Design Limits in Regime-Switching cases

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Abstract

This paper characterizes the derivation and the assessment of *design limits* of monetary policies in the case of a regime-switching economy. The object of the analysis on design limits is to derive the restrictions on how feedback rules, the Taylor-type rules typically used in monetary economics, affect the frequency fluctations underlying the state variable of interest. We extend the analysis in the context of a very structured type of model uncertainty where the uncertainty is described by the presence of different potential models whose probability of occurrence and switching is given by a known and ergodic Markov Chain transition matrix.

The presence of switching modifies the characteristics of design limits in two main aspects. First, contrary to the linear case, design limits are affected by the policy rule so that their role switches from a constraint to an externality that the policymaker may want to take into account. Second, frequency specific restrictions associated with a variance minizing rule appear more or less stringent with the respect to the linear case depending on the probability of switching: the higher it is, the worse is the performance in terms of frequency-specific fluctuations.

JEL Classification: C52, E6

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1 Non technical summary

The paper explores the restrictions on the effects of stabilizing policies on fluctuations from the perspective of the frequency domain. Those restrictions are called *design limits* by the control literature. The object of the analysis is the study of some fundamental limitations regarding frequency-specific effects that alternative monetary policy rules may imply. The object of interest underlying those limitations is the notion of the *Bode's integral* value, typically derived in the frequency domain[1], which represents an aggregate measure of design limits, in a way that will be clarified in the paper. The theory of design limits has been first introduced in economic contexts by Brock and Durlauf (2004). In this paper we extend the notion of design limits in a quite structured form of model uncertainty: the Markov Switching ARMA models (MSARMA) framework. The interest in the relation between the Bode's integral and the presence of uncertainty originates from the fact that the formula of the Bode's integral constraint can be interpreted by means of information entropy (or Shannon entropy) computation (Zang and Iglesias (2003)). The notion of entropy has recently received a renewed interest in macroeconomics due to the literature on robustness as a way to deal with model misspecification, rigorously developed in Hansen and Sargent (2008), in which the policymaker seeking to robustify against model misspeciifcation minimizes the entropy associated to the economic model under study rather than the familiar quadratic loss function typical of monetary policy exercises.

Design limits are relevant in the decision process of a monetary authority as long as there may be reasons to consider frequency-specific effects produced by the policy rule, in addition to the conventional object of monetary policy, namely, the minimization of the overall variability of the macroeconomic aggregates of interest. For instance, it seems plausible to suppose that the central bank is more interested in the business cycle performance of its decisions (medium frequencies) rather than the long run (or low-frequency) effects.

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The paper is structured as follows. Section 2 introduces the notion of design limits and their relevance in economics. Section 3 presents the framework of MSARMA models and summarizes the main findings of Pataracchia (2008a), which defines the frequency domain of the MSARMA models. Section 4 describes the derivation of the analogous of the Bode's integral constraint in the regime-switching case. In Section 5 we present some simulations and comment on the main characteristics of design limits in regime-switching cases. Section 6 concludes.

[1] A good reference for the notion of the Bode integral value in time domain is provided by Zang and Iglesias (2003) who use the notion of entropy for an information-theoretic interpretation of the costraint.

2 Introduction

Usual conventional monetary policy's objectives consist on the minimization of the overall unconditional variance of a vector of state variables of interest. A typical backward-looking model, usually considered in monetary policy literature is the following

$$x_t = A(L)\, x_{t-1} + B(L) u_t + \varepsilon_t \tag{1}$$

where x_t, the state variable of interest, may be a vector, composed, for instance, by the inflation rate and the output gap. $A(L)$ and $B(L)$ are two lag polynomials matrices, where L, the back shift operator, is such that $L^a x_t = x_{t-a}$. The control variable u_t, is set by the policymaker (for instance, the level of interest rate if the policymaker is a monetary authority). When $u_t = 0$ we are in the so called *free dynamics* case: the model evolves independently from the control of the policymaker and the autoregressive part of the system depends only on $A(L)$. For convenience, we label this case *NC*, standing for "no-control", C otherwise. The shocks vector, ε_t, while, in general, may have its own moving average structure, is supposed to be a vector of independent and identically normally distributed zero mean shocks with covariance matrix known and constant.

Suppose, for simplicity, x_t being scalar. There is an important relation between the unconditional variance of a stationary stochastic process, $var(x_t | C)$, and its spectral density $f_{x|C}(\omega)$:

$$var(x_t \mid C) = \int_{-\pi}^{\pi} f_{x|C}(\omega)\, d\omega \tag{2}$$

The area under the spectrum corresponds to the overall variance of the process. Formula (2) recovers the variance from the spectrum: the spectral density and the variance convey the same information about the second order moments of x_t. The frequency domain, however, reveals something more: the area under the spectrum between two frequencies represents the contributions of those frequencies to the overall variance. The total variability can indeed be considered as the weighted average of the spectral density across the frequencies. This is the main justification of the use of the spectral domain: if one believes that the policymaker may associate different losses to different frequency ranges, then the frequency domain studies enrich the policymaker's information set for better and more complete policy evaluations.

Let's provide a very simple example. Suppose a univariate AR(1) where $A(L) = 0.5$ and ε_t being a simple white noise with unit variance. A typical spectral representation of the free dynamics of such AR(1) is shown by the solid line in figure 1: when the autoregressive coefficients is positive (and less than one) the spectrum presents a

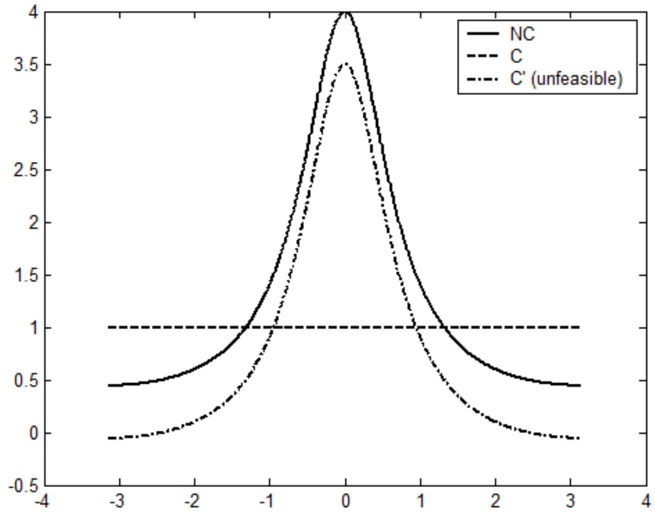

Figure 1: The spectral representation for the process $x_t = 0.5x_{t-1} + \varepsilon_t$ (solid line) and for the optimally controlled process which reduces to a white noise (dashed line). The dashed-dot line reproduces an unfeasible result.

peak at the low frequencies and decreases at the high frequencies[2]. Let's suppose now that $B(L) = 1$ and $u_t = -fx_{t-1}$. where f is set by the policy. If, as policymaker, we only care about the overall variance we try to kill off all the temporal dependences ($f = 0.5$) so to obtain a white noise (the flat dashed line in figure 1): we are reducing, by construction, the overall variance (the variance of the controlled process is 1, the part of the variation deriving from the shock process that the policymaker cannot control). However, as it may be seen from figure 1, while stabilizing the model, we are reducing low frequency components variance, but we are also increasing the contribution of variance deriving from high frequencies. We refer to those effects as *frequency trade-offs.*

Should we, as policymakers, care about frequency specific effects? There are several reasons to believe so.

First, nonseparable preferences for policymakers can lead to different losses for different frequency-specific fluctuations. Examples of this property are found in Otrok (2001) and Otrok, Ravikumar and Whiteman (2002): it seems reasonable to assume that individuals are more sensitive to fluctuations at high frequencies rather than low frequencies. For instance, consumers may have the following utility functions:

[2]Notice that the domain of the spectral representation is the close interval $[-\pi, \pi]$. The spectral representation is always symmetric with the respect to the frequency 0, so that, alternatively the spectrum can be completely defined just in the close interval $[0, \pi]$.

$$U\left(c_t - c_{t-1}\right)$$

This is a general idea that deviates from the usual way of consumption. If it is the correct specification of utility, then C in figure 1 may imply that people are worse off. In other words, if we introduce uncertainty about preferences, we may want to be sensitive about the frequency trade-offs implied by the minimization of the overall variance.

If we were able to control the process in the way represented by the dashed-dotted line, *C'*, in figure 1, then uncertainty about preferences would not be a great deal: no matter frequency specific trade-offs, *C'* would constitute an improvement. Unfortunately, in backward looking contexts, *C*' is not feasible. This has been the main message of Brock and Durlauf (2004) who first introduced the notion of the Bode's integral constraint in macroeconomic contexts. In what follows, we briefly explain its technical background and formal definition[3].

Suppse we consider the scalar version of (1):

$$x_t = A(L)\, x_{t-1} - B(L)\, u_t + \varepsilon_t \tag{3}$$

where x_t is supposed to be a zero mean , second order stationary process and ε_t is a mean-zero shock as a white noise with variance σ_ε^2. Suppose we want to stabilize the state variable: we consider a Taylor type feedback rule

$$u_t = F(L)\, x_{t-1}$$

The basic principle of all the stabilization policies requires the control to eliminate all the temporal dependences so that, in our case, x_t is shaped into a white noise, namely, $A(L)x_{t-1} - B(L)F(L)\, x_{t-1} = 0$. Every control rule, even if not optimal, allows to shape the autoregressive representation into a moving average one, so that every solution may be expressed on the form:

$$x_t = D^c(L)\, \varepsilon_t$$

where $D^c(L)$ is the *transfer function*[4] of the model under study. Equivalently, in the frequency domain, we can say that every control rule shapes the spectral representation of the unconstrained process

$$f_{x|NC} = \frac{\sigma_\varepsilon^2}{2\pi} D^{NC}\left(e^{-i\omega}\right) D^{NC}\left(e^{i\omega}\right)^{5} \tag{4}$$

[3]We invite the interested reader to refer to Brock and Durlauf (2004) and Brock, Durlauf and Rondina (2006) for an introduction to the control literature on the Bode's integral applied to economic contexts and for the extension to forward looking environments.

