Solution algorithm to a class of monetary rational equilibrium macromodels with optimal monetary policy design

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Abstract

In this paper an extended algorithm using well-known solution methods for monetary models characterized by rational expectations and optimal monetary policy design is given. The extension enables first the use of broad dynamic interdependencies within the structural model of the economy, second stochastic shocks on all endogenous variables and third commitment to a policy displaying no time inconsistency problem. All this points are not entirely new, but are seldom included into an operational solution algorithm. Furthermore a computational improvement concerning the splitting process for the stable and unstable part of the solution is proposed.

Keywords: multivariate rational equilibrium models, timeless perspective of optimal monetary policy, n-th order difference equation structural model

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1 Introduction

In this paper I present a generalization of the solution method to the policy optimization problem of a central bank confronted with a quadratic policy objective and a linear structural model of the underlying economy outlaid in Söderlind (1999). While Söderlind restricts himself to an economy, which first can be represented by a first-order difference equation and second includes only stochastic shocks on the state variables, I allow for a richer intertemporal interdependence by incorporating arbitrary sets of leads and lags and for a far more general pattern of uncertainty, since all variables may include stochastic components. Hence the structural model of the economy is given by an n-th order difference equation. The richer intertemporal structure does not impinge Söderlind's solution method fundamentally, since only the dimensions and structure of the first order condition of the central bank's problem need to be adjusted. But both the more general treatment of uncertainty and the system size implied by large economic systems calls for a change in the solution algorithm along the arguments in Sims (2001). Within this solution procedure I propose the use of a more robust reordering procedure for generalized eigenvalues based on the work of van Dooren (1981) and his successors instead of one particular algorithm often used in the monetary literature. The more flexible dynamic structure model permits the analysis of a broader class of economic models than the original method, e.g. the inclusion of debt contracts for financing investments in future capital. This enhancement is in line with Givens (2003) claiming a less burdensome method for more realistic models of the economy, but is far more general than his approach. The broader inclusion of uncertainty allows for shocks to non-predetermined variables, which may support the analysis of international or liquidity shocks, i.e. a unanticipated shock to the monetary policy instruments.

Additionally I take up the problem of inconsistency of optimal policy, first formulated in Kydland e.a. (1977), and a proposal for its solution from Woodford (2003). Tackling this problem, an alternative solution represents an optimal policy, which is characterized by the feature of consistency in the sense that it is not only optimal from the period, in which the policy is designed, onwards, but would have been also optimal from the perspective of an earlier period.

Despite theoretical links to the solution method of undetermined coefficients, as presented in McCallum (1998) and Uhlig (1999) among others, the solution presented hereafter does not call on these methods and diverge from those in not assuming a specific solution form, which is afterwards proofed to solve the problem. Nevertheless some relationships to the work of McCallum will be outpointed.

An implementation of the proposed solution based on Mathematica will be available online soon.

2 Model set-up

The structural model of the economy can be approximated by a system of difference equations up to n-th order, which can be written as

$$E_t\left(\mathbf{A}_n y_{t-i+n} + \mathbf{A}_{n-1} y_{t-i+n-1} + ... + \mathbf{A}_0 y_{t-i}\right) = -\mathbf{C} z_t. \tag{1}$$

The vector y^T,[1] can be divided into the components $(x_1, x_2, u)^T$, where the first subvector denotes predetermined, the second non-predetermined and the third one policy variables, respectively stochastic shocks to these variables. The dimensions of these subvectors are $(1, k)$, $(1, l)$ respectively $(1, j)$. The vector z_t denotes an exogenous stochastic shock vector with expectation value zero. The coefficients are square matrices of appropriate dimensions gathering the systems coefficients referring to the distinct periods. The coefficient matrix of the shock term $\mathbf{C}$ comprises all influences of the stochastic shock vector to endogenous variables. Regarding the time horizon of the model, this can include both lead and lag periods, since the maximal lag parameter i and the order parameter n can be chosen arbitrarily. Furthermore, with respect to the literature's classification of variables as predetermined and non-predetermined variables, one should recognize that each endogenous variable with a given initial value and an exogenously given prediction error is in fact a backward-looking and hence predetermined variable. Nevertheless, the difference between predetermined variables and non-predetermined variables is less clear than it seems to be, because in the major part of the literature the number of predetermined variables is related to the number of stable eigenvalues. If there are less predetermined variables than stable eigenvalues the resulting degrees of freedom, whose number is equivalent to the difference, can be used to impose exactly so many additional restrictions on the initial values of originally non-predetermined variables that the amounts of backward-looking variables and stable eigenvalues coincide. Using the distinction of predetermined variables and non-predetermined ones this paper conforms terminology with the major part of the existing literature despite the fact that, as we shall see later on, this distinction is not necessary for the solution method, which will be presented. In fact despite the use of the term predetermined variables the paper does not impose the assumption of an exogenously given expectational error.

Regarding the policy objective I assume that either it is given exogenously or can be derived from the utility representation underlying the economic model. It can be interpreted as a loss function of the central bank, which intends to minimize this loss given the structural model of the economy. The policy function is a quadratic form and can thus be written as

$$E_t \sum_{q}^{\infty} \beta^{t+q} \tilde{x}_{t+q}^T \mathbf{Q} \tilde{x}_{t+q}. \tag{2}$$

The vector $\tilde{x}^T$ is composed of n subvectors of the structure (x_1^T, x_2^T) with dimension $(1, k+l)$. Hence the policy function comprises the deviations of the endogenous variables

[1] y^T denotes the transpose of y.

in an arbitrary set of periods. The central bank is either able to commit to an announced policy or there is a policy alternative called the timeless policy perspective, which will presented in detail below.

