Stochastic Discount Factor Approach to International Risk-Sharing: A Trilateral Framework

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Abstract

This paper presents an extension of the stochastic discount factor approach to international (bilateral) risk-sharing given in Brandt, Cochrane, and Santa-Clara (2006). The discount factors in this bilateral setting are not uniquely determined for each country and crucially depend on the other country used in the calculations. We extend the bilateral into a three-country (trilateral) setting. In this way, we calculate discount factors that are unique for each of the three countries and simultaneously price all assets available to their residents. We conclude that risk-sharing measures based on the stochastic discount factor approach are quite robust to the number of countries used in their calculation.

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1 Introduction

Depending on the data sources and the theoretical framework used in order to quantify the degree of international risk-sharing, one arrives at very different conclusions. For example, methods that use consumption data and are based on specific underlying utility functions imply that there is not much risk to be shared (consumption growth is not very volatile) and that countries share a very small portion of this risk because cross-country consumption growth correlations are very low (Backus, Kehoe, and Kydland, 1992; Backus and Smith, 1993). Furthermore, portfolio calculations based on empirical risk-return profiles and certain specification(s) for the utility function find higher potential gains from international risk-sharing (more risk to be shared), but also very low degrees of actual risk diversification (Lewis, 2000). On the contrary, stochastic discount factor-based measures imply that there is a lot of risk to be shared (high volatility of the discount factors) and that a large portion of this risk is actually shared across countries. For the real exchange rate, which by definition equals the difference between domestic and foreign discount factors, is not very volatile (Brandt, Cochrane, and Santa-Clara, 2006).

In fact, the latter approach, developed by Brandt, Cochrane, and Santa-Clara (2006), calculates domestic and foreign marginal utility growth rates through stochastic discount factors derived from asset markets data (using the excess returns of the stock market index above the risk-free rate). In turn, they compare the volatility of these stochastic discount factors with the volatility of the real exchange rate. The main finding in their study is that the real exchange-rates (difference between marginal utility growth rates) are much less volatile than what the stochastic discount factors (proxies for marginal utility growth) of the corresponding countries would imply.

Therefore, they conclude that marginal utility growth rates must be very highly correlated across countries, i.e. a large portion of macroeconomic risk is shared internationally.

This paper presents an extension of the stochastic discount factor approach to measuring international risk-sharing given in Brandt, Cochrane, and Santa-Clara (2006). Hence, we extend their bilateral framework into a three-country setting. In this trilateral framework, domestic residents can invest in risky or risk-free assets at home or in each of the two foreign countries. In this way, we calculate stochastic discount factors that are consistent with the excess returns in each of the three markets (and not only in bilateral pairs) and show that all three stochastic discount factors are very strongly correlated. These results suggest that the asset markets-based approach to international risk-sharing is very robust to the number of countries used in the calculations: in fact, the results from this triangular framework differ only marginally from those derived using the bilateral setting. We conclude that the results of the asset markets-based approach to measuring international risk-sharing are quite robust to the number of countries used in the calculation of the stochastic discount factors.

The rest of this paper is organized as follows: section 2 develops the theoretical framework and presents the calculations of the stochastic discount factors and the risk-sharing index. Section 3 describes the data and shows replication of the results for the bilateral setting. Section 4 extends this approach to a three-country setting. We discuss the relevance of our findings in section 5. Section 6 concludes the paper.

2 Theoretical Framework

2.1 Pricing Kernels

In this section we derive the theoretical framework linking the change in the real exchange-rate with the domestic and foreign marginal utility growth rates (stochastic discount factors). Following the approach taken in Backus, Foresi, and Telmer (1996) and Backus, Foresi, and Telmer (2001), we model asset prices with pricing kernels, i.e. stochastic processes that govern the prices of state-contingent securities[1].

Let v_t represent the domestic currency value at time t of an uncertain, stochastic cash flow of d_{t+1} domestic currency units one period in the future. Then, the basic asset pricing relation relates v_t and d_{t+1} in the following way:

$$v_t = E_t(m_{t+1}d_{t+1}) \tag{1}$$

by dividing both sides of equation 1 by the initial investment v_t at time t, i.e. the value of the uncertain cash flow at time t, we get an expression in terms of returns:

$$1 = E_t(m_{t+1}R_{t+1}) \tag{2}$$

where $R_{t+1} = d_{t+1}/v_t$ is the gross return on this asset/investment between time t and $t+1$, and m_{t+1} is the domestic currency pricing kernel. This kernel m_{t+1} occupies a central place in this exposition since it gives the "rate" at which economic agents discount the uncertain payment d_{t+1} one

[1]Several conditions should be satisfied in order to derive a relationship between the (real) exchange rate and the stochastic discount factors in the two currencies. First, there should be free trade in assets denominated in each currency as well as free trade in each of the corresponding currencies. Second, no pure (zero initial investment) arbitrage opportunities should exist on any of the markets.

period in the future, i.e. it represents the (nominal) intertemporal marginal rate of substitution between time t and $t+1$ for all assets traded in the domestic economy[2].

Similar relations should hold for assets denominated in foreign currency and traded in the foreign economy. In fact, there are two equivalent ways to show these relations for foreign assets. First, through substitution of all domestic variables from equations 1 and 2 with their foreign counterparts we get the following equations for foreign assets:

$$v_t^* = E_t(m_{t+1}^* d_{t+1}^*) \tag{3}$$

and, in terms of gross returns:

$$1 = E_t(m_{t+1}^* R_{t+1}^*) \tag{4}$$

Second, the cash flows (or gross returns) received in foreign currency can be converted into domestic currency units at the expected future spot exchange rate, and then discounted using the domestic pricing kernel or domestic discount factor, just as in the case of domestic assets. According to this approach, we get the following relations:

$$v_t^* = E_t\Big[m_{t+1}(S_{t+1}/S_t)d_{t+1}^*\Big] \tag{5}$$

and, in terms of gross returns:

$$1 = E_t\Big[m_{t+1}(S_{t+1}/S_t)R_{t+1}^*\Big] \tag{6}$$

[2] m_{t+1} will be a unique solution of equations 1 and 2 only if the domestic economy has a complete set of state-contingent securities that can be freely traded. Otherwise, there are multiple solutions for m_{t+1}.

where S_t stands for the current spot nominal exchange rate (the price of foreign currency in domestic currency units) at time t, and S_{t+1}/S_t represents its gross rate of change between time t and $t+1$.

Because these two approaches must give equivalent results, we can equate 3 with 5:

$$E_t(m^*_{t+1}d^*_{t+1}) = E_t\Big[m_{t+1}(S_{t+1}/S_t)d^*_{t+1}\Big] \tag{7}$$

or 4 with 6, respectively:

$$E_t(m^*_{t+1}R^*_{t+1}) = E_t\Big[m_{t+1}(S_{t+1}/S_t)R^*_{t+1}\Big] \tag{8}$$

If no pure arbitrage opportunities exist and markets in both countries are complete, then the following should hold:

$$m^*_{t+1} = m_{t+1}(S_{t+1}/S_t) \tag{9}$$

which, in turn, gives the relation between the change of the exchange rate and the nominal discount factors in the two countries. Hence, the (nominal) exchange rate should move (depreciate/appreciate) exactly by the difference between the discount factors in the respective countries.