[4]Given the model $x_t = D^c(L)\, \varepsilon_t$, the *transfer function* is the mapping from the shock (input), ε_t, to the target vector (output), x_t.

[5]This expression represents the covariance generating function of the process in terms of the coefficients of $D^{NC}(L)$ and the variance of the white noise ε_t (Sargent (1987)).

into

$$f_{x|C} = \frac{\sigma_\varepsilon^2}{2\pi} D^C\left(e^{-i\omega}\right) D^C\left(e^{i\omega}\right) \tag{5}$$

where $D^{NC}\left(e^{-i\omega}\right)$ and $D^C\left(e^{-i\omega}\right)$ represent the analogous of the transfer functions in frequency domain, the complex number, $e^{-i\omega}$, ensures that the domain of the spectral density is the real line and the frequency ω belongs to the closed interval $[-\pi, \pi]$. The *Fourier transforms*[6] defined in formulas (4) and (5), may be equivalently described as follows:

$$f_{x|NC} = \frac{\sigma_\varepsilon^2}{2\pi}\left|D^{NC}\left(e^{-i\omega}\right)\right|^2$$

and

$$f_{x|C} = \frac{\sigma_\varepsilon^2}{2\pi}\left|D^{C}\left(e^{-i\omega}\right)\right|^2$$

where $|\cdot|^2$ denotes the complex and conjugate product of the transfer function.

We are now ready to define the object of interest of the Bode's integral, called *sensitivity function* in control literature:

$$S\left(e^{-i\omega}\right) \triangleq \frac{D^C\left(e^{-i\omega}\right)}{D^{NC}\left(e^{-i\omega}\right)} \tag{6}$$

From (4) and (5) it follows that

$$\left|S\left(e^{-i\omega}\right)\right|^2 \triangleq \frac{f_{x|C}}{f_{x|NC}} \tag{7}$$

which helps in understanding the role of $S\left(e^{-i\omega}\right)$: it describes how the spectrum of the unconstrained process is shaped into the controlled one. As a stabilizer, the policymakers wants to choose F so that $S\left(e^{-i\omega}\right) = 0$. This is naturally not possible because the realizations of the driving process do not belong to the policymaker's information set. Furthermore, there exists a more stringent feasibility constraint described by the celebrated Bode's integral Theorem[7] that we present after defining the Bode's integral.

Definition 1 *Given a model and a feasible, stabilizing rule F,with associated sensitivity function $S\left(e^{-i\omega}\right)$ defined as in (7), the Bode's integral (KB) is defined as follows*

$$KB = \int_{-\pi}^{\pi} \log\left|S\left(e^{-i\omega}\right)\right|^2 d\omega.$$

[6]The interested reader is invited to refer to chapter 13 of Sargent (1987) for a comprehensive introduction to the theory of Fourier transform.

[7]The original result is stated in Bode (1945)'s classical monograph.

Theorem 2 *Let's consider the process (3). If the roots of $A(L)$ – the eigenvalues of the free dynamics of* (3) *– are stable, then*

$$\int_{-\pi}^{\pi} \log \left| S\left(e^{-i\omega}\right) \right|^2 d\omega = 0$$

This theorem states that, in backward looking environments, even if the policy-maker is able to reduce the overall variance of the state of interest through the choice of F, the variance contributions at some frequencies will necessarily exacerbate, since, from Theorem 2, the sensitivity function, $S(\omega)$,[8] cannot be less than one at all the frequencies, in other words, it must be that

$$f_{x|C}(\omega) > f_{x|NC}(\omega) \text{ for some } \omega.$$

Notice that, for all the stable free dynamics, the value of the Bode's integral is always zero and independent of the particular model under exam. Further, it is independent of the policy rule. Figure 2 proposes again the spectral representation of a stable autoregressive process (the solid line) and the spectrum of the process when a variance minimized rule is adopted (the dashed line). As anticipated before, Theorem 2 ensures that *C'* is an unfeasible control. Figure 2 also shows that any random chosen feasible control, even if able to lower the overall variance, produces some frequency trade-offs.

This is the essence of the limitations of the control, or design limits, which seem particularly relevant when there is uncertainty about the correct specification of the frequency-specific preferences.

The second formulation of the Bode's integral theorem is the following:

Theorem 3 *Let's consider the process (3) and let's suppose that at least one root of $A(L)$ is greater than 1 (unstable). Then*

$$\int_{-\pi}^{\pi} \log |S(\omega)|^2 d\omega = 4\pi \sum_i |\log pi|$$

where p_i are the unstable roots of $A(L)$.

The underlying intuition is straighforward. When the process to stabilize has no finite variance, stabilization has some costs in terms of performance. In backward looking contexts Theorem 3 implies that the Bode's integral may take either zero or positive values, meaning that frequency-specific trade-offs are unavoidable. This is the reason why the Bode's integral is usually called "the Bode's integral *constraint*".

[8] Trough the paper, we will switch from the equivalent notations $S\left(e^{-i\omega}\right)$ and $S(\omega)$. The former is more formal, the latter is more intuitive and synthetic: in covariance stationary cases, the spectral representation and, consequently, the sensitivity function, are always real objects and functions of the frequency, ω.

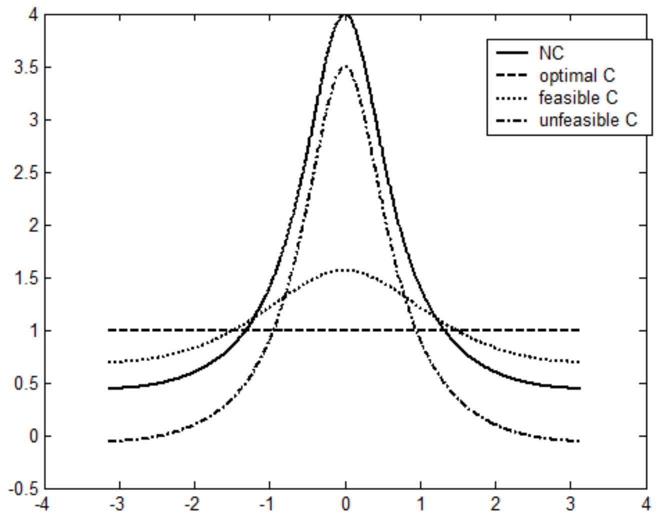

Figure 2: The spectral representation of the process $x_t = 0.5x_{t-1} + \varepsilon_t$ (solid line) and for the optimally controlled process which reduces to a white noise (dashed line). The dotted line represents the process resulting from a feasible control. The dashed-dot line reproduces an unfeasible result.

Brock and Durlauf (2005) provides an example of a formal control problem with the Bode's integral as a constraint. Notice that, even in the cases in which the Bode's integral is positive, its value is independent from the policy rule and given only by the unstable roots of the free dynamics.

Brock, Durlauf and Rondina (2006) describe how to quantify the Bode integral in forward looking models: they show that in these contests, Bode may also take negative values. This is due to agents' expectations which enrich the information set of the policymaker. However, the feasible rules which could, in principle, minimize the variance at all frequencies are, not always, the optimal response.

In this paper, we will provide an extension of the Bode's integral result to a context in which there is a specific form of model uncertainty: the policymaker knows that the economy may be represented by different potential models, whose probability of occurrence and switching are described by an ergodic, aperiodic Markov Chain. We suppose the policy is model independent and set prior to the realization of the Markov Chain. By that we mean that the Markov Chain, ξ_t, is not observed. However, we decide to analyze the case in which the transition probabilities are time invariant and known with certainty. In other words, to keep the exposition simple and clear, we decide not to consider the possibility that the policymaker can learn and update the transition probabilities as new observations are revealed. The procedure relies on the fact that the sensitivity function, the object of interest in the Bode integral constraint formulation, will be derived from the knowledge of the spectra as described in (7), rather than from the transfer function, as it is usually done in linear frameworks (that is, through Formula (6)). In regime switching cases, indeed, the transfer function is inherently non linear. We show that in this way we can still use the same linear framework powerful tools briefly described above.

3 The Spectral Density of MSARMA

Following Pataracchia(2008a), we consider the MSARMA(p,q) model of the following type:

$$x_t = \sum_{i=1}^{p} a_i(\xi_t)\, x_{t-i} + \varepsilon_t + \sum_{j=1}^{q} b_j(\xi_t)\, \varepsilon_{t-j} \tag{8}$$

where x_t is a zero mean purely indeterministic process in $\mathbb{R}^K$, $\varepsilon_t \sim WN\ (0,\ \Omega)$, $\xi_t = 1, 2$ is an irreducible, aperiodic and ergodic two states Markov Chain with finite space $\Xi = \{1, 2, ..., d\}$ with stationary transition probabilities denoted by $p(i, j) = pr\left(\xi_t = j \mid \xi_{t-1} = i\right)$ and unconditional (or steady state) probabilities , $\pi_i = pr(\xi_t = i)$, $1 \le i \le d$, where $\sum_{i=1}^{d} \pi_i = 1$[9]. Neither the noise (ε_t) nor the Markov Chain (ξ_t) are observed (the latter is said to be hidden).