3 Solution Algorithm

In order to solve the minimization problem of the central bank, i.e. min (2) s.t (1), the system of n-th order difference equations need to transformed into a system of first order difference equations. This can be done by using the standard technique of defining for each lead or lag of the endogenous variables a new variable consisting in a lead of its predecessor for all leads down to y_{t+q+1} and in a lag of its successor for all lags up to y_{t+q}, substituting this in the structural model equation and adding the definition as an additional constraint to the original problem.[2] While the newly defined variables corresponding to leads constitute non-predetermined variables, those for lags can be characterized as predetermined variables. In order to derive a so called solution of timeless perspective some constraints on the initial values of non-predetermined variables can be added.[3] Expressing the problem above in term of a Lagrange function, we obtain

$$E_t \sum_{q=0}^{\infty} \beta^{t+q} \left[-\tilde{x}_{t+q}^T \mathbf{Q} \tilde{x}_{t+q} + \varphi_{t+q+1}^T \left(\mathbf{A}_n y_{t+q}^n + ... + \mathbf{A}_{n-i+1} y_{t+q+1}^{n-i} + \mathbf{A}_{n-i} y_{t+q}^{n-i} .... + \mathbf{A}_0 y_{t+q}^0 \right.\right. \tag{3}$$

$$\left. + \mathbf{C} z_{t+q} \right) + \sum_{f=0}^{i-3} \varphi_{t+q+1}^{n-f\;T} \left(y_{t+q}^{n-f} - y_{t+q+1}^{n-f-1} \right) + \varphi_{t+q+1}^{n-i+2T} \left(y_{t+q}^{n-i+2} - y_{t+q+1}^{n-i} \right)$$

$$\left. + \sum_{f=i+1}^{n} \varphi_{t+q}^{n-fT} \left(y_{t+q}^{n-f} - y_{t+q-1}^{n-f+1} \right) + \varphi_t^T y_t \right],$$

which has to be maximized across each period's current endogenous variables and one period leads of Lagrange multipliers and the Lagrange multipliers of the initial period. Note that each Lagrange multiplier φ^l can be partitioned with respect to predetermined, non-predetermined and policy variables obtaining $(\varphi_1^l, \varphi_2^l, \varphi_3^l)^T$. Note additionally that the constraint on the initial non-predetermined variables can be written as

$$E_{t-1}(x_{2t}) = e = E_{t-1} \left[f + f_{x_1} x_{1(t-1)} + f_{x_2} x_{2(t-1)} + f_z z_t \right] \tag{4}$$

[2]Cf. Galor (2003), p.50ff.

[3]The term timeless perspective indicates that the expectations concerning non-predetermined variables are of a constant form independently from the period in which they are formed. This holds even in periods, which are prior to the first period of the optimization horizon. Economically this means that the central bank pays attention to the public's past expectations concerning the current situation, even if these have no influence on the decision of the individuals today (Cf. Woodford (2003), p. 540ff.). An alternative interpretation is given in McCallum (2000) according to which the central bank designs its optimal policy under the assumption that the current macroeconomic conditions are not the actual, but the expectation for those formed in earlier periods. Note also that, from technical point of view, doing this qualifies the regarding variables as predetermined ones.

indicating that the expectation for the first period's value of these variables are formed on the basis of past information. Trivially this holds for all lagged values for these variables as well. The consistency of the expectation formation implicates that its form will hold across the whole future time horizon, since only the information level will change to the information of the period t. Thus the solution itself will pin down the value for the initial period's non-predetermined variables.[4] In case of a steady state the expectation for the non-predetermined variables in the initial period depends exclusively on the expected stochastic shocks in this period.
Partitioning the matrix $\mathbf{Q}$ with respect to the n subvectors of $\tilde{x}^T$ referring to different periods yields

$$\begin{pmatrix} \mathbf{Q}_{11} & \dots & \mathbf{Q}_{1m} \\ \vdots & \ddots & \vdots \\ \mathbf{Q}_{m1} & \dots & \mathbf{Q}_{mm} \end{pmatrix}, \tag{5}$$

of which each blockmatrix can again be partitioned according to the subvector of predetermined and non-predetermined variables, i.e. x_1^T and x_2^T. These can be used to define the new matrices

$$\tilde{\mathbf{Q}}_{ij} \equiv \begin{pmatrix} \mathbf{Q}_{ij} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \forall\, i \wedge j \in \{1, ..., m\} \tag{6}$$

with dimension $(k+l+j, k+l+j)$. Joining these matrices into a matrix of the same structure than $\mathbf{Q}$ yields the matrix $\tilde{\mathbf{Q}}$ with dimension $(n(k+l+j), n(k+l+j))$. Hence we have constructed the coefficient matrix of a modified policy objective, in which the vector of endogenous variables consists of subvectors, referring to one period, of exactly the same dimensions as the vectors within the structural model of the economy. Equipped with

[4]Generally, an exclusive dependence of non-predetermined variables on predetermined ones is by no means a necessary condition for the existence of a solution for a linear model. As Sims (2001) shows, there may be solutions in which there is no such exclusive linear dependence. In this case the introduction of timeless policy perspective obscure the difference between predetermined and non-predetermined variables, since expectations for both are functions of prior experience.

these notational conventions solving the problem above yields the first order conditions

$$
\begin{pmatrix}
\mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{A}_n^T & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \vdots & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{A}_{n-i+2}^T & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{A}_{n-i}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \beta\mathbf{I} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{A}_{n-i-1}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \beta\mathbf{I} \\
\mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \vdots & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} & & \\
\mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{A}_0^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \cdots & \mathbf{0} & -\mathbf{A}_{n-i+1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{I} & & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & & & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \beta\mathbf{I} & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{pmatrix}
E_t
\begin{pmatrix}
y^n_{t+q+1} \\ \vdots \\ y^{n-i+2}_{t+q+1} \\ y^{n-i}_{t+q+1} \\ y^{n-i-1}_{t+q+1} \\ \vdots \\ y^0_{t+q+1} \\ \varphi_{t+q+1} \\ \varphi^n_{t+q+1} \\ \vdots \\ \varphi^{n-i+2}_{t+q+1} \\ \varphi^{n-i-1}_{t+q+1} \\ \vdots \\ \varphi^0_{t+q+1}
\end{pmatrix}
= \qquad (7)
$$

