Indeterminacy with Externalities and Capital Utilization*

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Abstract

In this paper, we present a dynamic general equilibrium model with two sectors: one aggregate firm produces consumption good and a second one investment good. We assume sector specific as well as aggregate externalities. Moreover, we account for variable capital utilization i.e. the depreciation rate is endogenously determined by the degree of capital exploitation. We show that under mild conditions, multiple equilibria occur. Indeed, following Grandmont, Pintus and de Vilder (JET, 1998), we can identify the necessary conditions for which the economic dynamics does change stability. In our framework, the capital utilization improves the imperfection in production sectors, resulted from the specific and aggregate externalities. We show that endogenous fluctuations are more likely to appear as long as the sensitivity of the capital utilization is sufficiently high with respect to the capital. Further, this is true for low value of elasticity of factor substitution.

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Key words: Externalities; Variable capital utilization; Indeterminacy; Bifurcations.

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1 Introduction

In recent times, indeterminacy and multiple equilibria have been widely appeared in business cycle models; there are infinite trajectories (paths) converging towards the steady-state equilibrium. Indeed, the role of production externality as a market imperfection in deriving the indeterminacy has been mentioned in macroeconomic studies and mainly considered in two types; through the existence of aggregate wide externality or by introducing a specific-sector external effect in a model with more than one sector of production.

Theoretical research has illustrated that in standard real business cycle model of a one sector such as Benhabib and Farmer (1994), Farmer and Guo (1994), the model exhibits multiple equilibria if and only if the degree of increasing returns-to-scale is sufficiently high. These models were criticized later as these values are not empirically plausible. More precisely, the increasing returns needed in this model to attain the indeterminacy are about 1.5. The perception for the indeterminacy appearance in one-sector model with production externality is relatively simple. Assume that agents expect that the capital return will get higher tomorrow; this will induce them to increase their capital tomorrow. Given that the increasing returns are adequately high, so that the marginal product is increasing with capital, the expectation will be fulfilled.

One of the earlier literature that initiated the sector-specific and aggregate external effects in standard real business cycle model is Benhabib and Farmer (1996). The authors proposed two-sector model and introduced sector-specific externality. However, they observed that the external effect needed to attain the multiple equilibria is quite smaller than that of Benhabib and Farmer (1994). The mechanism behind this type of models can be summarized as follows: once the agents believe that tomorrow the capital return will be higher; they reallocate factors of production between the two sectors (consumption and investment). This makes changing in the relative price of investment that affects the capital returns in such a way that the marginal products do not require to be increasing in capital for the belief to be fulfilled.

Furthermore, the concept of capital utilization has been applied widely in the real business cycle literature.[1] More recently, Wen (1998) introduced an endogenous depreciation rate in the model of Benhabib and Farmer (1994). In this case, the depreciation depends strictly on the capital utilization rate, which could be defined as the speed or intensity in which a given stock of capital can be operated. He concluded that once the capital utilization is taken into account, the model generates indeterminacy and endogenous fluctuations for mild enough increasing returns-to-scale. Wen found that the increasing returns are about 1.108, which is less than the model without capital utilization.

After that, endogenous depreciation rate in two-sector models starts to come out in the RBC models through the work of Guo and Harrison (2001), where they assumed that the representative firm is influenced by a sector-specific externality. They show that introducing a variable capital utilization rate into

[1] See among others, Greenwood, Hercowitz, and Huffman (1988), Burnside, Eichenbaum and Repelo (1996), Dejong, Ingram, Wen and Whiteman (1996), Bils and Cho (1994).

a two-sector model with externality will decline the increasing returns to scale needed to generate the endogenous fluctuation.

The model of Harrison and Weder (2000) is based on Benhabib and Farmer (1996) where they concluded that indeterminacy is obtained by the existence of modest sector-specific external effect. They found that the sector-specific externalities are enough to generate indeterminacy with a negligible departure from constant returns. Likewise, Harrison (2001) examined a discrete version of Benhabib and Farmer (1996) model with a general constant risk aversion utility function. He assumed that the degree of sector-specific externality in consumption and investment sectors are not the same. Multiple equilibria are resulted with minimum value of externality in investment sector, even if the externality in consumption sector does not exist.[2]

In this paper, we provide a less restrictive version of Benhabib and Farmer (1996) with both aggregate and sector-specific externalities. However, in this model we have some different aspects. From one hand, we use a general constant-returns to scale production function in both sectors in order to reveal that the elasticity of capital-labor substitution plays a main role in determining the stability of the economy (Garnier, Nishimura and Venditti (2007) and Pintus (2006)). From another hand the labor supply is assumed to be inelastic along with it is normalized to unity. Indeed, we use a general increasing utility function for the representative agents that depends on their current consumption to show the importance of the elasticity of intertemporal substitution in consumption in the local dynamics, as in Pintus.

In addition, we assume that the depreciation rate is variable and it is endogenously determined, which means that it depends strictly on the rate of capital utilization. This implies that higher the rate of capital utilization in the production, higher this capital will be depreciated. The capital utilization used by the economic agents depends on the sensitivity of the depreciation rate to the utilization rate of capital. However, this sensitivity will prevent agents from using all their stock of capital.

We apply the geometrical method for the discrete nonlinear two-dimensional economic system developed before by Grandmont, Pintus and de Vilder (1998). The purpose of this technique is to study how the capital utilization and the elasticity of factor substitution could affect the endogenous fluctuation and the stability of the steady state under two-sector model, and to identify the conditions under which the indeterminacy appears, as well as the local bifurcation.

The main results obtained in this paper are the following. As the elasticity of capital utilization with respect to consumption (or the elasticity of capacity utilization with respect to capital) increases, then the economic system is indeterminate. This is because given a certain level of income, once the demand of consumption goods declines, then the demand for capital gets higher. So that, the rate of its utilization would increase more, and so the output. Due to the existence of external effects, the marginal product of capital will be increasing

[2] This result is obtained previously by Weder (2000).

rather than decreasing like the standard RBC models (Benhabib and Farmer (1994)). This implies that the rational expectations equilibrium is indeterminate.

Secondly, we demonstrate that when the elasticity of capital-labor substitution is low, then the endogenous fluctuation appears. This could be interpreted as following: if this elasticity is low, then higher increase in capital meets with a slightly decline in labor, the total production rises. As a consequence of the presence of externality, the marginal production of labor will increase with labor, and so indeterminacy.

The paper is organized as follows. In the next section, we present the model and define the intertemporal equilibrium. Section 3 is devoted to the existence of the steady state. In section 4, we analyze the local dynamics. Discussing the results would be in section 5. Concluding remarks are provided in the section 6, while some computational details are gathered in the appendix.

2 The economy

This model integrates the capital utilization with a discrete-time version model of two sectors as in Benhabib and Farmer in which we assume that the agent's utility is drawn from their current consumption. Furthermore, the economy is populated by a large number of representative agents whose size is normalized to one. Firms use labor and capital as inputs to produce both consumption and investment goods using a constant-returns to scale production function, as well as taking the externalities as given. Each industry is consist of a large number of identical firms which are also normalized to unity.

2.1 Firms

In this economy, we assume a perfect competition in all markets. This means that the individual firms take the average capital and labor as given once they determine the optimal decisions. There are two types of goods; consumption c and investment goods I.[3] The consumption goods are produced according to the following production function

$$c_t = A\bar{A}_t F\left(u_t \mu_{k,t} k_t, \mu_{l,t} l_t\right) \tag{1}$$

while the production function of the investment goods is given by

$$I_t = B\bar{B}_t F\left[u_t\left(1-\mu_{k,t}\right)k_t, \left(1-\mu_{l,t}\right)l_t\right] \tag{2}$$

where k_t and l_t are the amount of capital stock and labor available in the economy. Moreover, $\mu_{k,t}$ and $\mu_{l,t}$ are respectively the share of capital and labor devoted to produce consumption goods, $u_t \in (0,1)$ is the capital utilization

[3] In the appendix, we derive the profit maximization conditions for the both sectors.

rate,[4] in addition to $F(.,.)$ is the production function and it is homogenous of degree one in both factors (labor and capital). We presume that A, B, $\bar{A}_t$ and $\bar{B}_t$ are scaling parameters taken as given at the perspective of the single firm. However, $\bar{A}_t$ and $\bar{B}_t$ are the social level of sector-specific as well as aggregate capital and labor existing in the economy. In order to be more clear, we specify -without loss in generality- that $\bar{A}_t$ and $\bar{B}_t$ take the form

$$\bar{A}_t = \left(\bar{u}_t \bar{\mu}_{k,t} \bar{k}_t\right)^{a\theta} \left(\bar{\mu}_{l,t} \bar{l}_t\right)^{b\theta} \left(\bar{u}_t \bar{k}_t\right)^{a\phi} \bar{l}_t^{b\rho} \tag{3}$$

and

$$\bar{B}_t = \left(\bar{u}_t \left(1 - \bar{\mu}_{k,t}\right) \bar{k}_t\right)^{a\theta} \left(\left(1 - \bar{\mu}_{l,t}\right) \bar{l}_t\right)^{b\theta} \left(\bar{u}_t \bar{k}_t\right)^{a\phi} \bar{l}_t^{b\rho} \tag{4}$$

The bar over a variable entails the economy-wide average, and it is given by the individual firm standpoint. Therefore, $\bar{\mu}_{k,t}\bar{k}_t$ is the average quantity of capital stock in the consumption sector, $\bar{\mu}_{l,t}\bar{l}_t$ is the average amount of labor devoted in producing consumption goods. Furthermore, $\bar{u}_t$ is the average capital utilization rate used in all the economy. Also, a, b, θ, ϕ and ρ are all positive parameters with $a + b = 1$. We assume for simplicity that the scaling factors A and B are equal, i.e. $A = B = Q$. The parameter θ measures the size of sector-specific externality,[5] while on another hand, ϕ and ρ determine the degree of aggregate external effect of capital and labor respectively. Additionally, $\bar{l}_t$ and $\bar{k}_t$ are the wide-average amount of labor and capital in the economy. Notice here that we simplify our analysis through supposing that the sector-specific and aggregate externalities are the same across the two sectors. Since we assume that the factor intensities are identical across sectors, then it is easy to show that the share of capital and labor in consumption sector are the same, that is $\mu_{k,t} = \mu_{l,t} = \mu_t$.