In our previous work, we showed that, given *global*[10] stationarity, we can derive the spectral density by simply applying the Riesz-Fisher Theorem.

Let's start the formal discussion by stating the necessary and sufficient for global stationarity. Model (8) is stationary if and only if, given the $dK^2(p+q)^2 \times dK^2(p+q)^2$ matrix P

$$P = \begin{bmatrix} p_{11}\{\Phi(1)\otimes\Phi(1)\} & p_{21}\{\Phi(1)\otimes\Phi(1)\} & \cdots & p_{d,1}\{\Phi(1)\otimes\Phi(1)\} \\ p_{12}\{\Phi(2)\otimes\Phi(2)\} & p_{22}\{\Phi(2)\otimes\Phi(2)\} & \cdots & p_{d2}\{\Phi(2)\otimes\Phi(2)\} \\ \vdots & \vdots & & \vdots \\ p_{1d}\{\Phi(d)\otimes\Phi(d)\} & p_{2d}\{\Phi(d)\otimes\Phi(d)\} & \cdots & p_{dd}\{\Phi(d)\otimes\Phi(d)\} \end{bmatrix},$$

[9] On the calculation of the ergodic probabilities π_i see Hamilton (2004), page 684. In the simple two-states case, given the Markov Chain $M = \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p22 \end{bmatrix}$, the steady state probabilities are such that $\pi_1 = \frac{(1-p_{22})}{(2-p_{11}-p_{22})}$ and $\pi_1 + \pi_2 = 1$.

[10] As in Pataracchia(2008), we use the adjective 'global' referred to the MSARMA model. For instance, by global stationarity we mean the stationary of the MSARMA to distinguish it from the stationarity of the underlying modes.

where each $\Phi\left(\xi_t\right)$ is a matrix $K\left(p+q\right)\times K\left(p+q\right)$ defined as follows

$$\Phi\left(\xi_t\right)=\begin{bmatrix}
a_1\left(\xi_t\right) & \cdots & & & a_p\left(\xi_t\right) & b_1\left(\xi_t\right) & \cdots & \cdots & & b_q\left(\xi_t\right)\\
I_K & 0 & & \cdots & 0 & 0 & \cdots & \cdots & & 0\\
0 & I_K & & \cdots & 0 & \vdots & & & & \vdots\\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & & \vdots\\
0 & \cdots & & I_K & 0 & 0 & \cdots & \cdots & & 0\\
0 & \cdots & & & 0 & 0 & \cdots & \cdots & & 0\\
0 & \cdots & & \cdots & 0 & I_K & 0 & \cdots & & 0\\
 & & & & 0 & 0 & I_K & & & 0\\
\vdots & & & & \vdots & \vdots & & \ddots & & \vdots\\
0 & \cdots & & 0 & 0 & 0 & \cdots & \cdots & I_K & 0
\end{bmatrix},$$

$$\sigma\left(P\right)<1$$

where $\sigma\left(P\right)$ is the spectral radius of the matrix P. In other words, model (8) is stationary if and only if all the eigenvalues of the associated matrix P are less than one in absolute value.

Given global stationarity, the spectral matrix of model (8) can be defined as follows

$$F_x\left(\omega\right)=\sum_{\tau=-\infty}^{\infty}\left(e'\otimes f'\right)P^{*\tau}W\left(0\right)fe^{-i\omega\tau}$$

where

$$P^{*}=\begin{bmatrix}
p_{11}\Phi\left(1\right) & p_{21}\Phi\left(1\right) & \cdots & p_{d1}\Phi\left(d\right)\\
p_{12}\Phi\left(2\right) & p_{22}\Phi\left(2\right) & \cdots & p_{d2}\Phi\left(d\right)\\
\vdots & \vdots & & \vdots\\
p_{1d}\Phi\left(d\right) & p_{2d}\Phi\left(d\right) & \cdots & p_{dd}\Phi\left(d\right)
\end{bmatrix}$$

is a $dK\left(p+q\right)\times dK\left(p+q\right)$ square matrix, $e=\left(1,...,1\right)'\in\mathbb{R}^d$ and $f'=\left(I_K,0,...,0\right)$ is a $K\times K\left(p+q\right)$ matrix. The matrix $W\left(0\right)$ is the $dK\left(p+q\right)\times K\left(p+q\right)$ matrix whose i^{th} block, for $i=1,..,d$, is given by $\pi_i E\left(z_t z_t'\left|\xi_t=i\right|\right)$, associated to the second order moments of (8)[11]. The spectral densities of each element of the state variable vector x_t correspond to the diagonal elements of $F_x\left(\omega\right).$

For instance, in the simple univariate case of a MSAR(1) of the form

$$x_t=a\left(\xi_t\right)x_{t-1}+\varepsilon_t \tag{9}$$

where ε_t is supposed to be a white noise with zero mean and known and constant variance σ_ε^2 and ξ_t is a 2×2 Markov Chain, the spectral density has the following form

$$f_x\left(e^{-i\omega}\right)=\frac{\sigma_\varepsilon^2}{2\pi}\begin{pmatrix} K\frac{1}{\left(1+\lambda_1^2-2\lambda_1\cos\omega\right)}+\\ H\frac{1}{\left(1+\lambda_2^2-2\lambda_2\cos\omega\right)}\end{pmatrix} \tag{10}$$

[11] See Pataracchia(2008) for further details on the construction of the matrix $W\left(0\right).$

where K and H are functions of $a(i)$ and $p_{ij}, \forall i, j = 1, 2$, explicitly derived in Pataracchia(2008a).

In order to appreciate the well behaving properties of (10), let's recall the linear framework. In the linear case, any stationary ARMA model with the following Wold representation

$$y_t = G(L)\eta_t \tag{11}$$

with η_t being a white noise with zero mean and known and constant variance σ_η^2, has the following spectral representation

$$f_y\left(e^{-i\omega}\right) = \frac{\sigma_\eta^2}{2\pi}\left|G\left(e^{-i\omega}\right)\right|^2 \tag{12}$$

Comparing (10) and (12) we notice that the structure of the spectral density of Markov Switching model (with a model independent shock process) is similar to any other linear stationary ARMA model. Indeed, rewriting (10) as follows

$$f_x\left(e^{-i\omega}\right) = \frac{\sigma_\varepsilon^2}{2\pi}A(\omega) \tag{13}$$

we can state that $A(\omega)$ "plays the role" of the complex and conjugate product in (12). This observation opens the way to possible extensions of the frequency domain criteria usually used in linear frameworks to the case of MSARMA models. One example may be the analysis of the frequency effects of the robust monetary policy rules, which we leave for future research.

In what follows we exploit (13) to characterize the design limits of regime switching models. While we recognize that the MSAR(1) is a very simple example which hardly allows a relevant economic application, we think it is important to consider it as a starting point because, while calculations remain tractable, it allows to derive quite general considerations.

4 Design Limits in MSAR(1)

The goal of this section is the derivation of the design limits in the case of the MSAR(1). Before going on, we need to make some assumptions regarding the control rule. When a feedback-type control is considered, (9) may be rewritten as

$$x_t = a(\xi_t)x_{t-1} - \varsigma(\xi_t)u_t + \varepsilon_t \tag{14}$$

For the sake of simplicity, we just assume $\varsigma(\xi_t)$ not only state-independent but also equal to one. The generic rule is on the form $u_t = F(L)x_{t-1}$, and we suppose $F(L) = F$ so that we can finally rewrite (14) as

$$\begin{aligned} x_t &= (a(\xi_t) - F)x_{t-1} + \varepsilon_t = \\ &= a^C(\xi_t)x_{t-1} + \varepsilon_t \end{aligned} \tag{15}$$

and we can still work in a MSAR(1) framework. Given (10), we compare:

$$f_{x|NC}\left(e^{-i\omega}\right)=\frac{\sigma_{\varepsilon}^{2}}{2\pi}\left(\begin{array}{c} K\frac{1}{(1-\lambda_1 e^{-i\omega})(1-\lambda_1 e^{i\omega})}+ \\ H\frac{1}{(1-\lambda_2 e^{-i\omega})(1-\lambda_2 e^{i\omega})} \end{array}\right) \tag{16}$$

with

$$f_{x|C}\left(e^{-i\omega}\right)=\frac{\sigma_{\varepsilon}^{2}}{2\pi}\left(\begin{array}{c} K^{c}\frac{1}{(1-\lambda_1^{c} e^{-i\omega})(1-\lambda_1^{c} e^{i\omega})}+ \\ H^{c}\frac{1}{(1-\lambda_2^{c} e^{-i\omega})(1-\lambda_2^{c} e^{i\omega})} \end{array}\right) \tag{17}$$

where, as in our previous work, c and nc correspond, respectively, to the case in which we suppose the policymaker intervenes through the choice of F and the case in which $u_t = 0$. We are interested in these two cases because, as we showed before, the Bode's integral represents an aggregate measure of design limitations the policymaker must face *with the respect* to the case without control.