$$
\begin{pmatrix}
 & & & & & & & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
 & & & & & & & \vdots & \multicolumn{2}{c}{-\beta^{-1}\mathbf{I}} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
 & & & & & & & \mathbf{0} & & & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
 & & & -2\tilde{\mathbf{Q}}^T & & & & \beta^{-1}\mathbf{A}_{n-i+1}^T+\gamma\mathbf{I} & \mathbf{0} & \cdots & -\beta^{-1}\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\
 & & & & & & & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & & & \\
 & & & & & & & \vdots & \mathbf{0} & \cdots & \mathbf{0} & & \mathbf{I} & \\
 & & & & & & & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & & & \\
\mathbf{A}_n & \cdots & \mathbf{A}_{n-i+2} & \mathbf{A}_{n-i} & \mathbf{A}_{n-i-1} & \cdots & \mathbf{A}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
 & & & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
 & \mathbf{I} & & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
 & & & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \cdots & \mathbf{0} & & & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \cdots & \mathbf{0} & & \beta\mathbf{I} & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \cdots & \mathbf{0} & & & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{pmatrix}
$$

$$
\cdot E_t \begin{pmatrix} y^n_{t+q} & \cdots & y^{n-i+2}_{t+q} & y^{n-i}_{t+q} & y^{n-i-1}_{t+q} & \cdots & y^0_{t+q} & \varphi_{t+q} & \varphi^n_{t+q} & \cdots & \varphi^{n-i+2}_{t+q} & \varphi^{n-i-1}_{t+q} & \cdots & \varphi^0_{t+q} \end{pmatrix}^T
$$
$$
+\tilde{\mathbf{C}}E_t(z_{t+q}),
$$

where $\gamma = 0 \; \forall \; q \in \mathbb{N}^{++}$ and $\gamma = 1$ iff $q = 0$.[5] The matrix $\tilde{\mathbf{C}}$ has dimension $((2n+1)(k+l+j), m)$, where m is the row dimension of z_{t+q}, but only one nonzero submatrix, which is exactly the matrix $\mathbf{C}$ and starts at position $(n(k+l+j)+1, 1)$. Furthermore, note that if $q - i < 0$, the according endogenous variables take on the value zero such that all corresponding entries in the matrices are without influence.

In order to gather all predetermined variables in the upper part of this system, it should be notified that the predetermined variables include all variables with given and known exogenous disturbance terms at the time of optimization and no further influences.[6] The

[5]The parameter γ can be detected in the fourth row of the coefficient matrix on the equation's right side. It implies, that the referring shadow price does influence the solution only in the case of the first period.

[6]Note that with respect to this principle the order of the variables can not be fixed completely, since the correct reordering depends on the characteristics of the underlying economic relationships, specifically

new system can be written compactly as

$$\mathbf{G}\begin{pmatrix} v_{t+q+1} \\ w_{t+q+1} \end{pmatrix} = \mathbf{H}\begin{pmatrix} v_{t+q} \\ w_{t+q} \end{pmatrix} + \tilde{\mathbf{C}}' z_{t+q} + \tilde{\mathbf{G}} e_{t+q+1} \tag{8}$$

with v defined as $(x_1^n, \ldots, x_1^0, x_2^{n-i-1}, \ldots, x_2^0, u^{n-i-1}, \ldots, u^0, \varphi_2^n, \ldots, \varphi_2^{n-i}, \varphi_3^n, \ldots, \varphi_3^{n-i})^T$ and the vector w as $(x_2^n, \ldots, x_2^{n-i}, u^n, \ldots, u^{n-i}, \varphi_1^n, \ldots, \varphi_1^0, \varphi_2^{n-i-1}, \ldots, \varphi_2^0, \varphi_3^{n-i-1}, \ldots, \varphi_3^0)^T$, $\tilde{\mathbf{G}}$ defined as the matrix consisting of the difference between the columns of $\mathbf{G}$ and $\mathbf{H}$ belonging to those variables for which there may be an expectation error, i.e. all variables from superindex $n-i$ upwards, sorted according to matrix $\mathbf{G}$ and e_{t+q+1} denoting the appropriate expectational error between period $t+q$ and $t+q+1$. The distinct partition for the vector of endogenous variables is used in order to exploit the fact that in the initial period the shadow prices on the budget constraint can be either set to zero, because there is no constraint on the choice of the non-predetermined variables, or set to a predetermined value, which incorporates the smoothness of expectations over time within the model.[7] Moreover, the non-predetermined variables in all prior periods obey to the same solution as that of the initial period, since in the initial steady state both the fundamentals and the expectation values with respect to shocks have been constant. Hence the referring lagrange multipliers can be chosen freely and are non-predetermined variables. Note additionally that all leads in the original structural model are non-predetermined variables, for which neither the shadow-prices nor the variables themselves respond to exogenously determined shocks. The coefficient $\tilde{\mathbf{C}}'$ is just reordered according to the new vector of endogenous variables.[8]

From now on we follow almost strictly the steps described in the fourth section of Sims (2001). [9] Hence use either the eigenvalue (for a non-singular matrix $\mathbf{G}$) or the generalized Schur method (for a singular matrix $\mathbf{G}$) to solve the generalized eigenvalue problem resulting from equation (8). The second method (the more general one) gives the four new matrices $\mathbf{T}$, $\mathbf{S}$, $\mathbf{R}$ and $\mathbf{P}$, for which the equations

$$\mathbf{G} = \mathbf{R} \cdot \mathbf{S} \cdot \mathbf{P}^H \tag{9}$$

$$\mathbf{H} = \mathbf{R} \cdot \mathbf{T} \cdot \mathbf{P}^H \tag{10}$$

the dependence of future state variables on current or lagged values or of non-predetermined variables. Especially in a model with many lags end leads this might become relatively complicated. The reordering below is based on the assumption that state variables are independent of non-predetermined variables.

[7]The second alternative arises in the case that a timeless optimal policy in the sense of Woodford (2003), i.e. the consistency of expectation formation before and after the policy decision, is derived. Under such a policy regime the non-predetermined endogenous variables in the initial period are given as function of both the predetermined variables and the stochastic shocks of the same period (Cf. (4)). This incorporates the conformation of the central bank to choose a policy, which should have been expected from the public, if the question of optimal policy had been considered at an earlier date. Cf. Woodford (2003), p. 538ff.

[8]In some model the coefficient $\tilde{\mathbf{C}}'$ may be a function of the coefficients $\mathbf{G}$ and $\mathbf{H}$.