2.2 Households

This model is in a certain environment where the representative household is endowed with a unit of labor in each period which is supplied inelastically. Each household has at the first period a quantity of capital $k_0 > 0$, and seek to maximize the discounted utility function $\sum_{t=0}^{\infty} \beta^t X(c_t)$, where $\beta \in (0, 1)$ is the discounted rate, and X is the instantaneous utility function that satisfies the following normal assumption.

Assumption 1 (Preferences). *The function X is continuous for $c \geq 0$, and C^r , $r \geq 2$, for $c > 0$, with $X'(c) > 0$, $X''(c) < 0$. Moreover, $\lim_{c \longrightarrow 0+} X'(c) = +\infty$. We denote $\sigma(c) \equiv |X''(c)c/X'(c)| \in (0, +\infty)$ the elasticity of the marginal utility (the inverse of the elasticity of intertemporal substitution in consumption).*

[4] Since the real interest rate in both sectors is the same, this entails that the rate of capacity utilization in both sectors is the same.

[5] See among others, Harrison (2001) in which he assumed that the sector-specific externality is different across sectors.

The boundary condition in assumption (1) is to guarantee the existence of interior solution for the household maximization problem. The agent chooses how much he will consume c_t, how much will invest in capital k_{t+1} and how much rate these capital would be operated u_t at each period subject to the following budget constraint

$$c_t + q_t \left(k_{t+1} - \left(1 - \delta\left(u_t\right)\right) k_t\right) = w_t l_t + r_t u_t k_t \tag{5}$$

where $q_t > 0$ is the price of investment goods in terms of consumption goods which means that the price of consumption is normalized to unity, k_t is the capital stock, w_t is the real price of labor in terms of consumption goods. Since the capacity utilization rate is decided by agents, then contrary to Guo and Harrison (2001) and analogous with Weder (2004), we suppose that the real rental rate of capital r_t is paid according to the amount of capital utilized not based on the total amount of capital hold by agents. Finally, the depreciation rate $\delta\left(u_t\right) \in (0, 1)$ is endogenous which could be described as[6]

$$\delta\left(u_t\right) = \frac{1}{\gamma} u_t^{\gamma}, \quad \gamma > 1 \tag{6}$$

where $u_t \in (0, 1)$ is the capital utilization rate. Equation (6) states that the depreciation rate depends on the degree of utilization of the capital stock, the more speed of capital utilization rate, the more this capital will be depreciated. The elasticity of capital depreciation with respect to the rate of capital utilization is denoted by γ, with $\gamma > 1$ in order to ensure that the depreciation rate is convex function. Otherwise, if $\gamma < 1$, then agents will use all their stock of capital. On the other side, agents choose the rate of capital utilization according to both capital gain (output) and capital loss (depreciation). In the standard RBC models where the depreciation rate is assumed to be constant, the marginal cost of capital utilization is zero, and this implies that agents will use all the stock of capital available for them. Furthermore, if $\gamma \to \infty$, then the sensitivity of depreciation rate to the capacity utilization decreases, thus declining the marginal cost of capital utilization.

The first-order conditions relating to the household's maximization problem subject to constraint (5) and (6) are:

$$q_t \frac{X'\left(c_t\right)}{X'\left(c_{t+1}\right)} = \beta \left[q_{t+1} \left(1 - \frac{1}{\gamma} u_{t+1}^{\gamma}\right) + u_{t+1} r_{t+1}\right] \tag{7}$$

$$q_t u_t^{\gamma - 1} = r_t \tag{8}$$

$$\lim_{t \longrightarrow \infty} \beta^t X'\left(c_t\right) k_{t+1} = 0 \tag{9}$$

Where equation (7) can be described as the augmented Euler equation, which economically says that the trade-off between current and future consumption

[6] Nishimura and Venditti (2001) did not assume that the capital utilization rate is variable. But, they have studied a two-sector model and demonstrated the effects of both full and partial depreciation rate on the indeterminacy appearance.

depends on the discount factor, capacity utilization rate, the capital rate of return and the price of investment good in terms of consumption goods. Likewise, we can express it as the consumption demand equation, in which it measures the impact of the time preference, the real interest rate and the relative price of investment goods on consumption smoothing. Furthermore, Equation (8) determines the optimal rate of capacity utilization by equating its marginal benefit to its marginal cost. The left-hand side of this equation represents the cost of using more capital which consists of higher depreciation rate, while the right-hand side denotes the benefit (gain) from increasing the rate of capacity utilization. Condition (9) is the well-known transversality condition.

2.3 Intertemporal equilibrium

Since the firms are identical, at the equilibrium we have $\bar{k}_t = k_t$, $\bar{l}_t = l_t$.We assume also that the labor is supplied inelastically, $l_t = 1$. Based on above equalities together with the assumption that the production function $F(.,.)$ is homogenous of degree one, the amount of consumption and investment goods produced in this economy are respectively

$$c_t = Q\mu_t^{\theta+1}\left(u_t k_t\right)^{\alpha} f\left(u_t k_t\right) \tag{10}$$

and

$$I_t = Q\left(1-\mu_t\right)^{\theta+1}\left(u_t k_t\right)^{\alpha} f\left(u_t k_t\right) \tag{11}$$

where $\alpha \equiv a(\theta+\phi)$, and f is the intensive production function which satisfies the following standard assumption.

Assumption 2 (Technology). *The reduced production function $f\colon R_+ \to R$ is smooth, strictly increasing and strictly concave. Furthermore, $f(0) = 0$, $\lim_{uk\to 0+} f'(uk) = +\infty$ and $\lim_{uk\to +\infty} f'(uk) = 0$. We denote $s(uk) \equiv f'(uk)\,uk/f(uk) \in (0,1)$ the share of capital in total income (equivalently, the elasticity of the intensive technology).*

Moreover, using the equilibrium conditions, the relative price of investment goods in terms of consumption goods could be written as:

$$q_t = \bar{A}_t/\bar{B}_t = \left(\mu_t/\left(1-\mu_t\right)\right)^{\theta} = q\left(\mu_t\right) \tag{12}$$

This result means that at the private level, holding constant the sectoral allocations of all other firms, the production possibility frontier (PPF) is linear in (c, I) plane and has a slope equal to $\left(-\bar{A}_t/\bar{B}_t\right)$, the relative price of investment and consumption good. But, at the social level and due to the existence of sector-specific externality, the PPF becomes convex. In a special case where there are no externalities, q_t would be constant and then the economy will collapse into one-sector economy.

Indeed, using equations (8), (10), (12) and the first-order condition of firms (24) we show implicitly that

$$\left\{\begin{array}{l} u_t = u\left(c_t, k_t\right) \\ q_t = q\left(c_t, k_t\right) \\ \mu_t = \mu\left(c_t, k_t\right) \end{array}\right. \tag{13}$$

Substituting these functions into Euler equation (7) with taking in account the first-order condition of the firms in the investment sector, we get[7]

$$q\left(k_t, c_t\right) \frac{X'\left(c_t\right)}{X'\left(c_{t+1}\right)} = \beta\left[q\left(k_{t+1}, c_{t+1}\right)\left(1 - \frac{1}{\gamma}\left[u\left(c_{t+1}, k_{t+1}\right)\right]^{\gamma}\right) + u\left(c_{t+1}, k_{t+1}\right) R\left(c_{t+1}, k_{t+1}\right)\right]$$

At this instant, in order to complete the description of the competitive equilibria, we take in consideration the law of motion of the capital stock

$k_{t+1} = Q\left(1-\mu_t\right)^{1+\theta}\left(u_t k_t\right)^{\alpha} f\left(u_t k_t\right) + \left(1-\delta_t\right) k_t$ as well as the transversality condition (9). So that, the intertemporal equilibrium with perfect foresight is a deterministic sequence $\{k_t, c_t\}_{t=0}^{\infty}$, $(k_t, c_t) >> 0$ for $t = 0, 1, ...$, satisfying for every t the following system of equations:

$$\left\{\begin{array}{l} q\left(k_t, c_t\right) \frac{X'\left(c_t\right)}{X'\left(c_{t+1}\right)} = \beta\left[q\left(k_{t+1}, c_{t+1}\right)\left(1 - \frac{1}{\gamma}\left[u\left(c_{t+1}, k_{t+1}\right)\right]^{\gamma}\right) + u\left(c_{t+1}, k_{t+1}\right) R\left(c_{t+1}, k_{t+1}\right)\right] \\ k_{t+1} = Q\left(1-\mu\left(c_t, k_t\right)\right)^{1+\theta}\left[u\left(c_t, k_t\right).k_t\right]^{\alpha} G\left(c_t, k_t\right) + \left(1 - \frac{1}{\gamma}\left[u\left(c_t, k_t\right)\right]^{\gamma}\right) k_t \end{array}\right. \tag{14}$$

with

$$G\left(c_t, k_t\right) = f\left(u\left(c_t, k_t\right).k_t\right)$$

subject to the initial stock of capital $k_0 > 0$, and the transversality condition.