In linear time invariant (LTI) frameworks (for instance, considering model (11)), we are used to consider $f_{y|nc}\left(e^{-i\omega}\right)$ and $f_{y|c}\left(e^{-i\omega}\right)$ in the following form:

$$f_{y|NC}\left(\omega\right)=f_{\eta}\left(\omega\right)\mid G\left(\omega\right)\mid^{2} \tag{18}$$

and

$$f_{y|C}\left(\omega\right)=f_{\eta}\left(\omega\right)\mid G^{c}\left(\omega\right)\mid^{2} \tag{19}$$

where $f_{\eta}\left(\omega\right)=\frac{\sigma_{\eta}^{2}}{2\pi}$ and $\mid G\left(\omega\right)\mid^{2}$and $\mid G^{c}\left(\omega\right)\mid^{2}$ are the complex and conjugate products of the transfer functions of the, respectively, unconstrained and constrained systems. As described in (6), in linear frameworks, the sensitivity function is derived by the knowledge of the transfer functions of the controlled and uncontrolled process. In our case, even if we deal with nonlinear objects, we can still define the complex and conjugate product of the sensitivity function via the knowledge of the spectra, as described in (7):

$$\mid S\left(\omega\right)\mid^{2}=\frac{f_{x|C}\left(\omega\right)}{f_{x|NC}\left(\omega\right)}$$

where

$$\begin{aligned}\frac{f_{x|C}\left(\omega\right)}{f_{x|NC}\left(\omega\right)} &= \left(\begin{array}{c} K^{c}\frac{1}{(1-\lambda_1^{c} e^{-i\omega})(1-\lambda_1^{c} e^{i\omega})}+ \\ H^{c}\frac{1}{(1-\lambda_2^{c} e^{-i\omega})(1-\lambda_2^{c} e^{i\omega})} \end{array}\right)\times \\ &\quad\times\left(\begin{array}{c} K\frac{1}{(1-\lambda_1 e^{-i\omega})(1-\lambda_1 e^{i\omega})}+ \\ H\frac{1}{(1-\lambda_2 e^{-i\omega})(1-\lambda_2 e^{i\omega})} \end{array}\right)^{-1}\end{aligned}$$

We are finally interested in the Bode's integral constraint

$$KB=\int_{-\pi}^{\pi}\log\mid S\left(\omega\right)\mid^{2}d\omega$$

Taking the log of $\mid S(\omega) \mid^2$, we end up with an expression on the form

$$\log \mid S(\omega) \mid^2 = \log X(\omega) + \log Y(\omega) - \log Z(\omega) - \log W(\omega)$$

where

$$\begin{aligned}
X(\omega) &= K^c \left(1 - \lambda_2^c e^{-i\omega}\right)\left(1 - \lambda_2^c e^{i\omega}\right) + \\
&\quad H^c \left(1 - \lambda_1^c e^{-i\omega}\right)\left(1 - \lambda_1^c e^{i\omega}\right) \\
Y(\omega) &= \left(1 - \lambda_1 e^{-i\omega}\right)\left(1 - \lambda_1 e^{i\omega}\right) \times \\
&\quad \left(1 - \lambda_2 e^{-i\omega}\right)\left(1 - \lambda_2 e^{i\omega}\right) \\
Z(\omega) &= \left(1 - \lambda_1^c e^{-i\omega}\right)\left(1 - \lambda_1^c e^{i\omega}\right) \times \\
&\quad \left(1 - \lambda_2^c e^{-i\omega}\right)\left(1 - \lambda_2^c e^{i\omega}\right) \\
W(\omega) &= K \left(1 - \lambda_2 e^{-i\omega}\right)\left(1 - \lambda_2 e^{i\omega}\right) + \\
&\quad H \left(1 - \lambda_1 e^{-i\omega}\right)\left(1 - \lambda_1 e^{i\omega}\right)
\end{aligned}$$

and

$$KB = \int_{-\pi}^{\pi} \log X(\omega)\, d\omega + \int_{-\pi}^{\pi} \log Y(\omega)\, d\omega - \int_{-\pi}^{\pi} \log Z(\omega)\, d\omega - \int_{-\pi}^{\pi} \log W(\omega)\, d\omega$$

Let's investigate each term. First, let's consider $Y(\omega)$. We may write it as

$$logY(\omega) = \log\left(\mid 1 - \lambda_1 e^{-i\omega} \mid^2 \mid 1 - \lambda_2 e^{-i\omega} \mid^2\right) = \log\left(\mid e^{i\omega} - \lambda_1 \mid^2 \mid e^{i\omega} - \lambda_2 \mid^2\right)$$

Following Brock et al.(2006), we can now exploit the Wu and Jonkheere lemma:

Lemma 4 $\int_{-\pi}^{\pi} \log \mid e^{i\omega} - r \mid^2 d\omega = 0$ *if* $\mid r \mid < 1, 2\pi \log \mid r \mid^2$ *otherwise.*

Now we can write

$$\int_{-\pi}^{\pi} \log Y(\omega)\, d\omega = 4\pi \sum_{v_i} \log \mid \lambda_{v_i} \mid, \; i \in v_i \;\; \text{if} \;\; \mid \lambda_i \mid > 1, \forall i = 1, 2$$

Similarly, we can do the same calculations for $Z(\omega)$ and finally get:

$$\int_{-\pi}^{\pi} \log Y(\omega)\, d\omega - \int_{-\pi}^{\pi} \log Z(\omega)\, d\omega =$$

$$4\pi \left(\sum_{v_i} \log \mid \lambda_{v_i} \mid - \sum_{r_i} \log \mid \lambda_{r_i}^c \mid \right), \tag{20}$$

$$i \in \{v_i\} \;\text{if}\; \mid \lambda_i \mid > 1, i \in \{r_i\} \;\text{if}\; \mid \lambda_i^c \mid > 1, \forall i = 1, 2$$

Some observations are needed at this point. From (20) one is tempted to conclude that one part of the Bode's integral formula resembles the linear framework's result: it will be zero in the case of a (global) stable free dynamics. Notice, however, that, contrary to Brock et al.(2006), we used the notion of the spectral representation to get the complex and conjugate product of the sensitivity function and the condition of stationarity is necessary to derive the spectral representation. Therefore, the condition of global stationarity must hold in both the uncontrolled and controlled model in order to conclude that expression (20) implies that the value of the Bode's integral is zero. This observation does not necessarily rule out interesting comparisons with positive Bode values in the underlying models: instabilities in each AR(1) model are neither a necessary nor a sufficient condition for global instability (Costa et al. (2005)).

We are now left with the terms $X(\omega)$ and $W(\omega)$. Let's start from the former.

$$\begin{aligned} X(\omega) &= K^c\left(1-\lambda_2^c e^{-i\omega}\right)\left(1-\lambda_2^c e^{i\omega}\right)+ \\ &\quad H^c\left(1-\lambda_1^c e^{-i\omega}\right)\left(1-\lambda_1^c e^{i\omega}\right) \end{aligned}$$

which is on the form

$$X(\omega)=A^c+B^c\cos\omega$$

so that

$$\int_{-\pi}^{\pi}\log X(\omega)\,d\omega=\int_{-\pi}^{\pi}\log\left(A^c+B^c\cos\omega\right)d\omega \tag{21}$$

where

$$\begin{aligned} A^c &= K^c(1+\lambda_2^{c2})+H^c\left(1+\lambda_1^{c2}\right) \\ B^c &= -2\left(K^c\lambda_2^c+H^c\lambda_1^c\right) \end{aligned} \tag{22}$$

Similarly

$$\int_{-\pi}^{\pi}\log W(\omega)\,d\omega=\int_{-\pi}^{\pi}\log\left(A+B\cos\omega\right)d\omega \tag{23}$$

From Gradshteyn and Ryzhik (1980)[12]

$$\int_{-\pi}^{\pi}\log\left(a+b\cos\omega\right)d\omega=2\pi\log\frac{a+\sqrt{a^2-b^2}}{2} \tag{24}$$

so that

$$\int_{-\pi}^{\pi}\log X(\omega)\,d\omega-\int_{-\pi}^{\pi}\log W(\omega)\,d\omega=$$

$$\int_{-\pi}^{\pi}\log\left(A^c+B^c\cos\omega\right)-\int_{-\pi}^{\pi}\log\left(A+B\cos\omega\right)d\omega=$$

[12]See Gradshteyn and Ryshink (1880), page 527.

$$= 2\pi \log \frac{A^c + \sqrt{A^{c2} - B^{c2}}}{A + \sqrt{A^2 - B^2}}$$[13]

We are now able to state the following Theorem.

Theorem 5 *Given model (15), the value of the Bode' integral corresponds to*

$$\begin{aligned} KB &= 2\pi \log \frac{A^c + \sqrt{A^{c2} - B^{c2}}}{A + \sqrt{A^2 - B^2}} + 4\pi \left(\sum_{v_i} \log \mid \lambda_{v_i} \mid - \sum_{r_i} \log \mid \lambda^c_{r_i} \mid \right) \qquad (25) \\ i &\in \{v_i\} \text{ } if \mid \lambda_i \mid > 1, i \in \{r_i\} \text{ } if \mid \lambda^c_i \mid > 1, \forall i = 1, 2 \end{aligned}$$

where A, A^c, B and B^c are defined as in (22).

Formula (25) provides a measure of design limits for MSAR(1). Its generic analytic formula depends on the policy rule, $a(i)$ and p_{ij} with $i, j = 1, 2$. This leads us to notice two important differences with the respect to the LTI cases. First, the Bode's integral is model specific, and therefore subject to model misspecifications, even in backward looking cases. Second, the presence of the terms A^c and B^c suggests that the value of the Bode's integral can be affected by the policy rule. The latter observation allows to associate to the Bode's integral the role of an *endogenous* constraint, similar to an externality effect, rather than the exogenous constraint that has been traditionally associated to it in linear frameworks. This certainly constitutes an additional reason to consider the analysis of design limits as important in any policy evaluation exercise.

