continuum of all producers belongs to the type with technology j, while the technology of the rest is of type l. Hence aggregating across all producers and recalling the definition of the general price level delivers

$$\frac{\eta-1}{\eta} = (\chi(\hat{s}_t^j)^{1-\eta} + (1-\chi)(\hat{s}_t^l)^{1-\eta})^{\frac{1}{1-\eta}}, \tag{39}$$

from which also follows that the real profits Π_t are actually equal to a fraction of real aggregate output, i.e. $\frac{1}{\eta}y_t$. The conditions for market equilibria are derived as follows. For the consumption good market aggregate demand is equated aggregate supply, while aggregate investment, which is the Dixit-Stiglitz aggregate of the individual investment functions I_{it}, is substituted by the capital stock dynamics. For the assets markets all capital stocks across all firms are aggregated and equated to the demand for bonds. Hence the following conditions must hold.

$$c_t + k_{t+1} - (\chi(1-\delta_j)k_{jt}^{\frac{\eta-1}{\eta}} + (1-\chi)(1-\delta_l)k_{lt}^{\frac{\eta-1}{\eta}})^{\frac{\eta}{\eta-1}} = y_t \tag{40}$$

$$\hat{b}_t = k_t, \tag{41}$$

where $y_t = \left(\chi(k_{jt}^{\alpha_j})^{\frac{\eta-1}{\eta}} + (1-\chi)(k_{lt}^{\alpha_j}))^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}$ denotes aggregate output and $k_t = (\chi k_{jt}^{\frac{\eta-1}{\eta}} + (1-\chi)k_{lt}^{\frac{\eta-1}{\eta}})^{\frac{\eta}{\eta-1}}$ stands for the aggregate capital stock in period t. Finally monetary policy is specified in form of a Taylor rule in order to close the economy by determining the interest rate as a function of the inflation rate and an unanticipated monetary policy shock component z_t

$$(1 + r_t) \quad = \quad \mu + \gamma(1 + \pi_t) + z_t. \tag{42}$$

Using (41) to eliminate $\hat{b}_t$ in (35) and (37) to eliminate $\hat{s}_{it}$ and $\hat{s}_{jt}$ in (39) and gathering the results together with (34),(40) and (42) yields the reduced set of equations determining the optimum of the model. This set forms a system of nonlinear equations in the endogenous variables $(k_j, k_l, c, (1+\pi), (1+r))$, which depend on the exogenous shock term z_t. Linearizing the system around any arbitrary steady state yields a system of form (2), where the vector of endogenous variables is $(\dot{k}_{jt}, \dot{k}_{lt}, \dot{c}_t, \dot{\pi}_t, \dot{r}_t)$.[5] The matrix $\mathbf{H}$ within this system can be represented as

$$\begin{pmatrix}
-\chi(\frac{k^{\frac{1}{\eta}} k_j^{\frac{\eta-1}{\eta}}}{\beta} + \frac{\alpha_j}{\eta} y^{\frac{1}{\eta}} k_j^{\alpha_j - 1 - \frac{\alpha_l}{\eta}}) & \frac{\frac{k^{\frac{1}{\eta}} k_l^{\frac{\eta-1}{\eta}}}{\beta} + \frac{\alpha_l}{\eta} y^{\frac{1}{\eta}} k_l^{\alpha_l - 1 - \frac{\alpha_l}{\eta}}}{(\chi - 1)^{-1}} & c & \beta^{-1} k & -\beta^{-1} k \\
(\alpha_j - 1)\frac{\chi}{1-\chi} & \frac{1}{(\alpha_l - 1)^{-1}}(\frac{\delta_l \alpha_j k_j^{\alpha_j - 1}}{\delta_j \alpha_l k_l^{\alpha_l - 1}})^{1-\eta} & 0 & (\frac{\delta_l \alpha_j k_j^{\alpha_j - 1}}{\delta_j \alpha_l k_l^{\alpha_l - 1}})^{1-\eta} - \frac{\chi}{\chi - 1} & \frac{\chi}{\chi - 1} - (\frac{\delta_l \alpha_j k_j^{\alpha_j - 1}}{\delta_j \alpha_l k_l^{\alpha_l - 1}})^{1-\eta} \\
0 & 0 & \frac{\sigma c^{-\sigma}}{\beta} & 0 & 0 \\
-\chi(\frac{k^{\frac{1}{\eta}} k_j^{\frac{\eta-1}{\eta}}}{(1-\delta_j)^{-1}} + y^{\frac{1}{\eta}} \alpha_j k_{jt}^{\alpha_j - 1 - \frac{\alpha_l}{\eta}}) & \frac{\frac{k^{\frac{1}{\eta}} k_l^{\frac{\eta-1}{\eta}}}{(1-\delta_l)^{-1}} + y^{\frac{1}{\eta}} \alpha_l k_l^{\alpha_l - 1 - \frac{\alpha_l}{\eta}}}{(\chi - 1)^{-1}} & c & 0 & 0 \\
0 & 0 & 0 & -\gamma(1 + \pi) & (1 + r).
\end{pmatrix} \tag{43}$$