3 Steady state analysis:

Our first task is to ensure the existence of a stationary solution of system (14) and to characterize its uniqueness. Moreover, we need to determine -at the steady state- the mechanism beyond the effect of the elasticity of depreciation rate γ on capacity utilization rate u and on the stock of capital k. In order to carry out these proposals, let us omit in (14) the time indexes and study the resulting system:

$$\left\{\begin{array}{l} \left(\eta + \frac{1}{\gamma} u^{\gamma}\right) u^{-1} = Q\left(1-\mu\right)^{\theta}\left(uk\right)^{\alpha} f'\left(uk\right) \\ \frac{1}{\gamma} u^{\gamma} k = Q\left(1-\mu\right)^{1+\theta}\left(uk\right)^{\alpha} f\left(uk\right) \end{array}\right. \tag{15}$$

with $\eta = 1/\beta - 1$. It is interesting to show the existence of the steady state numerically, in which we calibrate equations (15) using standard parameter values used before in macroeconomic and real business cycle literature. Additionally,

[7] See the appendix for the proof.

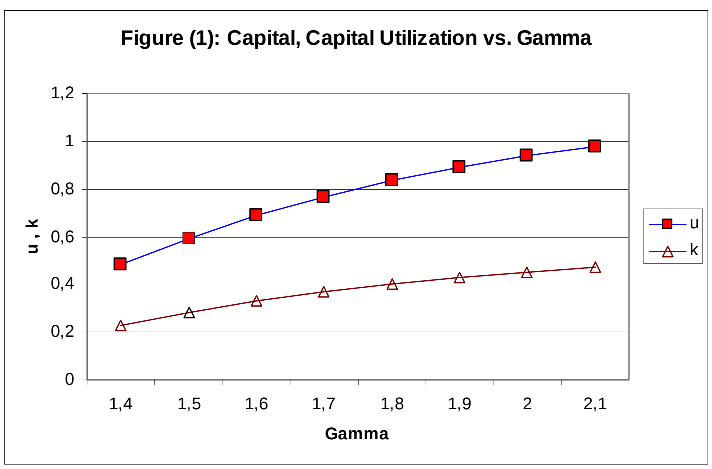
we consider in the steady state that the production function takes the form Cobb-Douglas $f(uk) = (uk)^a$. In table (1) below, we summarize the required parameter values in order to show that equations (15) exhibit unique steady state of capital stock k and capital utilization rate u. Therefore, after introducing these parameters into (15), we obtain a steady state value of u and k, call it $u^*$and k^*. If we take any value of $k \neq k^*$ and $u \neq u^*$ such that $u \in (0, 1)$, we show that the dynamic system of k and u converges -in the long term- to the same values u^*, k^*. This means that we have a unique steady state. So that, in the case where the production function takes a Cobb-Douglas form, we get a unique steady state consists of u^* and k^*.

In order to enrich our steady state analysis, we try to figure out the relationship between the rate of capacity utilization and the capital stock corresponding to different values of γ (the elasticity of depreciation rate with respect to the rate of capital utilization).[8] We suppose for simplicity that there is no aggregate externality, i.e. $\phi = 0$.

Parameters	θ	α	ϕ	μ	β	a	γ
Values	0.2	0.07	0	0.33	0.99	0.33	1.4

TABLE (1)

As we said before that γ is always greater than unity, so, based on equation (6), as $\gamma \longrightarrow \infty$, the sensitivity of depreciation rate to capital utilization declines, thereby reducing the marginal cost of capital utilization. So it is more profitable for agents to increase the amount of capital utilized in the production process. We know that the increasing in the capital utilization has two contrary effects. On one hand, as we increase the capital utilization rate, the amount of production gets higher, so the investment. On another hand, the depreciation rate of the existing capital increases, so the capital stock reduces.



[8] See equation (2) to know the mathematical relation.

As we show in figure (1) that at the steady state once γ increases, then the rate of capacity utilization and the capital stock get higher. This implies that the capital amount generated by production dominates the depreciated capital resulted from increasing the utilization rate. So that, for each value of γ we obtain particular values of both capital utilization rate u and capital stock k. Notice also that the increasing in the capital utilization rate has diminishing marginal effect on the capital stock, i.e. when the capacity utilization rate is low, its marginal increasing in capital stock is higher than that when the capital utilization is high.

Generally, we want to limit our analysis to economies with relatively modest externalities on aggregate capital and in particular we want to prevent the social marginal productivity of capital from increasing without bound as the capital intensity diverges to infinity. At the same time we want to rule out the case in which $\alpha + a = 1$. Indeed, in this eventuality, the economy could display long-run endogenous growth, so that, to avoid this phenomenon, we suppose that $\alpha + a < 1$.

We now try to determine analytically under which conditions we can obtain unique or several steady state. The steady state values of capital k and capacity utilization u should satisfy equations (15). If these equations intersect one time, then a unique steady state exists, while if we have multiple intersections, then multiple equilibria appears.

Total differentiation for (15) gives us:

$$\frac{\partial c}{\partial k} = \frac{Q\left(1-\mu\right)^{\theta}\left(uk\right)^{\alpha}\xi_1 + \frac{1}{\gamma}\left(u^{\gamma} + k\gamma u^{\gamma-1}u_k\right)}{Q\left(1-\mu\right)^{\theta}\left(uk\right)^{\alpha}\xi_2 - ku^{\gamma-1}u_c} \tag{16}$$

with

$$\zeta_1 \equiv f\left(u.k\right)\Big[\left(1+\theta\right)\mu_k - \left(1-\mu\right)\alpha\left(u.k\right)^{-1}\left(u+ku_k\right)\Big] - \left(1-\mu\right)f'\left(u.k\right)\left(ku_k+u\right)$$

$$\zeta_2 \equiv \left(1-\mu\right)f'\left(u.k\right)ku_c + \Big[\left(1-\mu\right)\alpha\left(u.k\right)^{-1}ku_c - \left(1+\theta\right)\mu_c\Big]f\left(u.k\right)$$

$$\frac{\partial c}{\partial k} = \frac{Q\left(1-\mu\right)^{\theta}u\left(uk\right)^{\alpha}\xi_3 + u^{\gamma-1}}{Q\left(1-\mu\right)^{\theta}\left(uk\right)^{\alpha}\xi_4 - u^{\gamma-1}} \tag{17}$$

with

$$\zeta_3 \equiv f'\left(u.k\right)\left(\theta\left(1-\mu\right)^{-1}\mu_k - \left(\alpha+1\right)u_k u^{-1} - \alpha k^{-1}\right) - f''\left(u.k\right)\left(u_k k + u\right)$$

$$\zeta_4 \equiv uf''\left(u.k\right).u_c k + f'\left(u.k\right)\left(\left(\alpha+1\right)u_c - u\theta\left(1-\mu\right)^{-1}\mu_c\right)$$

Therefore, the existence of unique or multiple steady state arises according to the following proposition.

Proposition 1 *(Existence of the steady state). If the sign of equation (16) and equation (17) are constant but opposite and satisfy the limit conditions, then the steady state is unique. Otherwise, if the sign of these equations are not constant, then we can have multiple equilibria and therefore global indeterminacy.*

Throughout the rest of the paper, proposition (1) will be supposed to hold and no longer referred.

4 Local Dynamics

In order to identify how capital utilization affects the appearance of indeterminacy and endogenous fluctuation, we study the local dynamics of system (14). We linearize the system in a neighborhood of the steady state (c^*, k^*). Straightforward computation yields us to the Jacobian Matrix J, where

$$J = \begin{bmatrix} \Sigma & \Pi \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -A_2\left(1+\varepsilon_{u,k}\right) & N - \varepsilon_{u,c}A_2 \\ \Gamma & -\omega \end{bmatrix}$$

With $\Sigma \equiv M\varepsilon_{u,k} + \left(\alpha - \frac{1-s}{\varphi}\right) A_5 + (\alpha + s) A_4 - A_2\left(1+\varepsilon_{u,k}\right)$ and $\Pi \equiv M\varepsilon_{u,c} - A_4 + N - \varepsilon_{u,c}A_2$. Notice that $\{A_2, A_4, A_5, M, N, \Gamma, \omega\}$ are blocks well defined in the appendix (A.4).