In the next Section, we present some simulations in order to understand the behaviour of the Bode's integral. Unless otherwise stated, we consider the variance minimizing rule as the candidate one. We think it is useful here to make a digression on the characteristics of the variability of our MSAR(1) and on the ability of any rule of the type described in (15) to control them. In regime-switching cases, total variability can be decomposed in two parts: the *variability-within* the underlying models and the *variability-between* the models, linked, respectively, to the shock process ε_t and the Markov Chain, ξ_t. The variability-within depends, of course, on the autoregressive coefficients of the underlying models, $a(1)$ and $a(2)$. The variability between the models may, in turn, be decomposed in two aspects. One is related to the *measure* or *amount* of the switching, once it occurs. In other words, the closer the models in terms of variability (in our simple case, the closer $|a(1)|$ and $|a(2)|$) the lower will be amount of the transition variability. There is, however, a second component of the transition variability, which is given by the *probability* of the switching. The higher p_{12} and p_{21} (or correspondingly, the lower p_{11} and p_{22}), the higher is the probability of the switching. The unconditional probability of switching is given by $\pi_1 p_{12} + \pi_2 p_{21}$. Notice that, by (15), the policy rule cannot affect (reduce) the transition probabilities and therefore, the switching probabilities. Those observations will turn out to be important for what follows.

[13] Notice that the explicit solution of the integral exists only for $A \geq |B|$ and $A^c \geq |B^c|$. This restriction, however, is not binding in any of the simulations presented next.

5 The dynamics of the Bode's integral

Given the complexity of analytical formulas, we present some simulations related to model (15), with F corresponding at the variance minimizing rule and $\sigma_{\varepsilon}^{2} = 1$. In Section 5.1.1 we consider the case in which the transition probabilities p_{11} and p_{22} are both set equal to 0.5 while we consider different combinations of $a\,(1)$ and $a\,(2)$. We call it the case of *symmetric* transition probabilities. In Section 5.1.2 we propose the same analysis in the case in which the transition probabilities are both greater and less than 0.5, so to investigate the effects of the Bode's integral constraint when the probability of the switching varies.

5.1 The Bode's integral across the models

5.1.1 The case of *symmetric* transition probabilities

In this section we suppose symmetric transition probabilitites ($p_{11} = p_{22} = p_{21} = p_{12} = 0.5$). The following figures represent three simulated cases. For each case we use three graphs: the first one refers to the global MSAR(1), while the other two refer to the respective underlying models. In all of them the solid line represent the spectral density of the free dynamics of the model while the dashed line illustrates the spectrum of the constrained processes when the variance minimizing rule is applied.

Figure (3) shows that the control is able to eliminate all the temporal dependences. In this case the two underlying models are close to each other and the variation due to the switching appears to be negligible: the total variability of the controlled process is only slightly above 1[14].

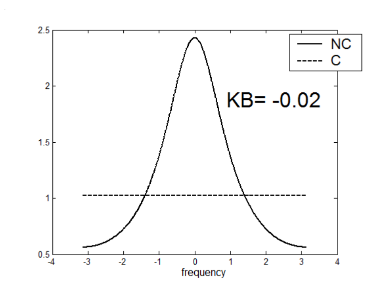

Figure 3: Spectrum of MSAR(1), $a\,(1) = 0.5$, $a\,(2) = 0.2$.

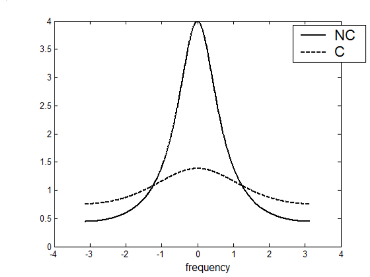

Figure 4: Spectrum of the underlying model 1: AR(1) with $a = 0.5$.

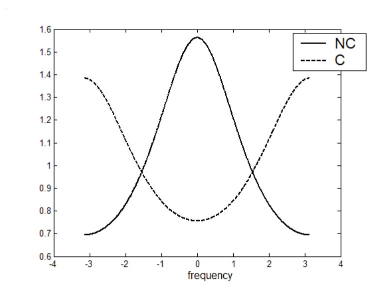

Figure 5: Spectrum of the underlying model 2: AR(1) with $a = 0.2$.

[14] In the limiting case, in which the models are identical (so that there is no variation due to the switching), the controlled process would result a white noise with unit variance. 1 constitutes, therefore, the lower bound for the variance of the controlled regime switching model.

The Bode's integral is negative, even if the controlled spectrum presents some frequency trade-offs (stabilization is improved at the low frequencies but exacerbated at the high ones). A negative Bode's integral implies that, even though some frequency trade-offs of the chisen policy appear, overall, the frequency-specific variability contributions are reduced in comparison to the free dynamics case, in the sense that, while we are able to reduce the overall variability, the frequency-specific trade-offs that result appear diminished.

In the underlying models we observe very different frequency-specific dynamics. Even if, in our set up, the policymaker cares only about global dynamics, he may want to take into account the spectra of the underlying models, both of which viable, given the assumption of ergodicity of the Markov Chain, in particular, if he associates different losses to different frequency ranges.

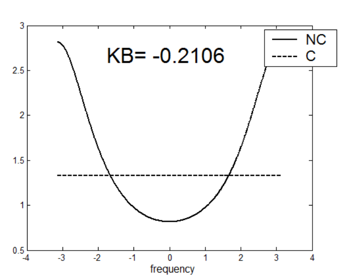

Figure 6: Spectrum of MSAR(1), $a\,(1) = -0.8$, $a\,(2) = 0.2$.

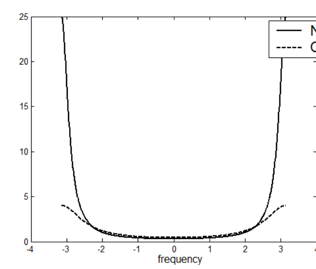

Figure 7: Spectrum of the underlying model1: AR(1) with $a = -0.8$.

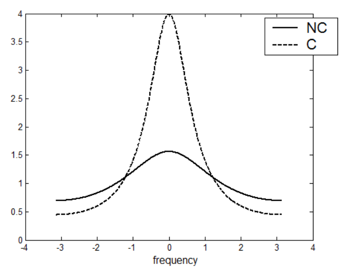

Figure 8: Spectrum of the underlying model 2: AR(1) with $a = 0.2$.

Figures 6, 7 and 8 refer to the second simulation. This time we consider a MSAR(1) composed by one model (model 1) quite persistent, while the second one is stabler. In this case the variation between the models is higher with the respect to the previous case and the variance of the controlled process, whose spectrum is depicted by the dashed line in figure 6, is 1.333. The Bode's integral is, again, negative and lower than the Bode's integral of the first case. Figure 6 shows how the control exacerbates the variability at the low frequencies. This is due, in particular to the bad performance in correspondence of the low frequencies in the model 2, as shown by Figure 8.

$a\,(2)$	-0.9	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	0.9
KB	-0.256	-0.064	0	-0.016	-0.03	-0.016	0	-0.064	-0.256

Table 1: The Bodes' Integral constraint with a(1)=0.5 and flat transition probabilities

$a\,(2)$	-0.9	-0.8	-0.6	-0.4	0	0.4	0.6	0.8	0.9
KB	-0.041	0	-0.061	-0.149	-0.232	-0.149	-0.061	0	-0.041

Table 2: The Bodes' Integral constraint with a(1)=0.8 and flat transition probabilities

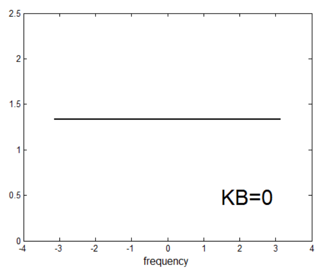

Figure 9: Spectrum of MSAR(1), $a\,(1) = 0.5$, $a\,(2) = -0.5$.

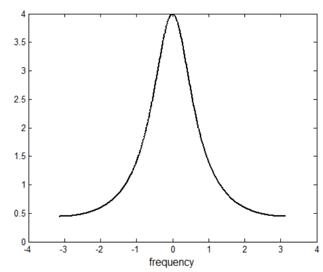

Figure 10: Spectrum of the underlying model 1: AR(1) with $a = 0.5$.

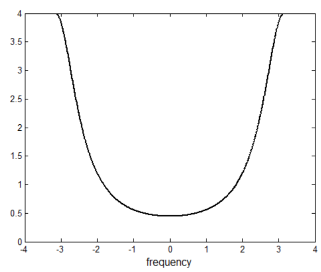

Figure 11: Spectrum of the underlying model 2: AR(1) with $a = -0.5$.

The last case, to which figures 9, 10 and 11 refer, deserves particular attention. The MSAR(1) behaves already as a white noise because the two underlying models have opposite coefficients and the transition probabilities are flat. Therefore, the best response in terms of the minimization of the variance is not to intervene and the Bode's integral is zero by definition. The variation between the two underlying models is not reduced at all, so that the variance of the controlled global process reaches quite high values compared to the first case.

Next we investigate the behavior of the Bode's integral value across different models: we keep fixed model 1's autoregressive coefficient, $a\,(1)$, and consider the dynamics of the Bode's integral constraint as the second model's coefficient $(a\,(2))$ changes.

Tables 1 and 2 show the values of the Bode's integral, given $p_{11} = p_{22} = 0.5$, and, respectively, $a\,(1) = 0.5$ and $a\,(1) = 0.8$.

Several considerations can be advanced. First, the Bode's integral is always negative or equal to zero. This is, at first glance, conterintuitive: the assumption of the switching regimes introduces an additional source of uncertainty and it is reasonable to think that this may cause an exacerbation to the limits of the policymaker's intervention. Second, in both tables, given $a\,(1)$, the values of the Bode's integral are symmetric around $a\,(2) = 0$: they vary depending on the absolute value of the difference of the two autoregressive coefficients. For instance, let's consider figure 12 to which Table 1 refers: from very negative values, moving from $a(2) = -0.9$ towards 0 the Bode's integral increases very fast till it reaches 0 when $a(2) = -0.5$. This is the case we discussed in figure 9: the policymaker does not intervene. Continuing moving rightwards, the Bode's integral slightly diminishes till $a(2) = 0$, after which it increases again until it reaches 0 at $a(2) = 0.5$. Here the two models coincide, no model uncertainty is present so that Bode is zero from Theorem 2.