Although the discussion in this section focused on *nominal* variables, similar condition can be stated in terms of *real* variables. Thus, taking logarithm of both sides of equation 9 and changing all nominal variables (exchange rates, gross returns, discount factors) with their real counterparts, we derive at a condition that equates the real exchange rate to the difference between changes in foreign and domestic intertemporal marginal rates of substitution between time t and $t+1$:

$$\ln\frac{e_{t+1}}{e_t} = \ln\frac{\lambda^*_{t+1}}{\lambda_{t+1}} = \ln\lambda^*_{t+1} - \ln\lambda_{t+1} \tag{10}$$

where e_t is the real exchange rate - the relative price of domestic in terms of foreign goods, λ_{t+1} is the change in domestic marginal utility between time t and $t+1$, λ^*_{t+1} is the change in foreign marginal utility between time t and $t+1$ (both measured in units of real, consumption goods). Rearranged in real terms, this condition states that in equilibrium the change in the relative price of domestic in terms of foreign goods (given by the real exchange rate) should equal the ratio between foreign and domestic marginal utility growth (log discount factors or pricing kernels). Derived through this simple asset pricing framework, equation 10 is of central importance for the stochastic discount factor approach to measuring international risk-sharing, elaborated in this study[3].

2.2 Risk-Sharing Index

The perfect international risk-sharing hypothesis implies complete equalization of marginal utility growth rates across countries. In our framework, given by equation 10, it means equality between λ_{t+1} and λ^*_{t+1} at any point in time. Thus, if this asset pricing condition holds *and* all country-specific risks are shares internationally, then the left-hand side of this equation should always be zero. In turn, the departures from this perfect situation can be measured by the deviations on the left-hand side, i.e. the fluctuations of the real exchange rate.

Brandt et al. (2006) use this intuition to propose a measure of international risk-sharing based on asset markets. First, they take variances of both sides of equation 10:

[3]For more extensive discussion on the application of this equation see Backus et al. (2001) and Brandt and Santa-Clara (2002) for example.

$$\sigma^2\Big(\ln\frac{e_{t+1}}{e_t}\Big) = \sigma^2\Big(\ln\lambda^*_{t+1} - \ln\lambda_{t+1}\Big) =$$
$$= \sigma^2\Big(\ln\lambda^*_{t+1}\Big) + \sigma^2\Big(\ln\lambda_{t+1}\Big) - 2\rho\sigma\Big(\ln\lambda^*_{t+1}\Big)\sigma\Big(\ln\lambda_{t+1}\Big) \qquad (11)$$

where σ^2 is variance, σ standard deviation, and ρ is the coefficient of correlation between the two discount factors λ_{t+1} and λ^*_{t+1}. Therefore, if the following two conditions hold: i) assets and currencies are priced according to equation 10 at any point in time; and ii) all risks are shared internationally, then: $\rho = 1, \lambda_{t+1} = \lambda^*_{t+1}$ and $\sigma^2\Big(\ln\frac{e_{t+1}}{e_t}\Big) = 0$. In general, the correlation between marginal utility growth rates will be given by:

$$\rho = \frac{\Big[\sigma^2\Big(\ln\lambda^*_{t+1}\Big) + \sigma^2\Big(\ln\lambda_{t+1}\Big) - \sigma^2\Big(\ln\frac{e_{t+1}}{e_t}\Big)\Big]}{2\sigma\Big(\ln\lambda^*_{t+1}\Big)\sigma\Big(\ln\lambda_{t+1}\Big)} \qquad (12)$$

indicating that risk-sharing across countries decreases in the variability of the real exchange rate. Based on this idea, Brandt et al (2006) construct the following risk-sharing index

$$RSI = 1 - \frac{\sigma^2\Big(\ln\frac{e_{t+1}}{e_t}\Big)}{\sigma^2\Big(\ln\lambda^*_{t+1}\Big) + \sigma^2\Big(\ln\lambda_{t+1}\Big)} \qquad (13)$$

where the numerator of the second term captures the variability in the real exchange rate (which, according to the argumentation above, measures the deviations from perfect risk-sharing), and the denominator is the sum of the variabilities in marginal utility growth in the two countries (the total risk that exists and can be shared across countries). Hence, this term gives a ratio between risk still not shared and total risk that can be shared between the two countries. In turn, Brandt et al. (2006) indicate that this index gives the portion of total (diversifiable) risk that is already shared by the two countries.

2.3 Basic Calculations

In order to calculate the risk-sharing index given in the previous section, first we have to recover the log discount factors (or marginal utility growth rates) from asset markets data in the corresponding countries[4]. For this purpose, we assume that the following assets are traded in a two-country setting:

$$\frac{dB^d}{B^d} = r^d dt \tag{14}$$

$$\frac{dS^d}{S^d} = \theta^d dt + dz^d \tag{15}$$

$$\frac{de}{e} = \theta^e dt + dz^e \tag{16}$$

$$\frac{dB^f}{B^f} = r^f dt \tag{17}$$

$$\frac{dS^f}{S^f} = \theta^f dt + dz^f \tag{18}$$

where B^d is domestic risk-free bond (with expected return r^d), S^d is domestic risky asset (expected return θ^d), e is the real exchange rate, i.e. the relative price of domestic in terms of foreign goods (expected return θ^e), B^f is foreign risk-free bond, and S^f is foreign risky asset (expected return θ^f). Thus, there are three sources of uncertainty in this setting related to the domestic asset, the real exchange rate, and the foreign asset. These shocks can be collected into a vector of shocks dz:

[4]For ease of exposition and manipulation in the further calculations (translating between levels and logarithms), the demonstration here uses continuous time formulation. All variables are calculated using the corresponding discrete time approximations, see section on data issues.

$$dz = \begin{bmatrix} dz^d \\ dz^e \\ dz^f \end{bmatrix}$$

with a corresponding variance-covariance matrix given by:

$$\Sigma = \frac{1}{dt}E(dzdz') = \begin{bmatrix} \Sigma^{dd'} & \Sigma^{de} & \Sigma^{df'} \\ \Sigma^{ed'} & \Sigma^{ee} & \Sigma^{ef'} \\ \Sigma^{fd'} & \Sigma^{fe} & \Sigma^{ff'} \end{bmatrix}$$

Furthermore, the calculation of the discount factor(s) from asset markets depends primarily on the variability of the excess returns on risky assets, driven by the shocks in vector dz[5]. We derive all excess returns in the appendix, and here present only their expected values. Thus, the domestic investor faces the following set of expected excess returns:

$$\mu^d = \begin{bmatrix} \theta^d - r^d \\ \theta^e + r^f - r^d \\ \theta^f - r^f + \Sigma^{ef} \end{bmatrix}$$

The first term in this vector gives the excess return that a domestic resident expects to get by investing on the domestic stock market. It equals the difference between the average real return on the domestic stock market index (θ^d) and the average real risk-free rate in the domestic economy (r^d) during the entire investment period. The expected excess return on the foreign exchange market is given by the second term in vector μ^d. It represents the average deviation from (uncovered) interest parity, calculated as borrowing in the domestic currency, converting the borrowed amount into

[5]Since we work with (expected) excess returns in this analysis, we do not make a real/nominal returns distinction.

the foreign currency, lending at the ongoing one-month foreign interest rate, and converting the proceeds back into domestic currency after one month. The last term in vector μ^d gives the expected excess return that a domestic investor expects to get by investing in the foreign stock market. Therefore, it represents a difference between the average return on the foreign stock market and the domestic one-month risk-free interest rate. The last part of this term Σ^{ef} results from the continuous-time formulation and gives the (average) co-movement between the returns on the foreign stock market and the exchange rate.