In the local dynamic analysis, we will restrict to the case in which the elasticity of capital utilization with respect to consumption is sufficiently high, the elasticity of capital utilization rate with respect to capital is satisfactorily low, and the elasticity of capital-labor substitution is adequately high.

Assumption (3): *$N > 0$, $\Gamma > 0$ and $\omega < 0$.*[9]

We know that the trace T and the determinant D of the Jacobian matrix are the sum and the product of the eigenvalues, respectively. Following Grandmont, Pintus and de Vilder (1998), the stability properties of the system, that is, the location of the eigenvalues with respect to the unit circle, will be characterized in the (T, D)-plane (see the figures below). This diagram is convenient for studying not only the local stability (or indeterminateness), but also local bifurcations, i.e., changing of stability of the steady state resulting from variations of parameters (hence, T, D) in the system. More precisely, we evaluate the characteristic polynomial $P(\lambda) \equiv \lambda^2 - T\lambda + D = 0$ at $-1, 0, 1$. On the line (AB), one eigenvalue is equal to -1, i.e., $P(-1) = 1 + T + D = 0$. On the line (AC), one eigenvalue is equal to 1, i.e., $P(1) = 1 - T + D = 0$. On the segment $[BC]$, the two eigenvalues are complex conjugates with a unit modulus, i.e., $D = 1$, and $|T| < 2$. The steady state is a sink when $D < 1$ and $|T| < 1 + D$. It is a saddle point when $|D + 1| < |T|$. It is a source otherwise. Therefore, the steady state is locally indeterminate if and only if (T, D) is inside the triangle (ABC), where there are infinitely many deterministic intertemporal paths that converge to the steady state, otherwise, it is locally determinate which means that there is a unique intertemporal path that converge towards the steady state. A transcritical bifurcation generically occurs when (T, D) crosses the line (AC),

[9] These blocks are available in the appendix.

a flip bifurcation generically occurs when (T, D) crosses the line (AB), whereas a Hopf bifurcation generically emerges when (T, D) crosses the segment $[BC]$.

The determinant D and the trace T of matrix J are given by:

$$\begin{aligned} D &= \varphi \frac{\Gamma (N - \varepsilon_{u,c} A_2) - (1 + \varepsilon_{u,k}) \omega A_2}{\varphi (N - \varepsilon_{u,c} A_2 - A_4 + \varepsilon_{u,c} A_1) - (1 - s) A_5 \varepsilon_{u,c}} & (18) \\ T &= 1 + D + \Lambda & (19) \end{aligned}$$

where

$$\Lambda = \frac{\omega}{\sigma} \frac{\varphi (A_4 (s + \alpha) + \alpha A_5 + A_1 \varepsilon_{u,k}) - (A_5 (1 - s) + \varphi A_2) (1 + \varepsilon_{u,k})}{\varphi (N - \varepsilon_{u,c} A_2 - A_4 + \varepsilon_{u,c} A_1) - (1 - s) A_5 \varepsilon_{u,c}} \quad (20)$$

represents the deviation of the point (T, D) from the line $T = 1 + D$. In this model, there are four elasticities : the elasticity of intertemporal substitution in consumption $\sigma \geq 0$, the elasticity of capital-labor substitution $\varphi \geq 0$, the elasticity of capital utilization rate with respect to consumption $\varepsilon_{u,c} \leq 0$ and the elasticity of capital utilization rate with respect to capital $\varepsilon_{u,k} \geq 0$. We choose $\varphi \geq 0$ as a bifurcation parameter. So, in the first case, we assume that $\varepsilon_{u,c}$ and $\varepsilon_{u,k}$ are fix, while in the second case $\varepsilon_{u,c}$ and σ are fix, and finally in the last case, we fix σ and $\varepsilon_{u,k}$. We characterize the stability properties of the steady state and the occurrence of the bifurcations in the $(T, D)-$ plane. When φ increases from 0 to $+\infty$, $(T(\varphi), D(\varphi))$ describes a half-line $\Delta \equiv \{(T, D) : \varphi \geq 0\}$, with origin (T_0, D_0) and slope S which is independent of φ.

$$\begin{aligned} D_0 &= 0 & (21) \\ T_0 &= 1 + D_0 + \frac{1}{\sigma} \frac{\omega (1 + \varepsilon_{u,k})}{\varepsilon_{u,c}} & (22) \end{aligned}$$

$$S = \left[1 + \frac{\omega}{\sigma \varepsilon_{u,c}} \frac{\varepsilon_{u,c} B + (1 + \varepsilon_{u,k}) (A_4 - N)}{\Gamma (N - \varepsilon_{u,c} A_2) - (1 + \varepsilon_{u,k}) \omega A_2} \right]^{-1} \quad (23)$$

where $B \equiv A_4 (s + \alpha) + \alpha A_5 - A_1 > 0$. Equation (21) and (22) give us that the origin (initial point) lies on the T-axis ($D = 0$), on the right-hand side of the line (AC). (See the graphics below).

Case (1): $\varepsilon_{u,c}$ and $\varepsilon_{u,k}$ are fix, but σ varies from 0 to $+\infty$.

In this paper, we have one predetermined variable, which means that we obtain indeterminacy if and only if the two eigenvalues belong to $(-1, 1)$. The movement of the origin depends strictly on the elasticity of intertemporal substitution of consumption parameter σ, and since $\omega < 0$, then the origin decreases with σ and it is still lies on the line $D = 0$. i.e.,

$$\frac{\partial T_0}{\partial \sigma} = -\frac{\omega (1 + \varepsilon_{u,k})}{\varepsilon_{u,c}} \left(\frac{1}{\sigma^2} \right) < 0$$

In this case we will explain the first case in the proposition (2) where we take in account that $\Upsilon\left(1+H\right)+\Psi<0$ and $\sigma^{2}\left(F_{2}F_{3}+F_{1}\Gamma\varepsilon_{u,c}\right)+\left(1+\varepsilon_{u,k}\right)\omega\left(2F_{3}\sigma-\Gamma\varepsilon_{u,c}\right)>0$. This analysis is related to figure (2.a). Using the previous inequalities together with assumption (3), we can confirm that the slope (23) of the half-line Δ is positive with $S'\left(\sigma\right)>0$. At the initial point, the slope $S\left(0\right)=0$. Then, when σ increases, the initial point $\left(T_{0},D_{0}\right)$ goes to the left and simultaneously the half-line Δ makes counter-clockwise rotation until arriving $S\left(\infty\right)$ which is negative.

Definition (1): We define the critical values σ^{C}, σ^{B} and σ^{A}such that the half-line Δ passes through points C, B and A respectively. Further, we consider σ' such that $S\left(\sigma\right)=1$, σ'' such that $S\left(\sigma\right)=-1$, and $\hat{\sigma}$ for $T=1$. (See the figures below)

Initially, the slope of the half-line Δ is positive and increasing $S'\left(\sigma\right)>0$. The slope continues increasing until $S\left(\infty\right)$ which is negative. According to (21) and (22), the starting point $\left(T_{0},D_{0}\right)$ lies on the line $D=0$. This initial point moves negatively with σ, $\left(\partial T_{0}/\partial\sigma<0\right)$ until arriving finally to the point $T=1$. So that, the indeterminacy appears in a case where the half-line Δ passes through the triangle $\left(ABC\right)$, where there are infinite trajectories that satisfy the transversality condition and converge toward the unique steady state. This line crosses the segment $\left[BC\right]$ for
$\varphi=-\left(1-s\right)A_{5}\varepsilon_{u,c}/\left[\left(\Gamma-1\right)\left(N-\varepsilon_{u,c}A_{2}\right)-\left(1+\varepsilon_{u,k}\right)\omega A_{2}+A_{4}-\varepsilon_{u,c}A_{1}\right]\equiv\varphi^{H}$, it crosses the line $\left(AB\right)$ for $\varphi=W_{1}/\left(2W_{2}+W_{3}\right)\equiv\varphi^{F}$ and passes through the line $\left(AC\right)$ for $\varphi=A_{5}\left(1-s\right)\left(1+\varepsilon_{u,k}\right)/\left[B+\left(1+\varepsilon_{u,k}\right)\left(A_{1}-A_{2}\right)\right]\equiv\varphi^{T}$.