In other words, as shown in the figures 12 and 13, the Bode's integral dynamics with the respect to $a(2)$ presents two peaks: one corresponds to the case in which the MSAR(1) is already perceived as a white noise, the other one represents the case in which there isn't any switching and the two underlying models are identical.

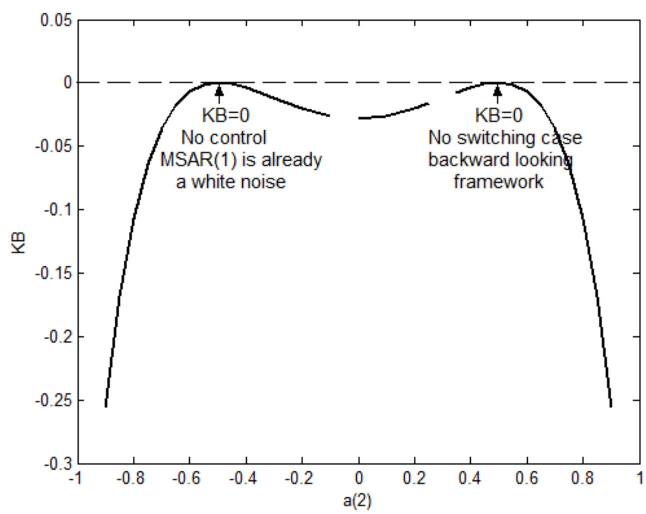

Figure 12: The dynamics of the Bode's integral constraint with $a(1) = 0.5$, $p_{11} = 0.5$ and $p_{22} = 0.5$.

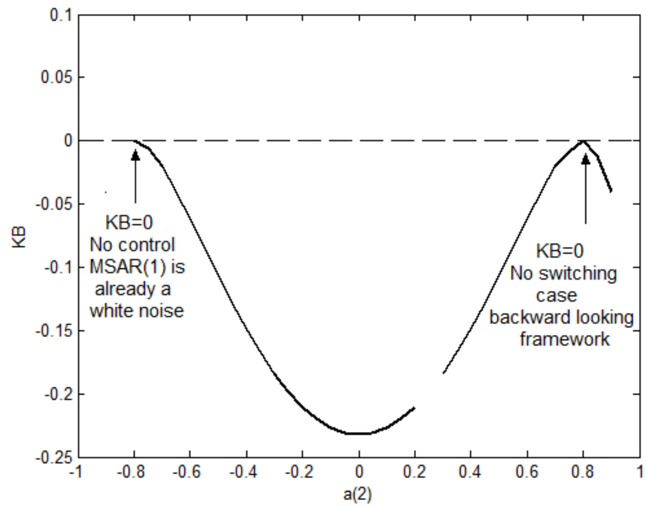

Figure 13: The dynamics of the Bode's integral constraint with $a\,(1) = 0.8$, $p_{11} = 0.5$ and $p_{22} = 0.5$.

The dynamics of the Bode's integral may have a meaningful interpretation. We already know that the lower is the value of the Bode's integral, the better is the control in the sense of producing reduced frequency specific trade-offs. Our job is to understand in which sense we may say that the introduction of model uncertainty, in the form of switching regimes, may reduce design limits when $p_{11} = p_{22} = 0.5$. The key point to understand the above dynamics is to recall the distinction of the two types of variability the policy maker is subject to: the variability-between and the variability-within the models.

One plausible interpretation of the qualitative dynamics of the figures 12 and 13, may be provided by answering the following question: how effective is the policy-maker's control in the reduction of the variability between the models, that is, the variability due to the transition from one model to the another one? A rough answer is provided by the dynamics of the Bode's integral, read with the opposite sign (or we may consider the symmetry with the respect to the $x-$axis). The combination of $a\,(1)$ and $a\,(2)$ for which KB is very negative corresponds to the situations in which the policymaker is quite effective in the reduction of the variability between the models. This is possible because the two models behave in a very different way and the fraction of the total variability due to the transition is important. When the two underlying models are close to each other, the control of the policymaker cannot be so effective in the reduction of the transition probability which is already limited. We can now reinterpret the two points in which Bode is zero: in the case of no switching, there is no transition probability to reduce, in the case in which $a\,(1) = -a\,(2)$ the models behave in the opposite way, the transition probability is very high but it cannot be reduced because the policymaker doesn't intervene. The fact that the Bode's integral can take negative values does not imply that the introduction of model uncertainty has improved the performance. The Bode' integral is a measure of *relative* performance (with the respect to the unconstrained case) and the negative values represent those cases in which the policymaker is able to reduce the frequency-specific trade-offs, even if such trade-offs are still present.

If the variation between the models seems a good candidate for the explanation of the qualitative dynamics of the Bode's integral in figure 12 (figure 13) , it cannot explain why Bode dynamics is not symmetric with the respect to the points $a(2) = -0.5$ $(a(2) = -0.8)$ and $a(2) = 0.5$ $(a(2) = 0.8)$. In other words, focusing on figure 12, when $a(1) = 0.5$, why do we observe different Bode values if $a(2) = 0.8$ or $a(2) = 0.2$? The reason is plausibly due to the reduction of the variability within the models. Technically, given the particular functional form of the spectra of the constrained system and the free dynamics, the effective reduction in variance is going to affect, through λ_i and λ_i^c, $i = 1, 2$, the quantity determining the Bode's integral because, given p_{11} and p_{22}, $2\pi \log \frac{A^c + \sqrt{A^{c2} - B^{c2}}}{A + \sqrt{A^2 - B^2}}$ is, ultimately, a function of the respective eigenvalues of P matrix, which determines the stability and the unconditional variance of the model. For instance, when $a(1) = 0.5$ and $a(2) = 0.2$ the uncontrolled process has an unconditional variance equal to 1.1696, while the variance-minimizing rule allows the policymaker to reduce the variance to 1.023. When, instead $a(2) = 0.8$, while the controlled process has the same variance as before, 1.023, the uncontrolled process has total variance equal to 1.8018, much higher than the previous case. In this sense we say that in the latter case the reduction of the variability within the models is more effective (compared to the uncontrolled case).

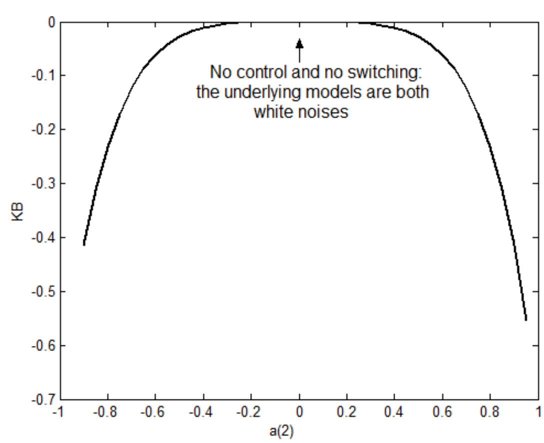

Figure 14: The dynamics of the Bode's integral constraint with $a(1) = 0, p_{11} = 0.5$ and $p_{22} = 0.5$.

To conclude this Section, we present in figure 14 the limiting case in which the combinations of models for which there is no switching occurring and at the same time no need for policymaker's intervention ($a(1) = a(2) = 0$), highlighted in the graphs above, coincide. This case will result important as a comparison for the following analysis.

5.1.2 The Bode's integral across the probabilities of switching

In this Section we analyze the three cases considered before ($a(1) = 0.8, 0.5, 0$) but we set different values for the transition probabilities. In particular we consider two cases. In the first, we diminish the overall unconditional probability of switching by setting $p_{11} = p_{22} > 0.5$ and equal to 0.8. In the second one the unconditional probability of switching is set to 0.2.

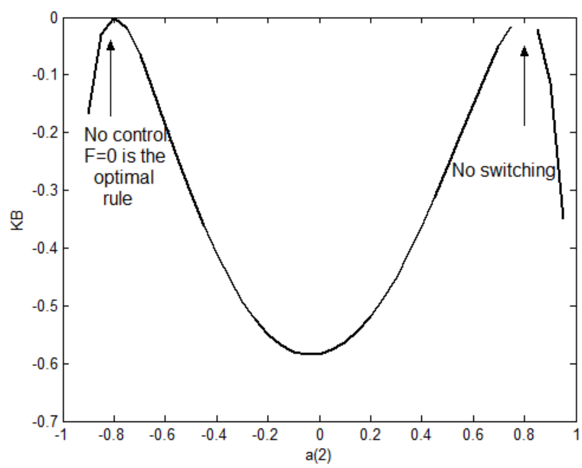

Figure 15: The dynamics of the Bode's integral constraint with $a(1) = 0.8, p_{11} = 0.8$ and $p_{22} = 0.8$.

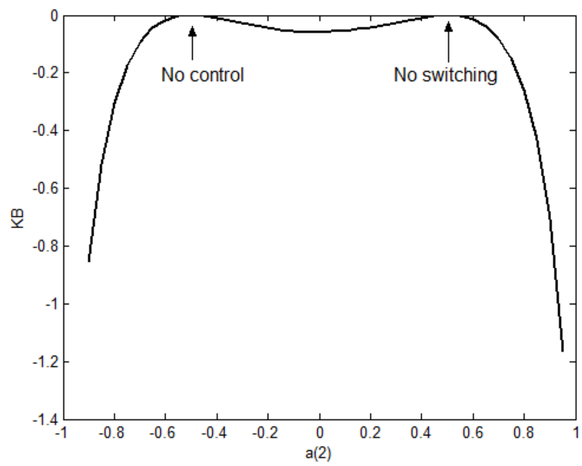

Figure 16: The dynamics of the Bode's integral constraint with $a\left(1\right)=0.5, p_{11}=0.8$ and $p_{22}=0.8$.