A similar vector of expected excess returns applies to the foreign investor:

$$\mu^f = \begin{bmatrix} \theta^d - r^d - \Sigma^{ed} \\ -(\theta^e + r^f - r^d - \Sigma^{ee}) \\ \theta^f - r^f \end{bmatrix}$$

The interpretation of the terms is analogous to that given for the domestic investor. The expected excess return on the foreign exchange market is exactly the opposite of the one for the domestic investor (corrected for the continuous-time term Σ^{ee}).

Then, the following discount factors price all assets according to the basic pricing conditions[6]:

$$\frac{d\Lambda^i}{\Lambda^i} = -r^i dt - \mu^{i\prime}\Sigma^{-1}dz, i = d, f \tag{19}$$

where $\frac{d\Lambda^i}{\Lambda^i} = \lambda^i$ in the basic asset pricing condition 10, r^i is the risk-free return, and μ^i is the vector of excess returns for risky assets in country i. In order to calculate the log discount factor $\ln \lambda^i$ required in equation 10, we use Ito's lemma and get the following expression:

[6]For more details on finding the discount factor in this setting see Brandt et al. (2006, p.675-677) or Chapter 4 in Cochrane (2004).

$$d\ln\Lambda = \frac{d\Lambda}{\Lambda} - \frac{1}{2}\frac{d\Lambda^2}{\Lambda^2} = -\Big(r + \frac{1}{2}\mu'\Sigma^{-1}\mu\Big)dt - \mu'\Sigma^{-1}dz \qquad (20)$$

and for its standard deviation:

$$\frac{1}{dt}\sigma^2(d\ln\Lambda^i) = \mu'\Sigma^{-1}\mu, i = d, f \qquad (21)$$

Therefore, the risk-sharing index given by 13 can be calculated directly from the second moments according to the following expression:

$$RSI = 1 - \frac{\sigma^2(d\ln\Lambda^d - d\ln\Lambda^f)}{\sigma^2(d\ln\Lambda^d) + \sigma^(d\ln\Lambda^f)} = 1 - \frac{\Sigma^{ee}}{\mu^{d\prime}\Sigma^{-1}\mu^d + \mu^{f\prime}\Sigma^{-1}\mu^f} \qquad (22)$$

In order to show the symmetric structure of our framework, we relate the shocks facing the domestic with those facing the foreign investor. The expected excess returns vectors μ^d and μ^f differ only by the exchange rate changes[7]:

$$\mu^d - \mu^f = \begin{bmatrix} \theta^d - r^d \\ \theta^e + r^f - r^d \\ \theta^f - r^f + \Sigma^{ef} \end{bmatrix} - \begin{bmatrix} \theta^d - r^d - \Sigma^{ed} \\ \theta^e + r^f - r^d - \Sigma^{ee} \\ \theta^f - r^f \end{bmatrix} = \begin{bmatrix} \Sigma^{ed} \\ \Sigma^{ee} \\ \Sigma^{ef} \end{bmatrix} \qquad (23)$$

From these formulae, it is clear that the expected excess return vectors differ exactly by the middle column of the common variance covariance matrix Σ^e:

$$\mu^d - \mu^f = \begin{bmatrix} \Sigma^{ed} \\ \Sigma^{ee} \\ \Sigma^{ef} \end{bmatrix} = \Sigma^e \qquad (24)$$

[7]In order to derive this relation, we disregard the change in sign before the foreign exchange excess returns when moving from domestic to foreign investor perspective.

In turn, we can derive a relationship between the domestic and foreign discount factor loadings (given by the last term of equation 20):

$$\mu^d \Sigma^{-1} = (\mu^f + \Sigma^e)\Sigma^{-1} = \mu^f \Sigma^{-1} + \Sigma^e \Sigma^{-1} = \mu^f \Sigma^{-1} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{25}$$

Equation 25 shows that domestic and foreign discount factors load equally on domestic and foreign stock market shocks, while their loadings on the foreign exchange shocks differ by exactly 1.

3 Data and Replication of Results

3.1 Data Description

In this section we replicate the results for the bilateral setting presented in Brandt et al. (2006). For that purpose, we construct a dataset that is as close as possible to the one used in the original study. In particular, we employ three types of time-series: for the risk-free rate we use interest rates on one-month Eurocurrency deposits, while for the return on the risky asset we use total returns on the stock market index for the corresponding country. We calculate inflation rates from the changes in the consumer price indices (CPI). The nominal exchange rates are expressed in terms of domestic currency per unit of foreign currency.

Our analysis includes three economies: USA, UK, and Japan. We use monthly data from January 1975 till June 1998 for the USA and the UK. For Japan interest rates on Eurocurrency deposits are not available before August 1978. Therefore, all data series for Japan start in August 1978 and go through June 1998. The series on Eurocurrency deposit interest rates, nominal exchange rates and total stock market index returns are measured at the beginning of the month, while the CPI series refer to mid-month values[8]. All data come from Datastream[9].

For stock market returns, we use the same indices employed in the original study[10]: S&P 500 for the USA, FTSE ALL for the UK, and NIKKEI 225 for Japan.

3.2 Summary Statistics

We use discrete time approximations of the continuous time formulae derived in section 2.3. The following sample counterparts are used in the calculation:

$$\theta^d - r^d = \tfrac{1}{\Delta} E_T R^d_{t+\Delta}$$
$$\theta^f - r^f = \tfrac{1}{\Delta} E_T R^f_{t+\Delta}$$
$$\theta^e + r^f - r^d = \tfrac{1}{\Delta} E_T \left(\tfrac{e_{t+\Delta} - e_t}{e_t} + r^f_{t+\Delta} - r^d_{t+\Delta} \right)$$
$$dz^d = R^d_{t+\Delta} - E_T R^d_{t+\Delta} \; dz^f = R^f - E_T R^f_{t+\Delta}$$
$$dz^e = \left(\tfrac{e_{t+\Delta} - e_t}{e_t} \right) - E_T \left(\tfrac{e_{t+\Delta} - e_t}{e_t} \right)$$
$$\Sigma = \tfrac{1}{\Delta} E_T (dz dz')$$

In these sample moments T is the sample size (281 monthly observations), E_T denotes the sample mean for the entire time period, $\Delta = \frac{1}{12}$years, $R^d_{t+\Delta}$ and $R^f_{t+\Delta}$ correspond to the domestic and foreign excess stock returns, and $r^d_{t+\Delta}$ and $r^f_{t+\Delta}$ refer to the domestic and foreign risk-free (Eurocurrency deposits) interest rates, respectively.