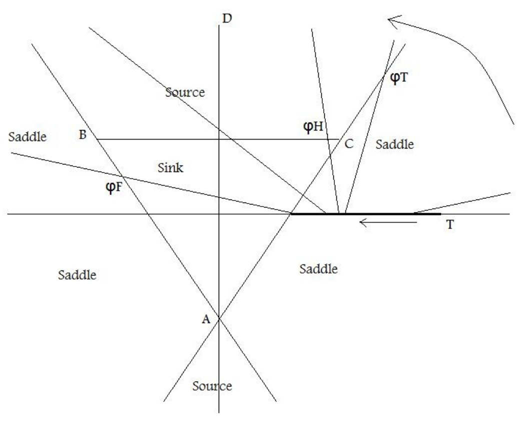


Figure (2.a)

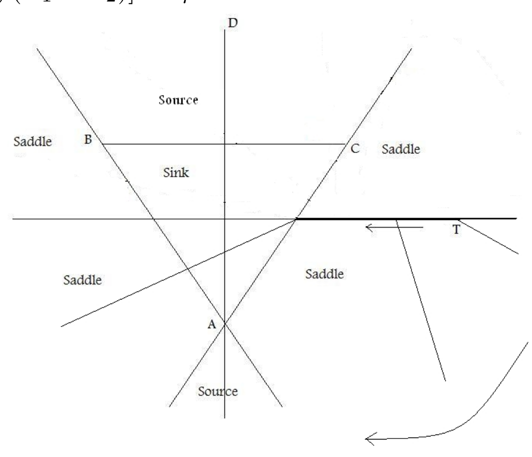


Figure (2.b)

If σ is very small and less than σ' then the steady state is unstable (saddle) for all values of φ. Once σ increases more than σ' but less than σ^{C}, then the steady state still unstable for small values of φ, and for high value of φ it becomes source. Moreover, Once $\sigma\in\left(\sigma^{C},\sigma''\right)$ the steady state is saddle (determinate) for small values of φ, but for modest values of φ, the steady state becomes stable (sink) and the equilibrium is indeterminate, for higher values of φ, the steady state returns to be unstable (source). In this case, we get transcritical as well as Hopf bifurcation for particular values of φ. Indeed, when σ lies between σ'' and σ^B, then we obtain flip, Hopf and transcritical bifurcation. If σ is higher than σ^B and lower than $\hat{\sigma}$, then there are both transcritical and flip bifurcation. Finally, once $\sigma \in (\hat{\sigma}, \infty)$, the economy is unstable for $\varphi < \varphi^F$. As we remark in figure (2.a) that the economic system might have more chance to get indeterminacy once the elasticity of intertemporal substitution in consumption σ gets high values.

The results are summarized in the following proposition.

Proposition 2 *If assumption (3) is satisfied, then the following generically holds.*

[1] if $\Upsilon(1+H)+\Psi < 0$ *and* $\sigma^2(F_2F_3 + F_1\Gamma\varepsilon_{u,c}) + (1+\varepsilon_{u,k})\,\omega\,(2F_3\sigma - \Gamma\varepsilon_{u,c}) > 0$.

(1.1) $0 < \sigma < \sigma'$ *, the steady state is saddle for all* φ*.*

(1.2) $\sigma' < \sigma < \sigma^C$*, the steady state is saddle for* $0 < \varphi < \varphi^T$*, undergoes a transcritical bifurcation for* $\varphi = \varphi^T$ *, is a source for* $\varphi > \varphi^T$*.*

(1.3) $\sigma^C < \sigma < \sigma''$*, the steady state is saddle for* $0 < \varphi < \varphi^T$*, undergoes a transcritical bifurcation for* $\varphi = \varphi^T$*, is a sink for* $\varphi^T < \varphi < \varphi^H$*, undergoes a Hopf bifurcation for* $\varphi = \varphi^H$*, is a source for* $\varphi > \varphi^H$*.*

(1.4) $\sigma'' < \sigma < \sigma^B$*, the steady state is saddle for* $0 < \varphi < \varphi^T$*, undergoes a transcritical bifurcation for* $\varphi = \varphi^T$*, is a sink for* $\varphi^T < \varphi < \varphi^H$*, undergoes a Hopf bifurcation for* $\varphi = \varphi^H$*, is a source for* $\varphi^H < \varphi < \varphi^F$*, undergoes a flip bifurcation for* $\varphi = \varphi^F$*, is a saddle for* $\varphi > \varphi^F$*.*

(1.5) $\sigma^B < \sigma < \hat{\sigma}$*, the steady state is saddle for* $0 < \varphi < \varphi^T$*, undergoes a transcritical bifurcation for* $\varphi = \varphi^T$*, is a sink for* $\varphi^T < \varphi < \varphi^F$*, undergoes a flip bifurcation for* $\varphi = \varphi^F$*, is a saddle for* $\varphi > \varphi^F$*.*

(1.6) $\hat{\sigma} < \sigma < \infty$*, the steady state is sink for* $0 < \varphi < \varphi^F$*, undergoes a flip bifurcation for* $\varphi = \varphi^F$*, is a saddle for* $\varphi > \varphi^F$*.*

[2] if $\Upsilon(1+H)+\Psi > 0$ *and* $\sigma^2(F_2F_3 + F_1\Gamma\varepsilon_{u,c}) + (1+\varepsilon_{u,k})\,\omega\,(2F_3\sigma - \Gamma\varepsilon_{u,c}) < 0$.[10]

(2.1) $0 < \sigma < \sigma''$*, then the steady state is saddle for all* φ*.*

(2.2) $\sigma'' < \sigma < \sigma^A$*, the steady state is saddle for* $0 < \varphi < \varphi^F$*, undergoes a flip bifurcation for* $\varphi = \varphi^F$*, is a source for* $\varphi > \varphi^F$*.*

(2.3) $\sigma^A < \sigma < \hat{\sigma}$*, the steady state is saddle for* $0 < \varphi < \varphi^T$*, undergoes a transcritical bifurcation for* $\varphi = \varphi^T$*, is sink for* $\varphi^T < \varphi < \varphi^F$*, undergoes a flip bifurcation for* $\varphi = \varphi^F$*, is a saddle for* $\varphi > \varphi^F$*.*

(2.4) $\hat{\sigma} < \sigma < \infty$*, the steady state is sink for* $0 < \varphi < \varphi^F$*, undergoes a flip bifurcation for* $\varphi = \varphi^F$*, is a saddle for* $\varphi > \varphi^F$*.*

So, supposing that the utility function is general allows us to identify the role of the elasticity of intertemporal substitution in consumption σ in the local dynamics. In particular, as we show in proposition (2) that for different values

[10] This case is related to figure (2.b).

of σ, the economic system might change its stability and hence the bifurcation emerges in corresponding to different values of the elasticity of capital-labor substitution φ.

In order to complete the characterization of the effects of σ on the local dynamics, figure (4.a) denotes that the elasticity of factor substitution has an important effect on endogenous fluctuation emergence. In particular, as we show above that the possibility of getting multiple equilibria increases when σ exceeds the critical value σ^C and start to decline once σ gets higher than σ^B. Likewise, for small values of σ, there is no indeterminacy and the economic system is stable.

Case (2): $\varepsilon_{u,c}$ and σ are fix, while $\varepsilon_{u,k}$ varies from 0 to $+\infty$.

Here, we study the role of the elasticity of capital utilization with respect to capital $\varepsilon_{u,k}$ in stabilizing the economy locally. Additionally, we evaluate how an increasing in $\varepsilon_{u,k}$ affect the stability of the steady state and the appearance of the local bifurcation.

In this case, we will study the first case in the proposition (3) where we suppose that $E_3E_2 > E_1E_4$ and $-E_2 < E_1 < -(E_3 + E_2 + E_4)$ with $E_4 > 0$.[11] Analyzing the second case is left for the reader.

Definition (2): We define the critical values $\varepsilon^A_{u,k}$, $\varepsilon^B_{u,k}$ and $\varepsilon^C_{u,k}$ such that the half-line Δ passes through the point A, B and C respectively. Further, we consider that $\varepsilon'_{u,k}$ such that $S(\varepsilon_{u,k}) = 1$ and $\varepsilon''_{u,k}$ such that $S(\varepsilon_{u,k}) = -1$. See Figures (3)

Since σ and $\varepsilon_{u,c}$ are fix, then the slops S depends only on $\varepsilon_{u,k} \geq 0$. Indeed, the initial point still lies on the line $(D = 0)$ but it is increasing as we rise $\varepsilon_{u,k}$ since[12]

$$\frac{\partial T_0}{\partial \varepsilon_{u,k}} = \frac{1}{\sigma}\frac{\omega}{\varepsilon_{u,c}} > 0$$

So that, the indeterminacy and the bifurcation appearance in this case would have completely different analysis from the previous one. It is clear that when $\varepsilon_{u,k}$ rises, the half-line Δ makes counter-clockwise rotation. Additionally, assuming that $-E_2 < E_1 < -(E_3 + E_2 + E_4)$ gives us that the slope gets higher with $\varepsilon_{u,k}$ until $S(\infty)$ which is positive and less than unity.

According to (21) and (22), the half-line Δ lies initially above the line $(D = 0)$ then the steady state is saddle (locally determinate) for which the elasticity of factor substitution φ is lower than φ^T. It crosses the line (AC) for $\varphi = \varphi^T$ and then the transcritical bifurcation occurs, and crosses the segment $[BC]$ for $\varphi = \varphi^H$ where the Hopf bifurcation emerges. When $\varepsilon_{u,k}$ is less than $\varepsilon''_{u,k}$, the steady state is unstable for small value of φ, while for a modest value of φ it becomes stable (sink), and for high value of φ the it returns to be unstable

[11] For more details, see the appendix.
[12] See figure (3.a)

(source). Once $\varepsilon_{u,k}$ exceeds $\varepsilon''_{u,k}$, for small value of φ, the steady state is saddle, for higher φ the steady state becomes sink, once φ is greater than φ^H it becomes source and finally, for higher φ we have a unique saddle path converges toward the steady state (locally determinate). In the case where $\varepsilon_{u,k} > \varepsilon^B_{u,k}$ the steady state initially is saddle for small value of φ, but for modest value of φ is less than φ^F we have a stable steady state, and for very high values of φ, we get a unique trajectory converges to the steady state. When the elasticity of capital utilization with respect to capital gets higher than $\varepsilon^A_{u,k}$, then the endogenous fluctuation disappears.