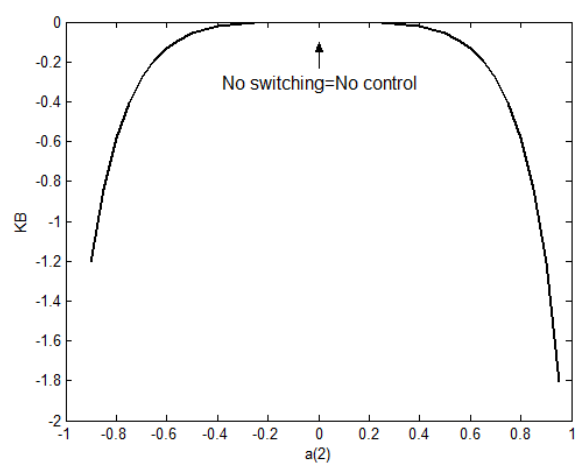

Figure 17: The dynamics of the Bode's integral constraint with $a\left(1\right)=0, p_{11}=0.8$ and $p_{22}=0.8$.

If we compare figures 15, 16 and 17 with the respective cases analyzed in the previous section, we realize that the qualitatively dynamics inside each graph does not vary, but, quantitatively, the values of the Bode's integrals are always reduced when the probability of switching is lower (except for the cases in which the Bode's integral is zero). In order to draw more general considerations, we propose next figures

18, 19 and 20 which depict our candidate three cases in presence of high probability of switching, 0.7 ($p_{11} = p_{22} = 0.3$).

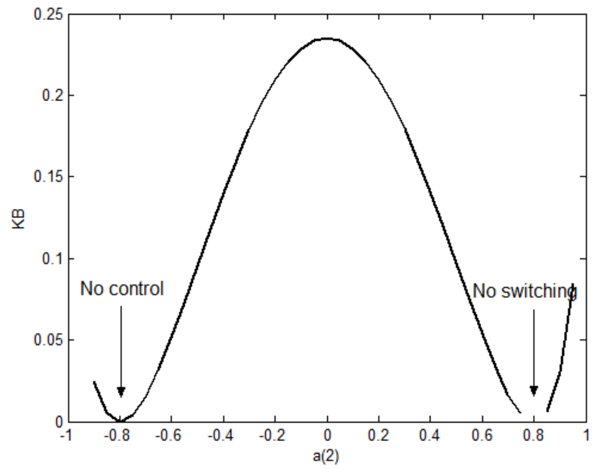

Figure 18: The dynamics of the Bode's integral constraint with $a\,(1) = 0.8, p_{11} = 0.3$ and $p_{22} = 0.3$.

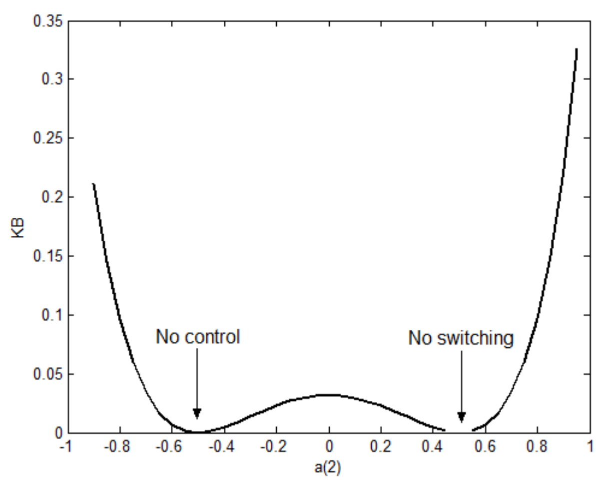

Figure 19: The dynamics of the Bodes' integral constraint with $a\,(1) = 0.5, p_{11} = 0.3$ and $p_{22} = 0.3$.

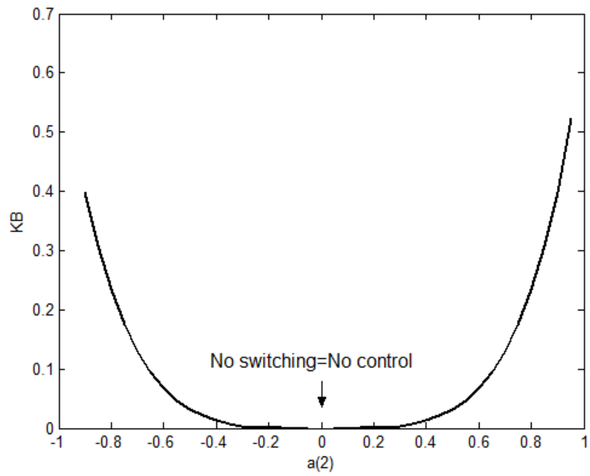

Figure 20: The dynamics of the Bode's integral constraint with $a\,(1) = 0$, $p_{11} = 0.3$ and $p_{22} = 0.3$.

When the probability of switching is increased, the Bode's integral value takes positive values. In other words, the presence of an highly uncertain state of the economy, in which there occurs frequent switching between one model and the other, impedes the action of the policymaker to be effective in reducing (or, at least, non exacerbating) the aggregate measure of the design limits, characterized by the value of the Bode's integral. This consideration appears very interesting, because it contributes in the understanding the links between the Bode's integral and the uncertainty faced by the policymaker, given, in our case, by the high probability of switching. In linear frameworks, it has been shown that the Bode's integral associated to a certain model corresponds to the difference of the information (or Shannon) entropy of the controlled model and the entropy of the free dynamics. It is possible to prove that the information entropy interpretation of the Bode's integral still holds in regime-switching cases (Pataracchia 2008b). In other words, high probability of switching corresponds to high level of *entropy* of the regime switching model as if the action of the policymaker is not only not able to reduce the uncertainty associated to it, but it exacerbates it.

5.2 The Bode's integral across the policy rules

In the previous sections we stressed how the dynamics of the design limits in Markov Switching contexts depends on the models' parameters (both in terms of autoregressive coefficients and transition probabilities).

A second important feature of design limits arising in switching regimes is the dependence on the particular rule decided by the policymaker: through the choice of the policy the policymaker not only shapes the spectral characteristics of the model, but he does it in a way which affects also the measure of frequency trade-offs, therefore the constraint he is subject to. Figure 21 shows the dynamics of the Bode's integral across several models when different policies of the type described in (15) are considered. We suppose $a(1) = 0.8$, $p_{11} = 0.2$ and $p_{22} = 0.8$ and we compare different policies with the variance minimizing rule. In absence of switching $(a(1) = a(2) = 0.8)$, the dynamics of the Bode' s integral converges to zero (we are back to the linear framework). As we move leftwards, the variability between the models becomes important and the dynamics of design limits varies substantially according to the policy rule. Notice that the variance minimizing rule (the solid line case) always dominates all the other proposed rules while the further is the policy rule from the optimal one, the worst the performance in terms of design limits.

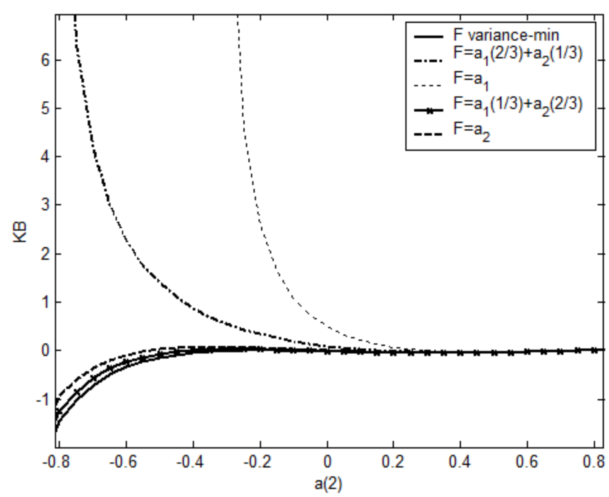

Figure 21: The dynamics of the Bode's integral constraint for different policies. $[a(1) = 0.8, p_{11} = 0.2, p_{22} = 0.8]$.

In general cases, however, there is no exact correspondence of the variance minimizing rule and the design limits minimizing one, as figures 22 and 23 show [15]. They depict the dynamics of the Bode's integral versus F.

[15] We do not make any precise statement on the eventual equivalence or relation between the policy rule which is optimal (variance-minimizing) and the rule which minimizes the Bode's Integral on purpose. Those links, are analyzed in details in a current work on progress of the author in which the above relation is considered in the much more general framework of the robust policies (*a' la* Hansen and Sargent(2008)) where the variance minimizing rule represents the limiting case where there is no concern for robustness.

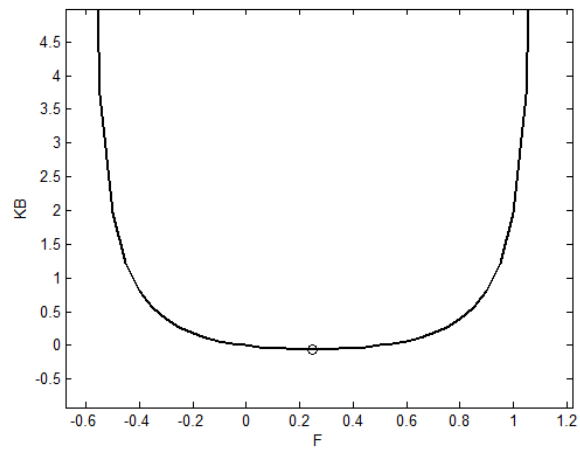

Figure 22: The dynamics of the Bode's integral across F with $a(1) = 0, a(2) = 0.5, p_{11} = 0.8$ and $p_{22} = 0.8$ (low switching probability case).