[8] The results are very robust with respect to the use of lag or lead values for the inflation rate

[9] CPI data is retrieved from Datastream and comes from the IMF International Financial Statistics (IFS) database.

[10] For the UK we do the same calculations using FTSE 100 index. The results change only slightly.

In accordance with the approach taken before, we use real variables: real (excess) stock returns, real risk-free interest rates and real exchange rates. Hence, we correct all data series by the inflation rate (measured by changes in the mid-month CPI). Moreover, we calculate stock market returns in two ways: i) assuming continuous-time specification and ii) with discrete time specification. Since the results are very similar, in the rest of the analysis we only present stock market returns using the discrete time framework.

The summary statistics are presented in Table 1. Its upper panel shows means and standard deviations for excess stock market returns (Stock) and for excess foreign exchange returns (X-rate). The latter are derived as deviations from the uncovered interest parity (UIP), calculated as excess returns one realizes by borrowing in the domestic currency (dollar), investing in one-month Eurocurrency deposits in the foreign country (pounds sterling or yen), and translating these yields back to the domestic currency at the end of the period. All entries in the table are annualized and reported in percentages.

The statistics in Table 1 are very similar to and convey the same message as the ones presented by Brandt et al. (2006)[11]. In fact, the mean excess returns given in the first row illustrate the high equity premium found in stock markets data. They range from 4.29 percent in Japan, 10.12 percent in the USA, to 15.46 percent in the UK, and all of them are statistically different from zero. Moreover, their associated standard errors, reported in the row beneath, are typically very high. Thus, they result in values for the Sharpe ratio between 0.22 for Japan and 0.72 for the USA and the UK. On

[11]The first moments are similar and normally keep the same ranking between different countries, but are not identical. On the other hand, the second moments are almost identical as the ones presented by Brandt et al. (2006). This is to be expected as the second moments are usually much less sensitive to the exact procedure used in the calculation.

the other hand, mean excess returns for foreign exchange are much smaller and not statistically different from zero. In fact, the mean excess return for the pound sterling/dollar exchange rate is negative (-0.88 percent), while it is slightly positive (2.06 percent) for the yen/dollar exchange rate. Furthermore, the annualized standard deviations for foreign exchange excess returns are about half the values for excess stock market returns (11.56 percent for the first and 12.67 percent for the second exchange rate).

Finally, the lower panel of this table presents a returns correlation matrix. Three conclusions are evident from this table. First, foreign exchange excess returns are very weakly correlated with excess returns on stock markets. Second, foreign exchange excess returns on one currency pair are highly correlated with excess return on the other currency pair (correlation of 0.507). Third, excess returns for different stock markets are highly correlated among themselves (correlations ranging from 0.32 between USA and Japan to 0.58 between USA and UK).

3.3 Replication of the Results for the Bilateral Setting

Using the dataset described in the previous section, here we present a replication of the results obtained by Brandt et al. (2006) for the bilateral setting. The most important result is presented in the first row of Table 2. The risk-sharing index obtains values higher than 0.98, which indicates that an extremely large portion of total macroeconomic risks faced by investors in different countries are shared internationally. This is the central result and the most important message from Brandt et al. (2006). In order to understand these high values for the risk-sharing index, we present its two components in the lower part of Table 2. The volatility of the real exchange rate (numerator in the second term of the risk-sharing index) is several times

Table 1: Summary Statistics (Annualized)

	USA	UK		Japan	
	Stock	Stock	X-Rate	Stock	X-Rate
			Returns (%)		
Mean	10.12	15.46	-0.88	4.29	2.06
Std Dev	14.09	21.37	11.56	19.52	12.67
Sharpe ratio	0.72	0.72	-0.07	0.22	0.16
			Return Correlations		
USA Stock	1				
UK Stock	0.5833	1			
X-Rate	0.01	-0.05	1		
Japan Stock	0.3245	0.342	0.07768	1	
X-Rate	-0.023	-0.063	0.507	0.1009	1

lower than the volatility of the stochastic discount factors, i.e. the volatility of the intertemporal marginal utility growth rates (denominator in the second term of the risk-sharing index). In turn, this implies low values for the second term and high value for the overall risk-sharing index.

The volatility of the stochastic discount factor (marginal utility growth rate) comes from three sources: domestic and foreign stock market excess return shocks and foreign exchange excess return shock. Table 3 shows the discount factors loadings on each of these underlying shocks $\mu^d\Sigma^{-1}$ and $\mu^f\Sigma^{-1}$. In line with equation 25, domestic and foreign discount factors load equally on each of the stock market shocks, and domestic discount factor loads on the exchange rate shocks by one more than the foreign discount factor. The last point implies that the difference between the two discount factors at each point in time equals the exchange rate. For the pair USA-

Table 2: Risk Sharing Index

	USA vs. UK	USA vs. Japan
Risk Sharing Index	0.9878	0.9857
Real X-Rate Volatility	0.1175	0.1247
(*Annualized Standard Deviation*)		
Volatility of Marginal Utility Growth:		
(*Annualized Standard Deviation*)		
Domestic (USA)	0.7549	0.7483
Foreign	0.7511	0.7309

UK, the values of the discount factor loadings on domestic (USA) and foreign (UK) stock excess returns are much higher than the value of discount factor loading for the exchange rate. This reflects the high equity premium on the stock markets in these countries given in Table 1. In fact, since the price of risk is very high on these stock markets (Sharpe ratios of 0.72), their excess return shocks matter more for the stochastic discount factor (marginal utility growth). For example, a positive (negative) shock on the US stock market (dz^d) leads to a much stronger decrease (increase) in domestic and foreign marginal utility growth rates (discount factor levels) than a shock of comparable magnitude (and direction) on the exchange rate (dz^e)[12]. The impact of the US excess return shock on the discount factors is even higher for the second country pair (USA-Japan). On the other hand, the discount factors negatively (and load much less in absolute value) on the Japanese excess return shocks. This partially reflects the low price of risk on the Japanese relative to the American stock market (Sharpe ratio of 0.22 for

[12] A favorable stock market shock leads to lower marginal utility growth rate as shown by the negative sign in front of the disturbance term in equation 20. Moreover, this shock is "scaled" by the loading coefficient $\mu'\Sigma^{-1}$.