Linearizing the household's transversality condition (36) after substituting the right side of (41) for $\hat{b}_t$ and linearizing and aggregating the terminal conditions of the firms of the same technological types yields the matrix $\mathbf{W}$ within the transversality condition (1).

$$\begin{pmatrix}
\chi k_j^{\frac{\eta-1}{\eta}} & (1 - \chi) k_l^{\frac{\eta-1}{\eta}} & 0 & 0 & 0 \\
\beta^{-1} k_j & 0 & 0 & 0 & 0 \\
0 & \beta^{-1} k_l & 0 & 0 & 0
\end{pmatrix} \tag{44}$$

The ranks of the matrices $\mathbf{H}$ (or equivalently of $\tilde{\mathbf{T}}$) and $\mathbf{W}$ depend clearly on the chosen parameters. But in a preliminary Monte Carlo simulation of the rank of matrix $\mathbf{H}$ for the standard parameters $\eta = 11$, $\beta = 0.995$, $1 + \pi = 1.005$, $\sigma = 5$, $\gamma = 0.5$ and the depreciation rates of both sectors variating between 0.005 and 0.025, while the capital coefficients of both technologies and the market share of these are drawn stochastically from the interval $[0, 1]$, not even the necessary conditions for a successful integration of the transversality condition can be matched.[6]

Due to computational problems so far results for large-scale models are not available or rather presentable.

to be completed for large-scale models

[5] Note that $\dot{r}_t$ and $\dot{\pi}_t$ denote here deviation rates of the gross interest rate and the gross inflation.

[6] This is a preliminary result, which is yet not citeable!

6 Conclusions

This paper analysis possibilities to use the transversality condition of macroeconomic DCGE models with rational expectations in order to limit the frequently appearing indeterminacy of the solution. In order to achieve this goal a simple method of integrating the transversality constraint into the standard solution is used. Basically, the solution path derived from standard algorithms using the stability criterion of saddle path convergence back to steady state is forced to fulfill additionally the transversality condition at the infinite horizon of the model. This is implemented by forcing the coefficients of the autoregressive solution path to lie within the column space of the coefficient of the transversality condition. General versions of the standard solution algorithms are adjusted in order to comply with this idea. Even for models, in which the standard stability criterion of a saddle path convergence back to steady state does not hold, the transversality condition can help to restrict the evolution of the model in such a way, that at least the discounted path of the endogenous variables displays a stable converging behaviour.

Using rank properties of the various coefficients of the model the relevance of the proposed method is assessed. It turns out that for small and relatively dense systems, which in general result from small scale models with a limited degree of intertemporal interdependencies, the proposed method fails. This is due to the fact that the high rank of the coefficients of the model's system of first order conditions forestall the possibility that the coefficients of the autoregressive solution can be linearly generated by the nullspace of the transversality condition. But for models of a larger scale and with a richer pattern of intertemporal interdependencies, i.e. higher number of lags or leads, this problem is alleviated. Hence the method promises to be useful for limiting the degree of indeterminacy in exactly those models, where this indeterminacy is expected to be particularly serious. These results are confirmed by the example of a small-scale macroeconomic model, which for the range of standard parameters does not fulfill even the necessary conditions for a solution, in which the transversality condition has been successfully integrated. So far no results for macroeconomic models of a larger scale are available, because computational problems prevented their derivation. To solve this problems will be the main focus within the near future of the research project.

7 Appendix

Theorem 1: The arbitrary vectors $\mathbf{Y}$ and $\mathbf{X}$ and a stochastic vector $\mathbf{V}$, which includes the nullvector within its domain and has a full-dimensional continuous distribution, fulfill for given matrices $\mathbf{N}$, $\mathbf{M}$ and $\mathbf{K}$ properties (1) $\mathbf{Y} \in \ker(\mathbf{N})$ and (2) $\mathbf{Y} = \mathbf{MX} + \mathbf{KV}$ if the following necessary and sufficient conditions hold.

a) $\mathbf{col}(\mathbf{K}) \subseteq \ker(\mathbf{N})$

b) $\mathbf{col}(\mathbf{M}) \subseteq \ker(\mathbf{N})$

Proof:

1. Sufficiency follows directly from a) and b).

2. Necessity:

    a. Suppose that $\mathbf{X} = \mathbf{0}$ and $\mathbf{col}(\mathbf{K}) \nsubseteq \ker(\mathbf{N})$; $\mathbf{Y} \in \ker(\mathbf{N})$ iff $\forall \mathbf{V} : \mathbf{KV} \in \ker(\mathbf{N})$ $\because$ $\mathbf{V}$ is stochastic $\Rightarrow \mathbf{col}(\mathbf{K}) \subseteq \ker(\mathbf{N}) \Rightarrow$ contradiction

    b. Suppose that $\mathbf{V} = \mathbf{0}$ and $\mathbf{col}(\mathbf{M}) \nsubseteq \ker(\mathbf{N}) \Rightarrow \mathbf{Y} \notin \ker(\mathbf{N}) \Rightarrow$ contradiction to (1)$\blacksquare$

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