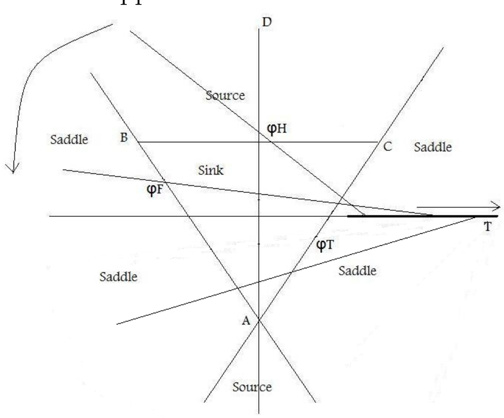


Figure (3.a)

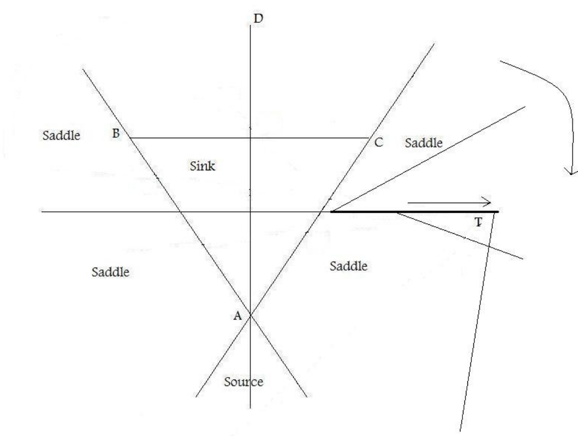


Figure (3.b)

We summarize the results in the following proposition.

Proposition 3 *If assumption (3) is satisfied, then the following generically holds.*

[1] if $E_3E_2 > E_1E_4$ *and* $-E_2 < E_1 < -(E_3 + E_2 + E_4)$ *with* $E_4 > 0$.

(1.1) $0 < \varepsilon_{uk} < \varepsilon''_{u,k}$, *the steady state is saddle for* $0 < \varphi < \varphi^T$, *undergoes a transcritical bifurcation for* $\varphi = \varphi^T$, *is a sink for* $\varphi^T < \varphi < \varphi^H$, *undergoes a Hopf bifurcation for* $\varphi = \varphi^H$, *is a source for* $\varphi > \varphi^H$.

(1.2) $\varepsilon''_{uk} < \varepsilon_{uk} < \varepsilon^B_{uk}$, *the steady state is saddle for* $0 < \varphi < \varphi^T$, *undergoes a transcritical bifurcation for* $\varphi = \varphi^T$, *is a sink for* $\varphi^T < \varphi < \varphi^H$, *undergoes a Hopf bifurcation for* $\varphi = \varphi^H$, *is a source for* $\varphi^H < \varphi < \varphi^F$, *undergoes a flip bifurcation for* $\varphi = \varphi^F$, *is a saddle for* $\varphi > \varphi^F$.

(1.3) $\varepsilon^B_{uk} < \varepsilon_{uk} < \varepsilon^A_{uk}$, *the steady state is saddle for* $0 < \varphi < \varphi^T$, *undergoes a transcritical bifurcation for* $\varphi = \varphi^T$, *is a sink for* $\varphi^T < \varphi < \varphi^F$, *undergoes a flip bifurcation for* $\varphi = \varphi^F$, *is a saddle for* $\varphi > \varphi^F$.

(1.4) $\varepsilon^A_{uk} < \varepsilon_{uk} < \infty$, *the steady state is saddle for* $0 < \varphi < \varphi^F$, *undergoes a flip bifurcation for* $\varphi = \varphi^F$, *is source for* $\varphi^F < \varphi < \varphi^T$, *undergoes a transcritical bifurcation for* $\varphi = \varphi^T$, *is a saddle for* $\varphi > \varphi^T$.

[2] If $E_3E_2 < E_1E_4$, *and* $E_1 > \max\{-E_2, -(E_2 + E_3 + E_4)\}$ *with* $E_4 <$

0.[13][14]

(2.1) $0 < \varepsilon_{uk} < \varepsilon^{C}_{u,k}$, *the steady state is saddle for* $0 < \varphi < \varphi^{T}$, *undergoes a transcritical bifurcation for* $\varphi = \varphi^{T}$, *is a sink for* $\varphi^{T} < \varphi < \varphi^{H}$, *undergoes a Hopf bifurcation for* $\varphi = \varphi^{H}$, *is a source for* $\varphi > \varphi^{H}$.

(2.2) $\varepsilon^{C}_{uk} < \varepsilon_{uk} < \varepsilon'_{uk}$, *the steady state is saddle for* $0 < \varphi < \varphi^{T}$, *undergoes a transcritical bifurcation for* $\varphi = \varphi^{T}$, *is a source for* $\varphi > \varphi^{T}$.

(2.3) $\varepsilon'_{u,k} < \varepsilon_{uk} < \varepsilon''_{uk}$, *the steady state is saddle for* φ.

(2.4) $\varepsilon''_{uk} < \varepsilon_{uk} < \infty$, *the steady state is saddle for* $0 < \varphi < \varphi^{F}$, *undergoes a flip bifurcation for* $\varphi = \varphi^{F}$, *is a source for* $\varphi > \varphi^{F}$.

In the above proposition, for a small value of the elasticity of capacity utilization with respect to capital ε_{uk}, the economy system is unstable in which there are infinite trajectories converging to a unique steady state. The unique equilibrium path is consistent with the anticipations of the agents. For higher value of ε_{uk} the indeterminacy disappears and we get a unique trajectory passes toward the steady state. However, the stability of the economy depends strictly on the value of ε_{uk}; for small values of ε_{uk}, the economy is stable for small value of φ, for higher value of φ, the economic system becomes unstable, and so the bifurcation emerges. For high values of ε_{uk}, the economy would be stable (determinate).

We can observe from figure (4.a) that the endogenous fluctuation is more likely to appear as the elasticity of capital utilization rate with respect to capital is less than $\varepsilon^{A}_{u,k}$. After that, the indeterminacy disappears when $\varepsilon_{u,k}$ gets higher than $\varepsilon^{A}_{u,k}$, thus a unique stable path occurs and so, the multiple equilibria vanishes.

Case (3): $\varepsilon_{u,k}$ and σ are fix, while $\varepsilon_{u,c}$ varies from $-\infty$ to 0.

In this case, we study the effect of the elasticity of capital utilization with respect to the consumption $\varepsilon_{u,c}$ on the stability of the economy locally, and on the appearance of bifurcations (changing the economic system from a stable system to an unstable one).

Definition (3): We define $\varepsilon'_{u,c}$ and $\varepsilon''_{u,c}$ such that the origin lies on the point $T = 1$ and $T = -1$. Moreover, we define $\varepsilon^{B}_{u,c}$, such that the half-line Δ passes through point B.

All the cases are treated in the same way, and every case studies separately the effect of a specific elasticity on the stability of the whole economy and on the appearance of the indeterminacy. Here, we will explore the first case of proposition (4) where we take in consideration that $G_4 - \frac{1}{\sigma} B A_3 < 0$, and $G_6 \left(G_3 + 2G_4 \varepsilon_{u,c} \right) - \left(G_3 \varepsilon_{u,c} + G_4 \varepsilon^{2}_{u,c} \right) G_7 < 0$, which is consistent with figure

[13] This case is related to figure (3.b).

[14] Notice that, this case is verified even if $E_1 < \min \{ -E_2, -(E_2 + E_3 + E_4) \}$.

(4.a). The second case which is related to figure (4.b) is left for the reader. We have

$$\frac{\partial T_0}{\partial \varepsilon_{u,c}} = -\frac{1}{\sigma}\frac{Z\left(1+\varepsilon_{u,k}\right)}{\varepsilon_{u,c}^2} < 0$$

This equation figures out that the origin (T_0, D_0) and the slope S decrease as $\varepsilon_{u,c}$ rises. More precisely, the slope is decreasing from $S(-\infty)$ that is positive to $S(0)$ which equals to Zero. See figures (4.a)

According to (21) and (22), the half-line Δ lies above of line $D = 0$. The steady state is always saddle for a small value of $\varepsilon_{u,c}$. Once the elasticity of capital utilization with respect to consumption rises, then the origin crosses the line $T = D + 1$, and so the indeterminacy appears for $\varphi < \varphi^T$, and for higher values of elasticity of factor substitution, the steady state changes it stability and becomes unstable (saddle), and so there exists a transcritical bifurcation for the critical value $\varphi = \varphi^T$. Notice that since the slope decreases with $\varepsilon_{u,c}$, then the half-line Δ does not cross the segment $[BC]$, and so the Hopf bifurcation does not appear. In the case where $\varepsilon_{u,c}$ increases more that $\varepsilon''_{u,c}$, then the steady state is saddle for a small value of φ, for modest values of φ, the steady state becomes stable (sink) and for $\varphi > \varphi^T$, we have a unique path converges toward the steady state (locally determinate).