Table 3: Discount Factor Loadings (Bilateral)

	USA	UK	USA	Japan
	Domestic	Foreign	Domestic	Foreign
dz^d	3.76	3.76	5.36	5.36
dz^e	-1.02	-2.02	1.51	0.51
dz^f	1.95	1.95	-0.13	-0.13

Japan compared to 0.72 for the USA). Hence, the development of each discount factor crucially depends on the excess return shocks in the country with higher-of-the-two Sharpe ratio. Finally, the exchange rate excess returns loadings are of similar magnitude for both country pairs (in absolute value terms).

In order to give a visual representation of the main result in our study, we present several plots for the discount factors. First, in Figure 1 we show time paths for the log discount factors in the two country pairs. We calculate the log level of the discount factor in line with equation 20. Therefore, they contain two components: a trend component given by the expected value of equation 20 (the term in brackets) and a disturbance component given by the loadings on the underlying excess return shocks. Therefore, the development of the log level discount factors can be best understood through the contribution of each of its components.

There are several interesting issues in this figure. First, the log level discount factors typically slope downward as a result of the trend component. In fact, as long as the sum of the average real risk-free rate and the discount factor volatility (the expected value of 20 given by the term in brackets) is positive (as normally observed), the log level discount factors will follow a downward trend. The easiest way to understand why this is usually the case is by looking at an economy with one only risk-free bond. If this economy experiences real growth over en extended period of time, then its average real risk-free interest rate will be positive (and the trend component will be negative). In turn, a downward trend in the log level discount factor corresponds with a decreasing trend in marginal utility growth rates or continual improvement in overall economic conditions. Second, it is clear from the figure that both discount factors follow a similar pattern and move very closely together. In fact, the only difference between them comes from the real exchange rate fluctuations (see equations 10 and 25). Based on this observation, we can conclude that marginal utility growth rates across countries follow very similar time paths, just as implied by the perfect risk-sharing condition.

Moreover, in Figure 2 we present scatterplots for the discount factor growth rates. We calculate these monthly growth rates according to equation 19. This figure just strengthens our conclusion from Figure1 : there is a very high positive correlation between the discount factor growth rates for each country pair. Most observations/points are literally lying on the 45 degree line, thereby indicating that asset markets imply nearly perfect levels of (bilateral) international risk-sharing.

4 Trilateral Setting

Section 3 demonstrated that measures based on the stochastic discount factor approach imply very high levels of international risk-sharing among two country-pairs: USA-UK and USA-Japan. In fact, we showed that discount factors for each country in the bilateral pair display very similar levels of volatility (Table 2), follow similar time paths (Figure 1), and have almost identical growth rates (Figure 2). However, all these calculations were con-

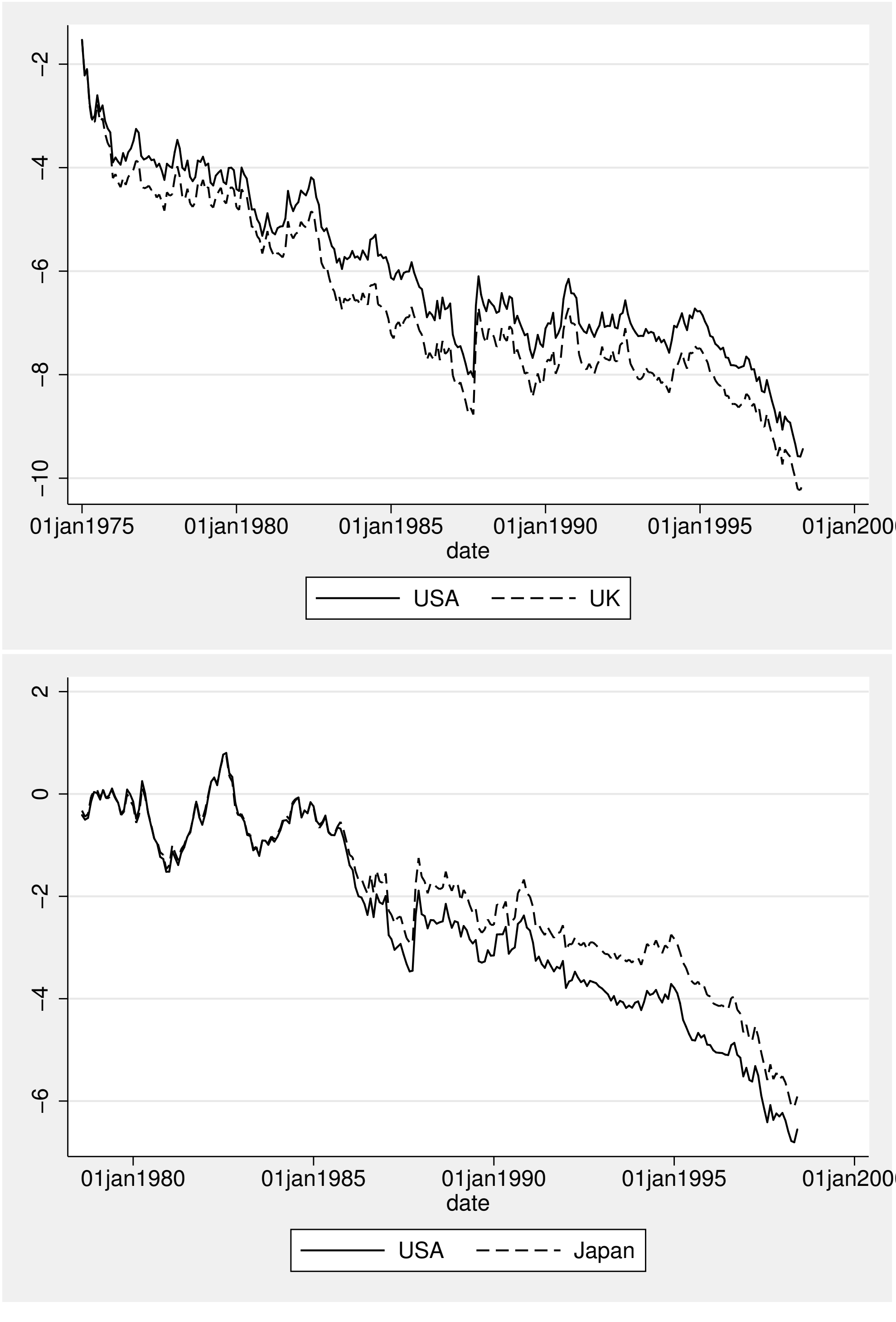

Figure 1: Log Level of Discount Factors (Bilateral)