[9]The method proposed in Blanchard e.a. (1981) can not be used, since matrix $\mathbf{G}$ is not necessarily inverse. While theoretically the procedures in the second section of Söderlind (1999) and the fifth section of Klein (2000) may be used, we will see that this procedure, mainly based on the work of Klein (2000) may not produce a solution, while there still may exist one.

hold. The two matrices $\mathbf{R}$ and $\mathbf{P}$ are unitary and the matrix $\mathbf{P}^H$ denotes the transpose of the complex conjugate of $\mathbf{P}$. Moreover the remaining two matrices are upper triangular. Hence premultiplying (8) with $\mathbf{R}^H$ yields

$$\mathbf{S} \cdot \mathbf{P}^H \begin{pmatrix} v_{t+q+1} \\ w_{t+q+1} \end{pmatrix} = \mathbf{T} \cdot \mathbf{P}^H \begin{pmatrix} v_{t+q} \\ w_{t+q} \end{pmatrix} + \mathbf{R}^H \tilde{\mathbf{C}}' z_{t+q} + \mathbf{R}^H \tilde{\mathbf{G}} e_{t+q+1}. \tag{11}$$

This system can be rewritten by gathering all stable generalized eigenvalues, which consist in the ratios between all corresponding diagonal entries of the matrices $\mathbf{T}$ and $\mathbf{S}$ above one. Therefore, just reshape the system according to the inverse order of the absolute value of the generalized eigenvalues.[10] Additionally defining the new vectors

$$\begin{pmatrix} \tilde{v}_{t+q} \\ \tilde{w}_{t+q} \end{pmatrix} = \tilde{\mathbf{P}}^H \begin{pmatrix} v_{t+q} \\ w_{t+q} \end{pmatrix}, \tag{12}$$

substituting them back into the reshaped system and partioning the new coefficient matrices $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{T}}$ according to their accompanying vectors yields

$$\begin{pmatrix} \tilde{\mathbf{S}}_{11} & \tilde{\mathbf{S}}_{12} \\ \mathbf{0} & \tilde{\mathbf{S}}_{22} \end{pmatrix} \begin{pmatrix} \tilde{v}_{t+q+1} \\ \tilde{w}_{t+q+1} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{T}}_{11} & \tilde{\mathbf{T}}_{12} \\ \mathbf{0} & \tilde{\mathbf{T}}_{22} \end{pmatrix} \begin{pmatrix} \tilde{v}_{t+q} \\ \tilde{w}_{t+q} \end{pmatrix} + \begin{pmatrix} \tilde{\mathbf{R}}_1^H \\ \tilde{\mathbf{R}}_2^H \end{pmatrix} \tilde{\mathbf{C}}' z_{t+q} + \begin{pmatrix} \tilde{\mathbf{R}}_1^H \\ \tilde{\mathbf{R}}_2^H \end{pmatrix} \tilde{\mathbf{G}} e_{t+q+1}.^{11} \tag{13}$$

Since all unstable generalized eigenvalues are gathered within the lower block of the system, stability requires the vector $\tilde{w}_{t+q}$ to balance the exogenous and endogenous fluctuations for all periods $q \in \mathbb{N}^+$. To yield a result for the non-predetermined variables the lower part can be solved forward to

$$\tilde{w}_t \;=\; -\sum_{i=0}^{\infty} (\tilde{\mathbf{T}}_{22}^{-1} \tilde{\mathbf{S}}_{22})^i \tilde{\mathbf{T}}_{22}^{-1} \tilde{\mathbf{R}}_2^H (\tilde{\mathbf{C}}' z_{t+i} + \tilde{\mathbf{G}} e_{t+i+1}). \tag{14}$$

Subtracting the according expectation value based on period t information we are left with

$$\sum_{i=0}^{\infty} (\tilde{\mathbf{T}}_{22}^{-1} \tilde{\mathbf{S}}_{22})^i \tilde{\mathbf{T}}_{22}^{-1} \tilde{\mathbf{R}}_2^H \tilde{\mathbf{G}} e_{t+i+1} = \sum_{i=0}^{\infty} (\tilde{\mathbf{T}}_{22}^{-1} \tilde{\mathbf{S}}_{22})^i \tilde{\mathbf{T}}_{22}^{-1} \tilde{\mathbf{R}}_2^H \tilde{\mathbf{C}}' (E_t(z_{t+i}) - z_{t+i}). \tag{15}$$

[10]In order to do this I propose the use of the Matlab version of an algorithm provided by Anderson e.a. (1996) instead of the widely used Matlab algorithm qzswitch originally proposed by Sims (2001). For large matrices the second one tends to produce a change in the absolute values of the generalized eigenvalues within the reordering process, if the absolute value is near the extremes of its domain. Furthermore, there are cases in which the algorithm can not assure a complete resorting despite the regularity of the underlying matrix-pencil. (Details with respect to these errors are given in the appendix.) Contrary the first one is an implementation of the methods outlined in van Dooren (1981) and Kagström e.a. (1994), which accounts for the invariance of the unordered subset of eigenvalues within the reordered Schur decompositions. Nevertheless, any algorithm should be carefully checked for its performance before usage, since many algorithms seem to display serious deficiencies.

[11]In the early literature on solution methods for RE models, e.g. Blanchard e.a. (1980), the coincidence of the numbers of predetermined variables and stable eigenvalues has been seen as necessary for the uniqueness of the solution. But the possibility of transferring non-predetermined variables to predetermined ones in order to assure the invertibility of $\tilde{\mathbf{S}}_{\mathbf{11}}$ mentioned in Klein (2000), the fact that the number of initial conditions contains all information for an unique solution as shown in Boyd e.a. (1990) and the missing influence of the distinction between predetermined variables and nonpredetermined ones on the problems's general solution, which becomes clear in Sims (2001), has shown that this root-counting does not pin down necessary conditions for uniqueness.

Differencing the result with its successor yields

$$\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}e_{t+1} = \sum_{i=1}^{\infty}(\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{S}}_{22})^i\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}}'(E_t(z_{t+i}) - E_{t+1}(z_{t+i})). \tag{16}$$

Recalling that the expectation value for all former exogenous stochastic shocks is zero, this expression simplifies to

$$\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}e_{t+1} = -\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}}'z_{t+1}. \tag{17}$$

Given the assumed first moment of the distribution of stochastic shocks this equation gives implicitly a necessary and sufficient condition for the existence of a converging solution to (8): if and only if the column space of the left contains that of the right side at least one solution exists.[12] If this basic condition does not hold, the unstable part can not be stabilized and there are only explosive solutions for the system.