Figure 22 shows the values of the Bode's integral constraint when $a(1) = 0, a(2) = 0.5$ and the transition probabilities are such that there is a low probability of switching (0.2). The set of rules which correspond to minimum values of Bode contains the minimizing variance rule, so that no trade-offs between the minimization of the variance of the state and the minimization of the design limits arise. In more general cases related to the low switching probabilities, but no necessarily with equal p_{11} and p_{22}, it is possible to see how, even if the optimal policy does not correspond to the minimization of frequency trade-offs, among the stabilizing policies the shape of the Bode's integral is quite flat ensuring good values of performance.

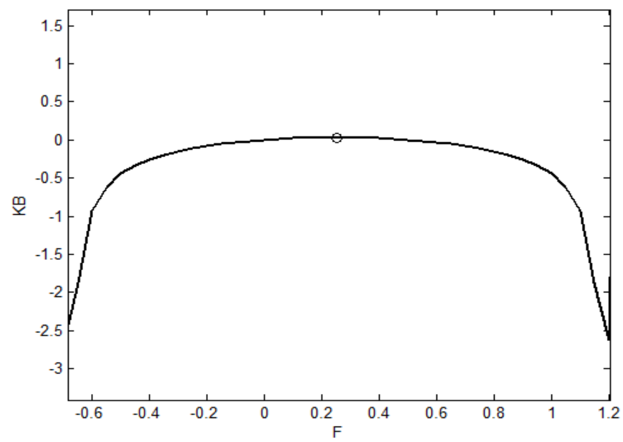

Figure 23: The dynamics of the Bode's integral across F with $a(1) = 0, a(2) = 0.5, p_{11} = 0.3$ and $p_{22} = 0.3$ (high switching probability case).

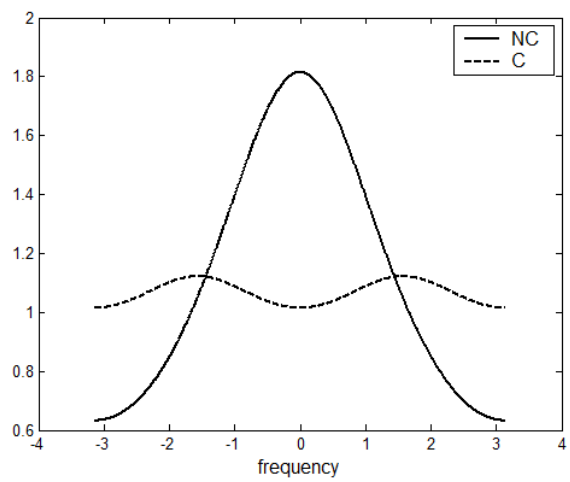

Figure 24: The spectral representation for the MSAR(1) with $a(1) = 0$, $a(2) = 0.5$, $p_{11} = 0.3$ and $p_{22} = 0.3$ (the solid line). The dashed line refers to the controlled case with $F = 0.25$ (variance minimizing rule).

Different is the case depicted by figure 23. In this high switching probability case, the optimal policy is among the ones which contribute most to increase design limits. To see why we compare the spectra of the controlled and uncontrolled cases when F corresponds to the optimal rule ($F = 0.25$) in figure 24. This case is important because it helps understanding the implications of conceiving the minimization of the design limits as an object. In low switching probability case, the minimization of the Bode's integral can be conceived as a plausible objective, since the optimal policy is always associated with the lowest values of the design limits (in figure 22 where the special case of equal p_{11} and p_{22} is considered, the two criteria indicate the same rule as optimal).

In presence of important amount of switching between the models, this is no more true. The main reason relies on the fact that, as already noticed, the stabilization has a price in terms of performance: figure 24 shows that the variance minimizing rule, while tends to flatten the frequency response of the model, it does so creating quite large frequency ranges where the contributions to the total fluctuations are exacerbated ($[1.2, \pi] \cup [-\pi, -1.2]$). This interval is larger than the one shown in figure 25 ($[2,\pi] \cup [-\pi, -2]$) where F is set to 1.2 which, according to figure 23, amongst the stabilizingones, implies low values of the Bode's integral.

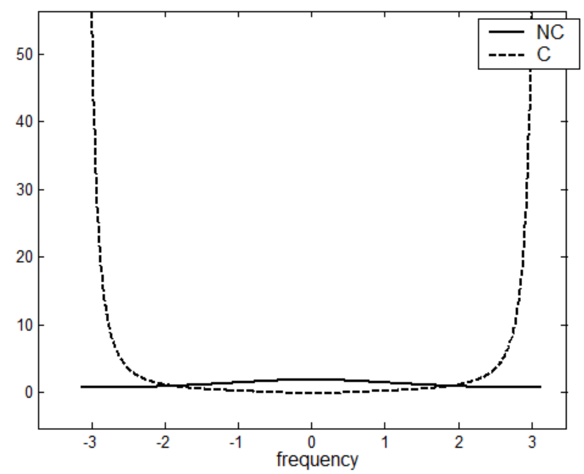

Figure 25: The spectral representation for the MSAR(1) with $a\,(1) = 0, a\,(2) = 0.5, p_{11} = 0.3$ and $p_{22} = 0.3$ (the solid line). The dashed line refers to the controlled case with $F = 1.2$ (design limits minimizing rule).

Recalling the general formula of the Bode's integral constraint (see Definition 1) and the fact that the derivative of the log function is much greater when the argument (of the log function) is smaller than one, we realize that, in general, the Bode's integral is a measure of frequency trade-offs which tends to overweight those intervals in which $\frac{f^c(\omega)}{f(\omega)} < 1$ and underweight the ones in which the relation is reversed.

However, general considerations on performance of the variance minimizing rule in terms of frequency trade-offs cannot be formulated and that its general characteristics depend, in particular, on the probability of the switching across the regimes.

This suggests that proposing the minimum Bode's integral as an object is certainly a too strong argument. Design limits may, nonetheless, constitute an important externality effect to take into consideration along with conventional monetary policy exercises.

6 Conclusion

In this paper we extend the theory of design limits in regime-switching contexts, deriving the analogous of the Bode's integral value, recently introduced in macroeconomic studies first by Brock and Durlauf (2004). The Bode's integral value quantifies the amount of the frequency trade-offs (also called design limits) the controller (typically, a central bank) has to face in stabilizing the economic system under exam. Positive values of the Bode's integral imply necessary exacerbations of the contribution of some frequency ranges to the total fluctuations of the model. Negative values do not imply the absence of frequency trade-offs, but denote less stringent limits.

The main message we want to convey is that the analysis of the design limits in regime switching cases can be viewed as a general framework in which, the linear case constitutes just a special case in which the potential models coincide.

While we described the general procedure, we show explicitly the computation for the simple case of a Markov Switching model with two possible AR(1) states where the policymaker has to chose a stabilizing, model-invariant rule before knowing the realization of the Markov Chain.

Two main features, peculiar to design limits in regime-switching cases are revealed. First, the Bode's integral value behavior strongly depends on the particular model considered. Second, different policy rules shape the spectral density of the process under exam affecting the measure of design limits. Therefore, contrary to the linear framework, design limits are not independent from the control of the policymaker.

Given the optimal variance-minimizing policy rule, the dynamics and the sign of the Bode's integral is strictly related to the probability of switching. When the transition probabilities of the Markov Switching model are such that, once the economy finds itself in one model is it very likely that it is going to stay rather than switch to the other model, the optimal policy is associated to less stringent frequency trade-offs and the value of the Bode's integral is typically negative. The key point is that there is an additional source of variability that the control may reduce: the variation-between the models. By that we refer to the variation due to the diverse characteristics of the underlying models. The more similar are those, the lower the variation-between. The more the control is able to reduce it, the lower the Bode integral value.

Different is the conclusion when the transition probabilities are such that there is an high unconditional probability of switching between the models. In those cases, the stabilization has a price in terms of frequency specific performance. The values of the Bode's integral are typically positive.

Model uncertainty in the form of Markov Switching regimes has, therefore, two main effects in terms of design limits. The introduction of an additional potential model enlarges the sources of variabilities that may be controlled so that the optimal rule is able to reduce the additional source of variability between the models and the Bode's integral is typically negative. The variance minimizing rule, however, cannot affect or reduce the variability due to the probability of switching because, by construction, it cannot modify the transition probabilities. It follows that in the cases in which the probability of switching is high, there is an important source of variation that cannot be controlled. In this case, the stabilization has a price in terms of frequency-specific performance.

The second peculiar characteristics of the Bode's integral value is its dependence on the policy rule. In this sense, the frequency-specific performance plays the role of an externality of the policymaker's action. This observation can open the way to a reconsideration of the Bode's integral constraint and a possible further extension in which the minimization of the frequency trade-offs may be associated a relative weight in the target vector of a monetary economic analysis. The examples shown make clear that general conclusions cannot be derived and that each particular case should be evaluated in order to derive policy relevant considerations.

In any case, we do not have the conceit to propose the minimization of the Bode's integral values as a pure object. However, we regard the communication of the frequency specific effects of any policy rule, including the knowledge of the design limits, as an important practise which should contribute to a more complete monetary authority's policy evaluation analysis.

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