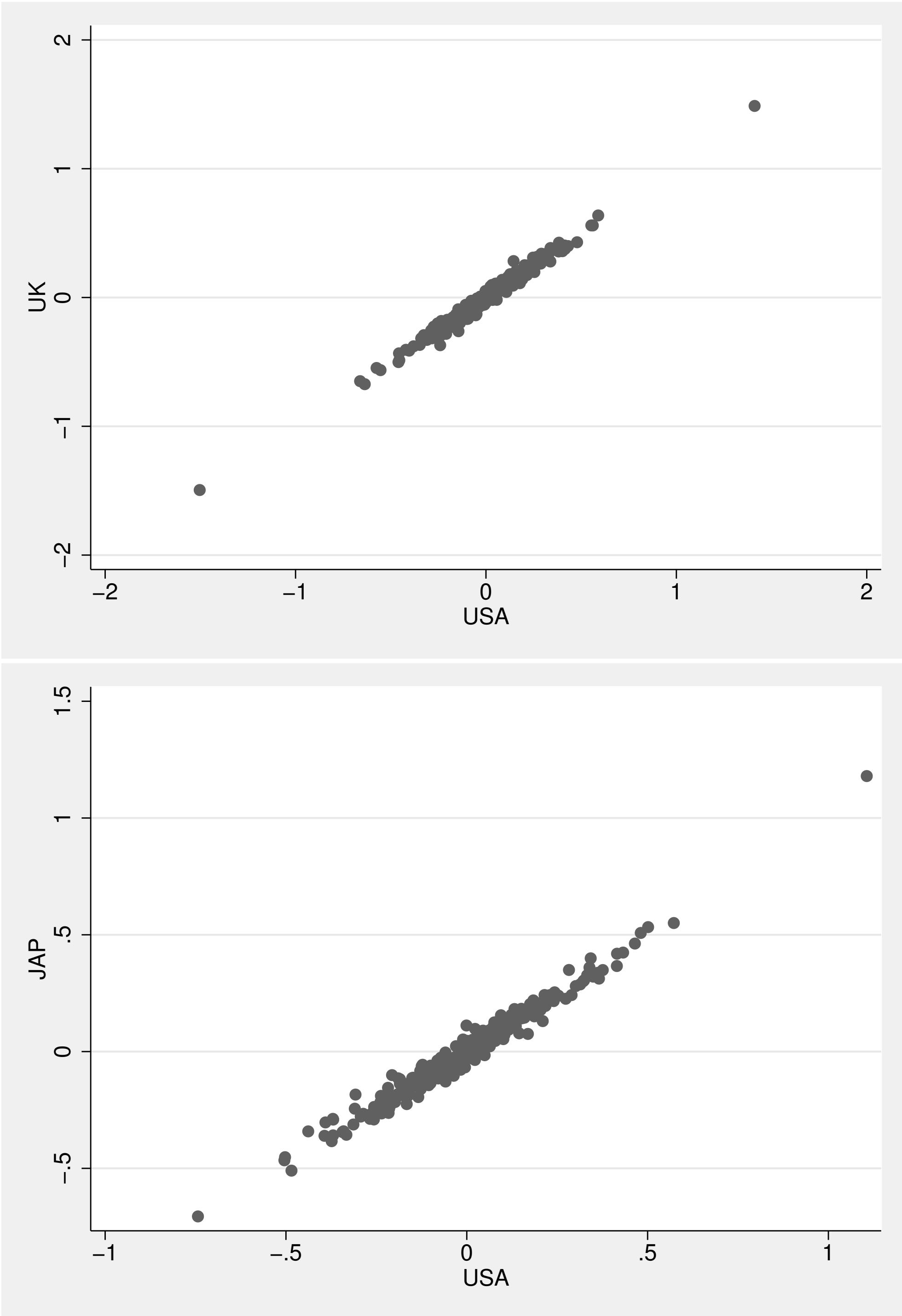

Figure 2: Growth of Discount Factors (Bilateral)

ducted within bilateral setting, i.e. treating only two countries at the time. Therefore, one possible criticism of this approach is that a country's discount factor obviously depends on the second country. In particular, the USA log discount factor displays a very similar behavior with the UK log discount factor (in the upper panel of Figure 1), and with the log discount factor for Japan (in the lower panel of Figure 1). However, the USA log discount factor from the upper panel is quite different from the USA log discount factor given in the lower panel. In other words, this shows that the discount factors in this framework are chosen in such a way as to satisfy the restrictions imposed by one bilateral country pair at the time.

Nevertheless, the discount factor for a certain country should be uniquely determined and incorporate all (direct) investment opportunities available to its residents (and therefore, should price all these assets). In order to investigate to what extent the results from section 3 depend on the specific, bilateral structure, we extend it into a three-country setting. Therefore, the discount factors calculated in this trilateral setting are *unique* for each country and *simultaneously* price all assets available to its residents (all risky assets in each of the three countries).

In this section we present a triangular (trilateral) extension to the stochastic discount factor approach of international risk-sharing. First, we adapt all calculations to the three-country setting. Second, we present results for the risk-sharing index, allowing for various importance levels to different partner countries. Third, we visualize our main results using several graphs.

4.1 Calculations

The discount factors in this trilateral setting can still be calculated according to equations 19 and 20:

$$\frac{d\Lambda^i}{\Lambda^i} = -r^i dt - \mu^{i\prime}\Sigma^{-1}dz, i = d, f_1, f_2$$

$$d\ln\Lambda = \frac{d\Lambda}{\Lambda} - \frac{1}{2}\frac{d\Lambda^2}{\Lambda^2} = -\Big(r + \frac{1}{2}\mu'\Sigma^{-1}\mu\Big)dt - \mu'\Sigma^{-1}dz$$

and their volatility according to equation 21:

$$\frac{1}{dt}\sigma^2(d\ln\Lambda^i) = \mu'\Sigma^{-1}\mu, i = d, f_1, f_2$$

where d refers to the domestic country, f_1 to the first foreign country, and f_2 to the second foreign country. In the calculations below, d stands for the USA, f_1 for the UK, and f_2 for Japan. In the trilateral setting, residents in each country are faced with five (instead of three) sources of uncertainty. Apart from shocks to domestic risky assets, they face two exchange rate shocks, and two foreign risky assets shocks. Thus, all these sources of uncertainty can be summarized in the following three vectors, each referring to residents of the corresponding country:

$$dz_d = \begin{bmatrix} dz^d \\ dz^{e_1} \\ dz^{e_2} \\ dz^{f_1} \\ dz^{f_2} \end{bmatrix}$$

$$dz_{f_1} = \begin{bmatrix} dz^d \\ dz^{e_1} \\ dz^{e_3} \\ dz^{f_1} \\ dz^{f_2} \end{bmatrix}$$

$$dz_{f_2} = \begin{bmatrix} dz^d \\ dz^{e_3} \\ dz^{e_2} \\ dz^{f_1} \\ dz^{f_2} \end{bmatrix}$$

with the following set of three variance-covariance matrices:

$$\Sigma_d = \frac{1}{dt}E(dz_d dz_d') = \begin{bmatrix} \Sigma^{dd'} & \Sigma^{de_1} & \Sigma^{de_2} & \Sigma^{df_1'} & \Sigma^{df_2'} \\ \Sigma^{e_1 d'} & \Sigma^{e_1' e_1} & \Sigma^{e_1' e_2} & \Sigma^{e_1 f_1'} & \Sigma^{e_1 f_2'} \\ \Sigma^{e_2 d'} & \Sigma^{e_2' e_1} & \Sigma^{e_2' e_2} & \Sigma^{e_2 f_1'} & \Sigma^{e_2 f_2'} \\ \Sigma^{f_1' d} & \Sigma^{f_1' e_1} & \Sigma^{f_1' e_2} & \Sigma^{f_1' f_1} & \Sigma^{f_1' f_2} \\ \Sigma^{f_2' d} & \Sigma^{f_2' e_1} & \Sigma^{f_2' e_2} & \Sigma^{f_2' f_1} & \Sigma^{f_2' f_2} \end{bmatrix}$$

$$\Sigma_{f_1} = \frac{1}{dt}E(dz_{f_1} dz_{f_1}') = \begin{bmatrix} \Sigma^{dd'} & \Sigma^{de_1} & \Sigma^{de_3} & \Sigma^{df_1'} & \Sigma^{df_2'} \\ \Sigma^{e_1 d'} & \Sigma^{e_1' e_1} & \Sigma^{e_1' e_3} & \Sigma^{e_1 f_1'} & \Sigma^{e_1 f_2'} \\ \Sigma^{e_3 d'} & \Sigma^{e_3' e_1} & \Sigma^{e_3' e_3} & \Sigma^{e_3 f_1'} & \Sigma^{e_3 f_2'} \\ \Sigma^{f_1' d} & \Sigma^{f_1' e_1} & \Sigma^{f_1' e_3} & \Sigma^{f_1' f_1} & \Sigma^{f_1' f_2} \\ \Sigma^{f_2' d} & \Sigma^{f_2' e_1} & \Sigma^{f_2' e_3} & \Sigma^{f_2' f_1} & \Sigma^{f_2' f_2} \end{bmatrix}$$

$$\Sigma_{f_2} = \frac{1}{dt}E(dz_{f_2}dz'_{f_2}) = \begin{bmatrix} \Sigma^{dd'} & \Sigma^{de_3} & \Sigma^{de_2} & \Sigma^{df'_1} & \Sigma^{df'_2} \\ \Sigma^{e_3d'} & \Sigma^{e'_3e_3} & \Sigma^{e'_3e_2} & \Sigma^{e_3f'_1} & \Sigma^{e_3f'_2} \\ \Sigma^{e_2d'} & \Sigma^{e'_2e_3} & \Sigma^{e'_2e_2} & \Sigma^{e_2f'_1} & \Sigma^{e_2f'_2} \\ \Sigma^{f'_1d} & \Sigma^{f'_1e_3} & \Sigma^{f'_1e_2} & \Sigma^{f'_1f_1} & \Sigma^{f'_1f_2} \\ \Sigma^{f'_2d} & \Sigma^{f'_2e_3} & \Sigma^{f'_2e_2} & \Sigma^{f'_2f_1} & \Sigma^{f'_2f_2} \end{bmatrix}$$

Moreover, we must impose an additional restriction in the calculation. Namely, we have to exclude the possibilities for triangular (cross-currency) arbitrage. In particular, if the exchange rate returns are given by:

$$\frac{de_1}{e_1} = \theta^e_1 dt + dz^e_1, \frac{de_2}{e_2} = \theta^e_2 dt + dz^e_2, \frac{de_3}{e_3} = \theta^e_3 dt + dz^e_3 \tag{26}$$

then the following cross-currency condition must hold:

$$\theta^e_3 dt + dz^e_3 = \theta^e_2 dt + dz^e_2 + \theta^e_1 dt + dz^e_1 \tag{27}$$

Finally, following a similar procedure as in the bilateral setting, we calculate the following vectors of expected excess returns for residents in each country[13]:

$$\mu^d = \begin{bmatrix} \theta^d - r^d \\ -(\theta^{e_1} + r^{f_1} - r^d - \Sigma^{e_1e_1}) \\ \theta^{e_2} + r^{f_2} - f^d \\ \theta^{f_1} - r^{f_1} + \Sigma^{e_1f_1} \\ \theta^{f_2} - r^{f_2} + \Sigma^{e_2f_2} \end{bmatrix}$$

[13]The appendix contains detailed calculations for all expected excess returns.

$$\mu^{f_1} = \begin{bmatrix} \theta^d - r^d - \Sigma^{e_1 d} \\ \theta^{e_1} + r^{f_1} - r^d \\ -(\theta^{e_3} + r^{f_1} - r^{f_2} - \Sigma^{e_3 e_3}) \\ \theta^{f_1} - r^{f_1} \\ \theta^{f_2} - r^{f_2} - \Sigma^{e_3 f_2} \end{bmatrix}$$

$$\mu^{f_2} = \begin{bmatrix} \theta^d - r^d - \Sigma^{e_2 d} \\ -(\theta^{e_2} + r^{f_2} - r^d - \Sigma^{e_2 e_2}) \\ \theta^{e_3} + r^{f_1} - f^{f_2} \\ \theta^{f_1} - r^{f_1} + \Sigma^{e_3 f_1} \\ \theta^{f_2} - r^{f_2} \end{bmatrix}$$

The interpretation of these excess returns is analogous to the one given for the bilateral setting. In fact, the main difference is that residents can invest in two (instead of one) foreign risk-free bonds and three (instead of two) stock markets.

The excess return vectors can be related using the restrictions imposed by the cross-currency condition 27 (no triangular arbitrage possibilities). For example, the excess returns for a domestic resident can be related with the excess returns for a resident in the first foreign country (f_1) as follows[14]:

$$\mu^{f_1} = A\mu^d \tag{28}$$

where the matrix A is defined as:

[14]For reasons of symmetry we use directly the discrete-time equivalents of the continuous-time formulae, just as implemented in the calculations. Thus, we disregard the continuous-time terms in the excess return vectors.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

Equation 28 shows that residents in both countries face the same expected excess returns on all three stock markets, while their foreign exchange excess returns form a linear combination. In turn, the variance convariance matrix with shocks facing the residents in the first foreign country is given by:

$$\Sigma_{f_1} = A\Sigma_d A' \tag{30}$$

and its inverse:

$$\Sigma_{f_1}^{-1} = (A\Sigma_d A')^{-1} = (A')^{-1}\Sigma_d^{-1}A^{-1} \tag{31}$$

Therefore the domestic and first foreign (f_1) discount factor loadings will be related as follows:

$$\mu^{f_1}\Sigma_{f_1}^{-1} = \mu^d A'(A')^{-1}\Sigma_d^{-1}A^{-1} = \mu^d\Sigma_d^{-1}A^{-1} \tag{32}$$

Equation 32 indicates that the only difference between domestic and foreign discount factors is given A^{-1}. In turn, it means that both discount factors load equally on all three stock market shocks, while their foreign exchange loadings differ by a linear combination of the exchange rate shocks.