From now one principally two possibilities to solve for the stable part of the system exist. The first one arises, if the equation

$$\tilde{\mathbf{R}}_1^H\tilde{\mathbf{G}}e_{t+q+1} = \mathbf{\Phi}\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}e_{t+q+1} \tag{18}$$

implies consistent solutions for the matrix $\mathbf{\Phi}$. In this case we can again follow Sims (2001) and premultiply (13) by the matrix $\begin{pmatrix}\mathbf{I} & -\mathbf{\Phi}\end{pmatrix}$. Joining the result with the one period updated version of (14)and substituting the endogenous expectation error in the latter by (15) we obtain a new system which can be solved to

$$\begin{aligned}
\begin{pmatrix} v_{t+q+1} \\ w_{t+q+1} \end{pmatrix} \quad = \quad & \begin{pmatrix} \tilde{\mathbf{P}}_{11}\tilde{\mathbf{S}}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{P}}_{21}\tilde{\mathbf{S}}_{11}^{-1} \end{pmatrix} \cdot \\
& \begin{pmatrix} \tilde{\mathbf{T}}_{11}\tilde{\mathbf{P}}_{11}^H + \left(\tilde{\mathbf{T}}_{12} - \mathbf{\Phi}\tilde{\mathbf{T}}_{22}\right)\tilde{\mathbf{P}}_{21}^H & \tilde{\mathbf{T}}_{11}\tilde{\mathbf{P}}_{12}^H + \left(\tilde{\mathbf{T}}_{12} - \mathbf{\Phi}\tilde{\mathbf{T}}_{22}\right)\tilde{\mathbf{P}}_{22}^H \\ \tilde{\mathbf{T}}_{11}\tilde{\mathbf{P}}_{11}^H + \left(\tilde{\mathbf{T}}_{12} - \mathbf{\Phi}\tilde{\mathbf{T}}_{22}\right)\tilde{\mathbf{P}}_{21}^H & \tilde{\mathbf{T}}_{11}\tilde{\mathbf{P}}_{12}^H + \left(\tilde{\mathbf{T}}_{12} - \mathbf{\Phi}\tilde{\mathbf{T}}_{22}\right)\tilde{\mathbf{P}}_{22}^H \end{pmatrix} \begin{pmatrix} v_{t+q} \\ w_{t+q} \end{pmatrix} \\
& + \begin{pmatrix} \tilde{\mathbf{P}}_{11}\tilde{\mathbf{S}}_{11}^{-1}\left(\tilde{\mathbf{R}}_1^H - \mathbf{\Phi}\tilde{\mathbf{R}}_2^H\right) \\ \tilde{\mathbf{P}}_{21}\tilde{\mathbf{S}}_{11}^{-1}\left(\tilde{\mathbf{R}}_1^H - \mathbf{\Phi}\tilde{\mathbf{R}}_2^H\right) \end{pmatrix} \tilde{\mathbf{C}}z_{t+q} \\
& + \begin{pmatrix} \tilde{\mathbf{P}}_{11}\tilde{\mathbf{S}}_{11}^{-1}\left(\tilde{\mathbf{S}}_{12} - \mathbf{\Phi}\tilde{\mathbf{S}}_{22}\right) - \tilde{\mathbf{P}}_{12} \\ \tilde{\mathbf{P}}_{21}\tilde{\mathbf{S}}_{11}^{-1}\left(\tilde{\mathbf{S}}_{12} - \mathbf{\Phi}\tilde{\mathbf{S}}_{22}\right) - \tilde{\mathbf{P}}_{22} \end{pmatrix} \sum_{s=1}^{\infty}(\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{S}}_{22})^{s-1}\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}}'E_{t+1}(z_{t+q+s}).
\end{aligned} \tag{19}$$

For unique $\mathbf{\Phi}$ this solution is also unique, otherwise there is a infinite number of possible solutions. For uniqueness again the row space of the right side of must contain the row

[12] Cf. Sims (2001), pp.11,13. Note that the relationship of this criterion to the winding-number criterion in Onatski (2006) remains somewhat obscure, because the criterion for existence of a solution presented above uses the information contained in the shock term, while the method of Onatski (2006) relies exclusively on information contained in the non-stochastic summands of the model equation. Of course, ultimately the coefficient of the stochastic summand is derived from the information contained in the non-stochastic summands. Nevertheless there remains a difference to Onatski (2006) in using this information a second time.

space of the left side. Sims (2001) presents a method for testing for uniqueness and solving for the unique solution.[13] If the matrix $\mathbf{\Phi}$ should be not fully determined, we can nevertheless construct the set of its solutions by using the concept of the pseudoinverse, which yields

$$\mathbf{\Phi} = (\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+\tilde{\mathbf{R}}_1^H\tilde{\mathbf{G}} + (\mathbf{I} - (\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})^+\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}})\mathbf{Z}, \tag{20}$$

where $\mathbf{Z}$ is an arbitrary matrix of appropriate dimensions. Is is worthy to note that the condition for a solution within the solution method in Klein (2000), i.e. invertibility of $\mathbf{P}_{11}$, is reflected in this formula. In order to see this, recall the definition of predetermined variables, which states that those variables have an expectational error of zero. Hence the last term within the coefficient for the stochastic shock would vanish and premultiplying the upper part of the system with the mentioned inverse reveals that the non-predetermined variables can be given as a function of the predetermined ones.[14] Furthermore, in the mentioned case the general solution of Sims (2001) collapses to the solution in Klein (2000). Another interesting special case of this solution is characterized by the condition

$$\tilde{\mathbf{R}}_1^H\tilde{\mathbf{G}}e_{t+q+1} = 0. \tag{21}$$

If this holds the unique solution to $\mathbf{\Phi}$ is the zeromatrix with appropriate dimensions. Hence, again, the solution collapses to a simpler form.
The second solution method is the more general one, which does reveal the full set of potential multiple solutions. Assuming

$$e_{t+q+1} = \mathbf{\Lambda} z_{t+q}, \tag{22}$$

one can write (17) as

$$\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}\mathbf{\Lambda} = -\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}}. \tag{23}$$