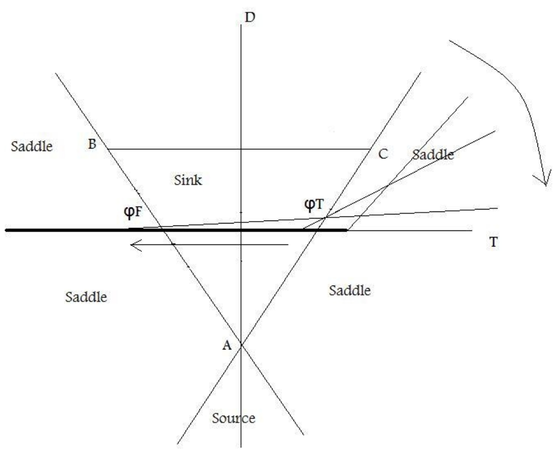


Figure (4.a)

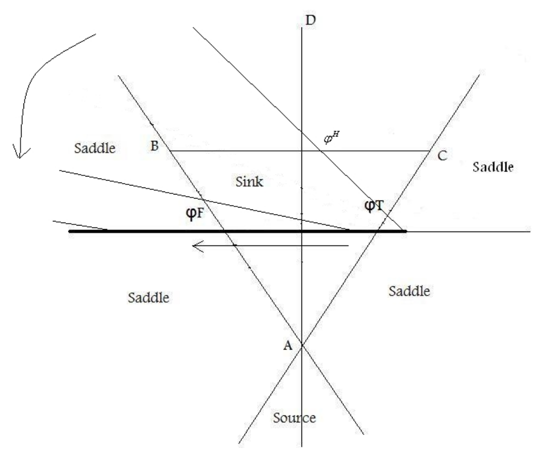


Figure (4.b)

We summarize the results in the following proposition.

Proposition 4 *If assumptions (3) is satisfied, the following generically holds.*

[1] If $G_4 - \frac{1}{\sigma}BA_3 < 0$, *and* $G_6\left(G_3 + 2G_4\varepsilon_{u,c}\right) - \left(G_3\varepsilon_{u,c} + G_4\varepsilon_{u,c}^2\right)G_7 < 0$.

(1.1) $-\infty < \varepsilon_{uc} < \varepsilon'_{uc}$, *the steady state is saddle for all* φ.

(1.2) $\varepsilon'_{uc} < \varepsilon_{uc} < \varepsilon''_{uc}$, *the steady state is sink for* $0 < \varphi < \varphi^T$, *undergoes a transcritical bifurcation for* $\varphi = \varphi^T$, *is a saddle for* $\varphi > \varphi^T$.

(1.3) $\varepsilon''_{uc} < \varepsilon_{uc} < 0$, the steady state is saddle for $0 < \varphi < \varphi^F$, undergoes a flip bifurcation for $\varphi = \varphi^F$, is a sink for $\varphi^F < \varphi < \varphi^T$, undergoes a transcritical bifurcation for $\varphi = \varphi^T$, is a saddle for $\varphi > \varphi^T$.

[2] If $G_4 - \frac{1}{\sigma} BA_3 > 0$, and $\varepsilon^2_{u,c}\left(G_4 G_5 + \frac{1}{\sigma} G_3 BA_3\right) + \left(2G_4 \varepsilon_{u,c} + G_3\right) \frac{1}{\sigma} ZG_2 > 0$.[15]

(2.1) $-\infty < \varepsilon_{uc} < \varepsilon'_{uc}$, the steady state is saddle for all $0 < \varphi < \varphi^T$. undergoes a transcritical bifurcation for $\varphi = \varphi^T$, is a sink for $\varphi^T < \varphi < \varphi^H$, undergoes a Hopf bifurcation for $\varphi = \varphi^H$, is a source for $\varphi^H < \varphi < \varphi^F$, undergoes a flip bifurcation for $\varphi = \varphi^F$, is a saddle for $\varphi > \varphi^F$.

(2.2) $\varepsilon'_{uc} < \varepsilon_{uc} < \varepsilon^B_{uc}$, the steady state is sink for $0 < \varphi < \varphi^H$, undergoes a Hopf bifurcation for $\varphi = \varphi^H$, is a source for $\varphi^H < \varphi < \varphi^F$, undergoes a flip bifurcation for $\varphi = \varphi^F$, is a saddle for $\varphi > \varphi^F$.

(2.3) $\varepsilon^B_{uc} < \varepsilon_{uc} < \varepsilon''_{uc}$, the steady state is sink for $0 < \varphi < \varphi^F$, undergoes a flip bifurcation for $\varphi = \varphi^F$, is a saddle for $\varphi > \varphi^F$.

(2.4) $\varepsilon''_{uc} < \varepsilon_{uc} < 0$, the steady state is saddle for all φ.

From the above proposition, we show that when the ε_{uk}, and σ are fix, while ε_{uc} increases, then the Hopf bifurcation does not occur. Additionally, the economic system is stable for a small value of ε_{uc}, where the saddle path emerges. When ε_{uc} exceeds the critical value ε'_{uc}, then there exists a multiple equilibrium paths that converge to a unique steady state, the equilibrium trajectory is chosen according to the anticipation of the agents.

5 Discussion

In this section, we provide an interpretation for our main results and then we compare it with the related literature. Contrary to Benhabib and Farmer (1996), Guo and Harrison (2001), Wen (1998) and Nishimura and Venditti (2001), we consider a general constant return-to-scale production function, as well as the utility function is general too. This is because the elasticity of capital-labor substitution plays an important role in determining the local stability of the economy (Garnier Nishimura and Venditti (2007), Pintus (2006), Grandmont Pintus and de Vilder (1998)). Additionally, we show also that the elasticity of intertemporal substitution in consumption plays a role in determining the indeterminacy.

At the same time, similar to Wen, Guo and Harrison, we show that the emergence of the endogenous fluctuation is affected by the capital utilization rate. We confirm that for sufficiently high value of elasticity of capital utilization rate with respect to consumption (capital), the steady state is stable and the economy is unstable where there are infinite trajectories passe toward the steady state. For a given level of income, once the demand of consumption goods decreases, then the demand of capital will get higher, this leads to rise the capacity utilization of this capital, and so the output will increase. Additionally, we show that for certain values of elasticity of capacity utilization with respect

[15] This case is related to figure (4.b).

to consumption (capital), we get flip, Hopf as well as transcritical bifurcation according with different values of elasticity of capital-labor substitution.

The intuition is that once the capital utilization increases, the production will rise due to the more intensively used of available capital, and it rises the elasticity of output with respect to capital. So, if there exists an external increasing returns to scale, this phenomenon will be satisfactory to allow the capital elasticity of output above one. Besides, this justification could be applied also in the case where the elasticity of capital utilization with respect to consumption is high. This implies that the rational-expectations equilibrium is indeterminate.

We show also that the endogenous fluctuation emerges for low elasticity of capital-labor substitution. This occurs because when the elasticity of capital-labor substitution is low, this means that for higher amount of capital, the labor will decline slightly, making total production higher. Since there exists external effects consistent with the specific and aggregate externalities, this induces that the marginal product of labor is increasing in labors. This can not be happened in the RBC models without externality, where the marginal product of labor is always decreasing.

Notice that the case of Benhabib-Farmer (1996) is recovered whenever we put the elasticity of capacity utilization with respect to consumption (and capital) equals zero, together with setting the elasticity of factor substitution equals unity (Cobb-Douglas case).

6 Conclusion

We have presented a two-sector model; consumption and investment sectors with variable capital utilization rate. We suppose that there are two types of externalities, sectoral and aggregate externalities together with inelastic labor supply. The production function is general and constant return-to-scale, the utility function is general too. We supposed this since the elasticity of capital-labor substitution and the elasticity of intertemporal substitution in consumption play important roles in determining the endogenous fluctuation. Applying geometrical method allows us to determine the conditions under which the local indeterminacy and changing in the stability (bifurcation) occur.

We assume as well that the depreciation rate is endogenously determined and depends on the rate of capital utilization. Our purpose was to study the effects of capital utilization rate on the endogenous fluctuation in two-sector model and to identify the conditions under which the indeterminacy and bifurcation appear. The results obtained here can reply to our questions mentioned in the introduction, and are convenient with what we were waiting.

We have analyzed the role of capital utilization rate on the stability properties of the steady state and local bifurcation. We have identified that for some values of elasticity of capital utilization rate, the steady state is sink (stable), and for other values the steady state becomes unstable. For particular values of elasticity of capital-labor substitution, we get flip, Hopf as well as transcritical bifurcation. This bifurcation appears in a case where the steady state changes its stability; from stable to unstable steady state and vice versa.