Since the possible case that $\dim(e_{t+q+1}) > \dim(z_{t+q})$ indicates potential indeterminacy, we can follow the arguments in Lubik e.a. (2003) and split the endogenous expectation error in a part purely determined by exogenous fluctuations of fundamentals and in a sunspot part. Therefore we get

$$e_{t+q+1} = e_{1(t+q+1)}(z_{t+q}) + e_{2(t+q+1)} = \mathbf{\Lambda}_1 z_{t+q} + \mathbf{\Lambda}_2\zeta_{t+q}. \tag{24}$$

Since the second term on the right side is purely sunspot is does not contribute to the explanation of (23) and therefore it is orthogonal to $\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}$. Hence

$$\tilde{\mathbf{R}}_2^H\tilde{\mathbf{G}}(\mathbf{\Lambda} z_{t+q} + \mathbf{\Lambda}_2\zeta_{t+q}) = -\tilde{\mathbf{R}}_2^H\mathbf{C}z_{t+q}, \tag{25}$$

[13] Cf. Sims (2001), p.13.

[14] For these reasons the conditions for the existence of an unique equilibrium given in Klein (2000) are actually sufficient but not necessary. Cf. Klein (2000), p. 418f. For the same reasons the so called minimal-state-variables solution of McCallum (1998) does not necessarily provide a method to find the bubble-free solution.

where $\mathbf{\Lambda}_1 = \mathbf{\Lambda}$ holds due to the mentioned orthogonality. Ignoring the second term on the left side for the moment, it is easy to solve for $\mathbf{\Lambda}$ by using once again the concept of the pseudoinverse. We get

$$\mathbf{\Lambda} = -(\tilde{\mathbf{R}}_2^H \tilde{\mathbf{G}})^+ \tilde{\mathbf{R}}_2^H \tilde{\mathbf{C}} - (\mathbf{I} - (\tilde{\mathbf{R}}_2^H \tilde{\mathbf{G}})^+ \tilde{\mathbf{R}}_2^H \tilde{\mathbf{G}})\mathbf{Z}, \tag{26}$$

where $\mathbf{Z}$ is an arbitrary matrix of appropriate dimensions. Recalling the second term on the left side of (25) and multiplying the last result by $\tilde{\mathbf{R}}_2^H \tilde{\mathbf{G}}$ we observe that this operation yields exactly (25), whereby the second term on the right side fulfills the same function as the second term on the left side of (25). Hence we can interpret the second term of the right side of our last result as a pure sunspot term, which does not contribute to the explanation of the existence of a solution, but nevertheless influences the reaction form of one specific out of the continuum of multiple solutions. Thus the solution are formed by a systemic component and a purely random component $\mathbf{Z}$, which stands for self-fulfilling expectations of the economic agents.
Following the same steps as above while substituting $\tilde{\mathbf{S}}_{12}$ for $\mathbf{\Phi}$, obtains the full set of potential solutions

$$\begin{aligned}
\begin{pmatrix} v_{t+q+1} \\ w_{t+q+1} \end{pmatrix} = &\begin{pmatrix} \tilde{\mathbf{P}}_{11}\tilde{\mathbf{S}}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{P}}_{21}\tilde{\mathbf{S}}_{11}^{-1} \end{pmatrix} \cdot \\
&\begin{pmatrix} \tilde{\mathbf{T}}_{11}\tilde{\mathbf{P}}_{11}^H + \left(\tilde{\mathbf{T}}_{12} - \tilde{\mathbf{S}}_{12}\tilde{\mathbf{T}}_{22}\right)\tilde{\mathbf{P}}_{21}^H & \tilde{\mathbf{T}}_{11}\tilde{\mathbf{P}}_{12}^H + \left(\tilde{\mathbf{T}}_{12} - \tilde{\mathbf{S}}_{12}\tilde{\mathbf{T}}_{22}\right)\tilde{\mathbf{P}}_{22}^H \\ \tilde{\mathbf{T}}_{11}\tilde{\mathbf{P}}_{11}^H + \left(\tilde{\mathbf{T}}_{12} - \tilde{\mathbf{S}}_{12}\tilde{\mathbf{T}}_{22}\right)\tilde{\mathbf{P}}_{21}^H & \tilde{\mathbf{T}}_{11}\tilde{\mathbf{P}}_{12}^H + \left(\tilde{\mathbf{T}}_{12} - \tilde{\mathbf{S}}_{12}\tilde{\mathbf{T}}_{22}\right)\tilde{\mathbf{P}}_{22}^H \end{pmatrix} \begin{pmatrix} v_{t+q} \\ w_{t+q} \end{pmatrix} \\
&+ \begin{pmatrix} \tilde{\mathbf{P}}_{11}\tilde{\mathbf{S}}_{11}^{-1}\tilde{\mathbf{R}}_1^H(\tilde{\mathbf{C}} + \tilde{\mathbf{G}}\mathbf{\Lambda}) \\ \tilde{\mathbf{P}}_{21}\tilde{\mathbf{S}}_{11}^{-1}\tilde{\mathbf{R}}_1^H(\tilde{\mathbf{C}} + \tilde{\mathbf{G}}\mathbf{\Lambda}) \end{pmatrix} z_{t+q} \\
&+ \begin{pmatrix} \tilde{\mathbf{P}}_{11}\tilde{\mathbf{S}}_{11}^{-1}\left(\tilde{\mathbf{S}}_{12} - \tilde{\mathbf{S}}_{12}\tilde{\mathbf{S}}_{22}\right) - \tilde{\mathbf{P}}_{12} \\ \tilde{\mathbf{P}}_{21}\tilde{\mathbf{S}}_{11}^{-1}\left(\tilde{\mathbf{S}}_{12} - \tilde{\mathbf{S}}_{12}\tilde{\mathbf{S}}_{22}\right) - \tilde{\mathbf{P}}_{22} \end{pmatrix} \sum_{s=1}^{\infty} (\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{S}}_{22})^{s-1}\tilde{\mathbf{T}}_{22}^{-1}\tilde{\mathbf{R}}_2^H\tilde{\mathbf{C}}\prime E_{t+1}(z_{t+q+s}).
\end{aligned} \tag{27}$$