7 Appendix:

(A.1) In this section, we derive the FOCs for the firms in both sectors:

- The consumption sector:

Since q_t is the price of investment goods in terms of consumption goods, then the price of consumption goods is normalized to unity, r_t is the real interest rate, w_t is the real wage. So that, the profit function for this sector can be characterized as:

$$r_t = A\bar{A}_t f'\left(u_t \frac{\mu_{k,t}}{\mu_{l,t}} k_t\right) \tag{24}$$

$$w_t = A\bar{A}_t f\left(u_t \frac{\mu_{k,t} k_t}{\mu_{l,t}}\right) - A\bar{A}_t f'\left(u_t \frac{\mu_{k,t} k_t}{\mu_{l,t}}\right) u_t \frac{\mu_{k,t} k_t}{\mu_{l,t}} \tag{25}$$

- The investment sector:

$$r_t = q_t B\bar{B}_t f'\left(u_t \frac{(1-\mu_{k,t})}{(1-\mu_{l,t})} k_t\right) \tag{26}$$

$$w_t = q_t B\bar{B}_t f\left(\frac{(1-\mu_{k,t})}{(1-\mu_{l,t})} u_t k_t\right) - q_t B\bar{B}_t f'\left(\frac{(1-\mu_{k,t})}{(1-\mu_{l,t})} u_t k_t\right) \frac{(1-\mu_{k,t})}{(1-\mu_{l,t})} u_t k_t \tag{27}$$

(A.2) In this section, we want to prove that $\mu_{k,t} = \mu_{l,t}$:

Using the above FOCs (24) and (25) yields:

$$\frac{w_t}{r_t} = \frac{f\left(u_t \frac{\mu_{k,t} k_t}{\mu_{l,t}}\right)}{u_t f'\left(u_t \frac{\mu_{k,t}}{\mu_{l,t}} k_t\right)} - k_t \frac{\mu_{k,t}}{\mu_{l,t}} \tag{28}$$

and dividing (26) by (27) gives:

$$\frac{w_t}{r_t} = \frac{f\left(u_t \frac{(1-\mu_{k,t})}{(1-\mu_{l,t})} k_t\right)}{u_t f'\left(u_t \frac{(1-\mu_{k,t})}{(1-\mu_{l,t})} k_t\right)} - k_t \frac{(1-\mu_{k,t})}{(1-\mu_{l,t})} \tag{29}$$

From (28) and (29), we obtain

$$\frac{f\left(u_t \frac{(1-\mu_{k,t})}{(1-\mu_{l,t})} k_t\right)}{u_t f'\left(u_t \frac{(1-\mu_{k,t})}{(1-\mu_{l,t})} k_t\right)} - \frac{(1-\mu_{k,t})}{(1-\mu_{l,t})} k_t = \frac{f\left(u_t \frac{\mu_{k,t}}{\mu_{l,t}} k_t\right)}{u_t f'\left(u_t \frac{\mu_{k,t}}{\mu_{l,t}} k_t\right)} - \frac{\mu_{k,t}}{\mu_{l,t}} k_t \tag{30}$$

So, the left and the right hand side of equation (30) are equal if and only if $\mu_{k,t} = \mu_{l,t}$.

(A.3) In this section, we derive the Euler equation:

If we take equation (26), together with equations (13) and (4) then, simply we attain:

$$R(c_t, k_t) = Qq(k_t, c_t)(1 - \mu(c_t, k_t))^{\theta} (u(c_t, k_t).k_t)^{\alpha} f'(u(c_t, k_t).k_t)$$

Substituting this value into the Euler equation previously obtained from the FOCs for the households, we finally get:

$$q(k_t, c_t) \frac{X'(c_t)}{X'(c_{t+1})} = \beta \left[q(k_{t+1}, c_{t+1}) \left(1 - \frac{1}{\gamma} [u(c_{t+1}, k_{t+1})]^{\gamma}\right) + u(c_{t+1}, k_{t+1}) R(c_{t+1}, k_{t+1}) \right]$$

(A.4) The blocks:

Since the model we study is so big, so to simplify more the exposition, we use suitable blocks that are well verified.

Main blocks:

$$\begin{aligned}
A_1 &\equiv \left(\frac{\theta}{1+\theta}\frac{\mu}{1-\mu}(\alpha+s)+\alpha+1\right)\left(1-\beta\left(1-\frac{1}{\gamma}u^{\gamma}\right)\right)-\beta u^{\gamma} \\
A_2 &\equiv \frac{\theta}{1+\theta}\frac{1}{1-\mu}(\alpha+s) \\
A_3 &\equiv \frac{1}{\gamma}u^{\gamma}\left((s+\alpha)\frac{1}{1-\mu}-\gamma\right) \\
A_4 &\equiv \frac{\theta}{1+\theta}\frac{\mu}{1-\mu}\left(1-\beta\left(1-\frac{1}{\gamma}u^{\gamma}\right)\right) \\
A_5 &\equiv 1-\beta\left(1-\frac{1}{\gamma}u^{\gamma}\right)
\end{aligned}$$

and

$$\begin{aligned}
M &\equiv A_1 - \frac{1-s}{\varphi}A_5 \\
Z &\equiv \frac{1}{\gamma}u^{\gamma}\frac{\mu}{1-\mu} \\
H &\equiv u^{\gamma}\left(1-\frac{1}{\gamma}\right)
\end{aligned}$$

Furthermore, in order to simplify more and more the blocks, we denote

Minor blocks:

$$\begin{aligned}
N &\equiv \frac{1}{\alpha+s}A_2 - \frac{1}{\sigma} \\
\omega &\equiv Z - A_3\varepsilon_{u,c} \\
\Gamma &\equiv 1 + H + (1+\varepsilon_{u,k})A_3
\end{aligned}$$

- Case (1): $\varepsilon_{u,c}$ and $\varepsilon_{u,k}$ are fix, while σ moves from 0 to ∞.
We supposed for simplifying the calculations that:

$$
\begin{aligned}
F_1 &\equiv \varepsilon_{u,c}\left(\Upsilon\left(1+H\right)+\Psi\right)\\
F_2 &\equiv \omega P-\Gamma\varepsilon_{u,c}\\
F_3 &\equiv \Gamma\Upsilon\varepsilon_{u,c}-V
\end{aligned}
$$

with

$$
\begin{aligned}
\Upsilon &\equiv \left(\frac{1}{\alpha+s}-\varepsilon_{u,c}\right)A_2\\
\Psi &\equiv A_2\left(1+\varepsilon_{u,k}\right)\left(\frac{1}{\alpha+s}A_3-Z\right)
\end{aligned}
$$

For obtaining the bifurcation values, we simply assume that:

$$
\begin{aligned}
W_1 &\equiv (1-s)\left(\varepsilon_{u,c}+\frac{1}{\sigma}\omega\left(1+\varepsilon_{u,k}\right)\right)A_5\\
W_2 &\equiv \left(N-\varepsilon_{u,c}A_2\right)\left(1+\Gamma\right)-A_4+\varepsilon_{u,c}A_1-\left(1+\varepsilon_{u,k}\right)\omega A_2\\
W_3 &\equiv \frac{1}{\sigma}\omega\left(B+\left(1+\varepsilon_{u,k}\right)\left(A_1-A_2\right)\right)
\end{aligned}
$$

- Case (2): $\varepsilon_{u,c}$ and σ are fix, while $\varepsilon_{u,k}$ moves from 0 to ∞.
We supposed for simplifying the calculations that:

$$
\begin{aligned}
E_1 &\equiv \left(1+H\right)\left(N-A_2\varepsilon_{u,c}\right)\varepsilon_{u,c}\\
E_2 &\equiv \varepsilon_{u,c}\left(A_3N-A_2Z\right)\\
E_3 &\equiv \varepsilon_{u,c}\frac{1}{\sigma}\left(Z-A_3\varepsilon_{u,c}\right)B\\
E_4 &\equiv \frac{1}{\sigma}\left(A_4-N\right)\left(Z-1\right)
\end{aligned}
$$

- Case (3): $\varepsilon_{u,k}$ and σ are fix, while $\varepsilon_{u,c}$ moves from $-\infty$ to 0.
We supposed for simplifying the calculations that:

$$
\begin{aligned}
G_2 &\equiv \left(1+\varepsilon_{u,k}\right)\left(A_4-N\right)\\
G_3 &\equiv \Gamma N-A_2\left(1+\varepsilon_{u,k}\right)Z\\
G_4 &\equiv A_2A_3\left(1+\varepsilon_{u,k}\right)-\Gamma A_2\\
G_5 &\equiv \frac{1}{\sigma}\left(BZ-A_3G_2\right)\\
G_6 &\equiv \left(G_3+G_5\right)\varepsilon_{u,c}+\left(G_4-\frac{1}{\sigma}BA_3\right)\varepsilon_{u,c}^2+\frac{1}{\sigma}ZG_2\\
G_7 &\equiv G_3+G_5+2\left(G_4-\frac{1}{\sigma}BA_3\right)\varepsilon_{u,c}
\end{aligned}
$$

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