The indeterminacy of the general solution is due to the undetermined matrix $\mathbf{\Lambda}$, which incorporates random fluctuations in the expectations of economic agents into the solution. As it can bee seen in the different solution possibilities, the question of removing indeterminacy boils down to altering the number of rows in one of the equations (18) or (23) in order to secure one of them to be fully determined and consistent. Hence a necessary and sufficient condition for uniqueness is the full rank property of the matrix $\tilde{\mathbf{R}}_2^H \tilde{\mathbf{G}}$. Despite the potential indeterminacy in the solution of the model, we can nevertheless use the central idea of choosing a bubble-free solution propagated in McCallum (1981, 1998).[15] As shown above the coefficients for the stochastic shock terms in the presented solution can be split in a systemic part, which secures the existence of the solution, and a part, which does not contribute to the explanation of the existence of an solution but includes some additional fluctuation in the solution. As it have been argued, this part can be

[15] Note that despite using the idea of the bubble-free solution the presented solution algorithm does not employ the picking mechanism proposed by McCallum (1981, 1998), i.e. the minimal-state-variable solution, since this mechanism will not work with general matrices as already discussed in footnote 14.

interpreted a pure sunspot shock representing self-fulfilling expectations without fundamental economic reasoning. Eliminating this part by choosing a zeromatrix for **Z** yields an unique bubble-free solution, which depends only on economic fundamentals.

4 Conclusions

In this paper a solution method for a rational expectation equilibrium of an economy, in which a central authority minimizes the loss from a policy objective given some structural equations for the rest of the economy, has been outlaid. While this approach is already well known in the literature, the paper extends the standard method to the case that there is a high degree of intertemporal interdependecies within the structural model of the economy. It therefore comprises the proposal of Givens (2003) for a more realistic structural model than in Söderlind (1999). The approach also avoids the restriction in Söderlind (1999), that stochastic shocks exclusively meet predetermined variables. Additionally by employing the algorithm of Sims (2001) it uses more general existence and uniqueness conditions as Klein (2000) and therefore enables the solution of a larger set of economic systems. Furthermore, it accounts for the fact that an optimal policy as discussed in the standard literature is characterized by an inconsistency problem. This problem can be resolved by altering the optimal policy into an optimal policy from a timeless perspective described in Woodford (2003), which incorporates the rational expectations of the public for non-predetermined variables beyond the initial period, hence assures the smoothness of expectations across the entire time horizon and pays attention to the history dependency of expectations.

As an additional technical result the evaluation of the widely spread matlab-code qzswitch showed that this algorithm does not secure a complete reordering of the generalized eigenvalues for all regular matrix pencils and, additionally, imposes some changes on the generalized eigenvalues relatively close to the extremes of the stability continuum. At least the first finding imposes a serious obstacle for the analysis of large economic systems. Therefore the use of competing algorithms might facilitate economic analysis of elaborate economic models.

A Appendix - evaluation of the algorithm qzswitch

In order to show the source of the malfunction within the algorithm qzswitch we first notify that the whole reordering process is a loop consisting in repetitions of the same step, in which it is first checked, if the general eigenvalue formed by the main diagonal elements of the same row within the matrices **S** and **T** is greater than its successor, i.e. the generalized eigenvalue according to the next row. If this is true the mentioned algorithm is performed and the loop returns to the former step. If not the loop goes forward to the next step. Each step itself is formed by the mentioned algorithm.

The algorithm seems to follows the idea to preserve the original matrices $\mathbf{G}$ and $\mathbf{H}$ while modifying their generalized schur decompositions. Therefore it starts with the decompositions

$$\mathbf{G} = \mathbf{R} \cdot \mathbf{S} \cdot \mathbf{P}^H \tag{28}$$
$$\mathbf{H} = \mathbf{R} \cdot \mathbf{T} \cdot \mathbf{P}^H. \tag{29}$$

Next two unitary matrices $\mathbf{xy}$ and $\mathbf{wz}$ are defined, which enable to write the above expressions as

$$\mathbf{G} = \mathbf{R} \cdot \mathbf{xy}^H \cdot \mathbf{xy} \cdot \mathbf{S} \cdot \mathbf{wz} \cdot \mathbf{wz}^H \cdot \mathbf{P}^H \tag{30}$$
$$\mathbf{H} = \mathbf{R} \cdot \mathbf{xy}^H \cdot \mathbf{xy} \cdot \mathbf{T} \cdot \mathbf{wz} \cdot \mathbf{wz}^H \cdot \mathbf{P}^H. \tag{31}$$

Defining the new matrices $\tilde{\mathbf{R}}^H = \mathbf{xy} \cdot \mathbf{R}^H$, $\tilde{\mathbf{S}} = \mathbf{xy} \cdot \mathbf{S} \cdot \mathbf{wz}$, $\tilde{\mathbf{T}} = \mathbf{xy} \cdot \mathbf{T} \cdot \mathbf{wz}$ and $\tilde{\mathbf{P}} = \mathbf{P} \cdot \mathbf{wz}$ these expressions can be further reduced to

$$\mathbf{G} = \tilde{\mathbf{R}} \cdot \tilde{\mathbf{S}} \cdot \tilde{\mathbf{P}} \tag{32}$$
$$\mathbf{H} = \tilde{\mathbf{R}} \cdot \tilde{\mathbf{T}} \cdot \tilde{\mathbf{P}}. \tag{33}$$

Of course the new triangular matrices in the middle of the right sides should be characterized by the same set of generalized eigenvalues as the original decomposition (Cf. van Dooren (1981), p.123).
So far we only dealt with mathematical identities. But now the algorithm operationalizes this concept by manipulating some submatrices of the original decomposition. The starting point is to extract the matrices $\mathbf{S}_{i,1}$ and $\mathbf{T}_{i,1}$, i.e. the two $(2, \dim(\mathbf{S}))$-dimensional[16] submatrices of $\mathbf{S}$, respectively $\mathbf{T}$, starting with element $s_{i,1}$, respectively $t_{i,1}$, from the original triangular matrices. Then $\mathbf{xy1}_{i,i}$ and $\mathbf{wz1}_{i,i}$ are defined as the two following $(2,2)$-dimensional unitary matrices

$$\mathbf{xy1}_{i,i} = \begin{pmatrix} s_{i,i+1}t_{i,i} - t_{i,i+1}s_{i,i} & s_{i+1,i+1}t_{i,i} - t_{i+1,i+1}s_{i,i} \\ -s_{i+1,i+1}t_{i,i} + t_{i+1,i+1}s_{i,i} & s_{i,i+1}t_{i,i} - t_{i,i+1}s_{i,i} \end{pmatrix} \cdot \frac{1}{((s_{i+1,i+1}t_{i,i} - t_{i+1,i+1}s_{i,i})^2 + (s_{i,i+1}t_{i,i} - t_{i,i+1}s_{i,i})^2)^{0.5}} \tag{34}$$

$$\mathbf{wz1}_{i,i} = \begin{pmatrix} s_{i+1,i+1}t_{i,i+1} - t_{i+1,i+1}s_{i,i+1} & s_{i+1,i+1}t_{i,i} - t_{i+1,i+1}s_{i,i} \\ -s_{i+1,i+1}t_{i,i} + t_{i+1,i+1}s_{i,i} & s_{i+1,i+1}t_{i,i+1} - t_{i+1,i+1}s_{i,i+1} \end{pmatrix} \cdot \frac{1}{((s_{i+1,i+1}t_{i,i} - t_{i+1,i+1}s_{i,i})^2 + (s_{i+1,i+1}t_{i,i+1} - t_{i+1,i+1}s_{i,i+1})^2)^{0.5}}. \tag{35}$$

Premultiplying the extracted submatrices with the matrix $\mathbf{xy1}_{i,i}$ and substituting back the results in exactly the same positions as $\mathbf{S}_{i,1}$ and $\mathbf{T}_{i,1}$ into the original triangular matrices yields the starting matrices for a new extraction operation. This time the two $(\dim(\mathbf{S}), 2)$-dimensional submatrices $\mathbf{S}_{1,i}$ and $\mathbf{T}_{1,i}$ are extracted out of the intermediate results. These

[16] Note that the notation dim is used here in order to denote the length of a quadratic matrix. Therefore it is a one-dimensional measure.

matrices are then postmultiplied with $\mathbf{wz1}_{i,i}$. The results are again substituted back into exactly the same positions, where the extracted parts have been in the intermediate versions of the triangular matrices. The such obtained triangular matrices are taken as the triangular matrices $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{T}}$ of the new decomposition.
Expressing matrix products as sums over the products of submatrices, (28), or (32) can be written as

$$\begin{aligned}\mathbf{G} &= \sum_{j}^{n/2}\left(\sum_{l}^{n/2}\tilde{\mathbf{R}}_{1,k}\cdot\mathbf{xy1}_{k,k}\cdot\mathbf{S}_{k,1}\right)_{1,i}\cdot\mathbf{wz1}_{i,i}\cdot\tilde{\mathbf{P}}_{i,1}^{H} \qquad (36)\\ &= \sum_{j}^{n/2}\sum_{l}^{n/2}\left(\tilde{\mathbf{R}}_{1,k}\cdot\mathbf{xy1}_{k,k}\cdot\mathbf{S}_{k,1}\right)_{1,i}\cdot\mathbf{wz1}_{i,i}\cdot\tilde{\mathbf{P}}_{i,1}^{H},\end{aligned}$$

where i and k are now defined as $i = 2j-1$, respectively $k = 2l-1$. This expression seems to clarify the function of the algorithm described above. It splits the whole decomposition in an array of additive terms and decomposes each of those into new multiplicative components without changing their result. Therefore, it implicitly calculates within a given step

$$\tilde{\mathbf{G}} = \mathbf{G} + \left(\tilde{\mathbf{R}}_{1,i}\cdot\mathbf{xy1}_{i,i}\cdot\mathbf{S}_{i,1}\right)_{1,i}\cdot\mathbf{wz1}_{i,i}\cdot\tilde{\mathbf{P}}_{i,1}^{H} - \left(\mathbf{R}_{1,i}\cdot\mathbf{S}_{i,1}\right)_{1,i}\cdot\mathbf{P}_{i,1}^{H} = \mathbf{G}. \qquad (37)$$

The problem of the algorithm arises within the two matrices $\mathbf{xy1}_{i,i}$ and $\mathbf{wz1}_{i,i}$. Of course these matrices collapse to zero matrices, if the matrix-pencil $\mathbf{G}-\mathbf{H}$ should be non-regular, since this case is reflected by coincident zeros in the same row of the main diagonals of both matrices $\mathbf{S}$ and $\mathbf{T}$. So far this result is standard, since also competing algorithms are restricted to regular matrix-pencils. But additionally the two matrices collapse to zero matrices, if either $t_{i,i} = t_{i,i+1} = t_{i+1,i+1} = 0$ or $s_{i,i} = s_{i,i+1} = s_{i+1,i+1} = 0$ hold. But these conditions do not necessarily correspond to non-regularity for the underlying matrix-pencil. Nonetheless, the algorithm fails to achieve a resorting result.
This argumentation corresponds to the findings that, running qzswitch in the mentioned case, one obtains a only partially resorted set of generalized eigenvalues. Of course the probability of this error rises with the size of the original system, especially for the reason that economic systems are often relatively sparse. Furthermore one can obtain, in some cases, a set of generalized eigenvalues $\{\frac{\tilde{t}_{i,i}}{\tilde{s}_{i,i}}\}_{i\le\dim(\tilde{\mathbf{S}})}$, which does not coincide with the set $\{\frac{t_{i,i}}{s_{i,i}}\}_{i\le\dim(\mathbf{S})}$. Fortunately, the changes seem to be restricted to the upper (∞) and lower (0) extremes of the stability continuum and may therefore be caused by round-off errors. This would imply that stability issues were not affected. Additionally, there can be found cases, in which the algorithm changes an originally singular matrix $\mathbf{S}$ into a nonsingular matrix $\tilde{\mathbf{S}}$. But since the solvability of the model depends critically on the invertibility of a submatrix of $\tilde{\mathbf{S}}$, this may contrive solutions, where there are none. For these reasons I propose the use of alternative algorithms, which follow the mathematical arguments of van Dooren (1981) and Kagström e.a. (1996). To my knowledge several competing codes in different computer languages can be found in the web.

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