Table 4: Real X-Rate and Discount Factor Volatility (Annualized)

Real X-Rate		**Discount Factor**	
e_1 (UK/USA)	0.115816	USA	0.775553
e_2 (JAP/USA)	0.124724	UK	0.791909
e_3 (UK/JAP)	0.12052	JAP	0.760496

4.2 Results from the Trilateral Setting

Table 4 shows the real exchange rates and discount factor volatilities for the trilateral setting. The figures are very similar to those calculated for the bilateral setting. In fact, three major points stay unchanged. First, discount factor display comparable volatility levels across different countries. Second, real exchange rate volatilities are also very similar. Third, marginal utility growth volatility (measured by the discount factor volatility) is several times larger than real exchange rate volatility. In fact, the last observation is crucial for the measurement of international risk-sharing. It indicates that the differences between marginal utility growth rates across these three countries are very low, suggesting that a lot of risk-sharing takes place among them.

We modify the risk-sharing index given by equation 13 in order to adapt it to our trilateral framework. Hence, we include all three countries in its calculation. For example, for the domestic country (USA), we include both real exchange rates (with respect to the UK and with respect to Japan) and all three discount factor volatilities. Moreover, we allow for differences between partner countries by assigning them specific weights α and $(1-\alpha)$, respectively. In this way, all foreign partner weights for a certain country must sum up to 1. Therefore, the easiest way to think about this approach is as an "effective, trade-weighted" combination of foreign partners.

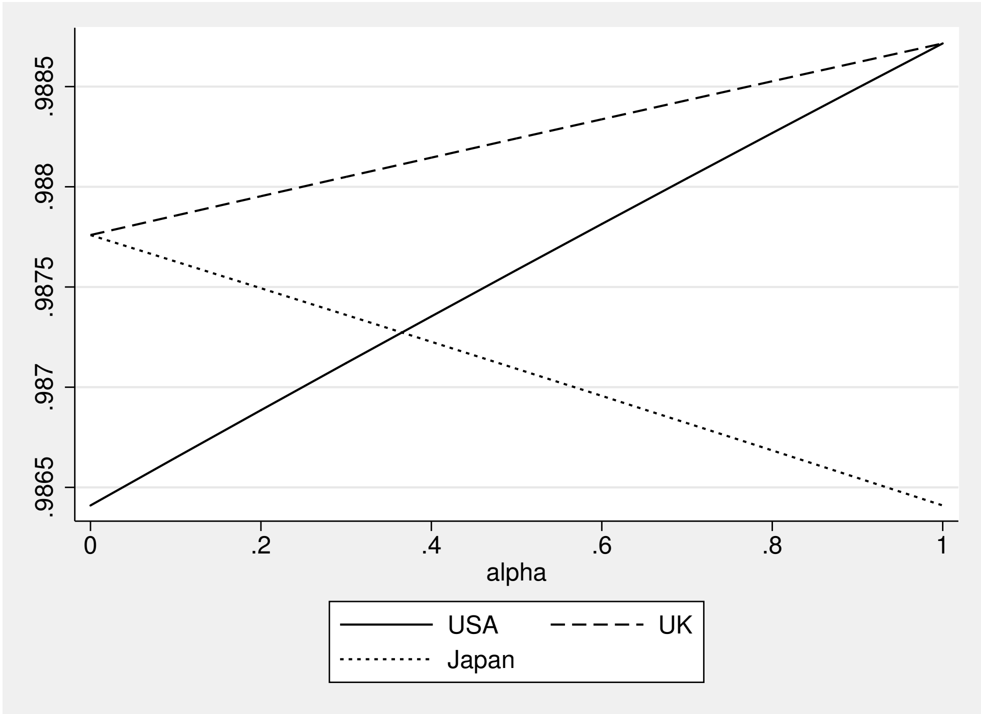

Figure 3: Risk-Sharing Index (Trilateral)

$$RSI = 1 - \frac{\alpha \Sigma^{e_1 e_1} + (1-\alpha)\Sigma^{e_2 e_2}}{\mu^{d\prime}\Sigma_d^{-1}\mu^d + \alpha \mu^{f_1'}\Sigma_{f_1}^{-1}\mu^{f_1} + (1-\alpha)\mu^{f_2'}\Sigma_{f_2}^{-1}\mu^{f_2}} \tag{33}$$

In fact, these weights should correspond to the relative importance of specific partner countries for international risk-sharing. Hence, there is no specific theoretical way to derive them. Rather, in this study we allow the value for α to fluctuate anywhere between 0 and 1, thereby covering all possible combinations.

Figure 3 shows results for the risk-sharing index for each country when different weights are assigned to its other two partners. In fact, the value for α, indicated on the horizontal axis, goes from one extreme (0) to the other (1) (where at each extreme only one of the partner countries matters for risk-sharing) and covers all possible intermediate cases.

For example, the line for the USA represents different values for the USA risk-sharing index going from $\alpha = 0$ (all risk-sharing is done with Japan) to $\alpha = 1$ (all risk-sharing takes place with the UK). The upward slope of this line with respect to α suggests that USA achieves a higher level of international risk-sharing when UK becomes the relatively more important partner. The similar logic applies to the calculations for the other two countries: the upward line for the UK indicates increasing risk-sharing levels when USA becomes relatively more important partner (compared to Japan), and the downward sloping line for Japan indicates decreasing risk-sharing levels when USA becomes relatively more important partner (compared to the UK).

In turn, we can derive two conclusions from this figure. First, though differences exist, the risk-sharing index does not vary a lot with respect to the specific combination of partner countries. Second, irrespective of the relative importance of different partner countries, the risk-sharing index for each country is higher than in the bilateral setting. This is the central result from our trilateral setting: measures of risk-sharing based on the stochastic discount factor approach are not sensitive to the number of countries used in their calculation.

4.3 Graphs

The evolution of the stochastic discount factors in the trilateral framework depends on five excess return shocks: three associated with the stock markets in each country plus two associated with the exchange rates. Table 5 presents the discount factor loadings on these five shocks for each of the three countries. Several findings in this table deserve attention. First, in line with the results for the bilateral setting and equation 32, all discount fac-

Table 5: Discount Factor Loadings (Trilateral)

	USA	UK	Japan
dz^d	4.07	4.07	4.07
dz^{e_1}	-0.23	1.74	
dz^{e_2}	1.51		0.23
dz^{e_3}		-1.51	-1.74
dz^{f_1}	1.86	1.86	1.86
dz^{f_2}	-0.45	-0.45	-0.45

tors load equally on the stock market excess return shocks in each country. Second, these loadings differs across stock markets, being the the strongest for the USA, and weakest for Japan. In fact, the magnitude and relative importance of these loadings on stock markets in the trilateral setting closely resemble those for the two bilateral pairs in section 3. Third, as pointed out in equations 28 and 32, each exchange rate loading forms a linear combination of the other two. For example, condition 32 and the definition of matrix A given in 29 imply the following relation between the loadings on the exchange rate excess return shocks for the domestic (USA) and the first foreign country (UK): $dz^{e_2}_{f_1} = dz^{e_2}_{d} - dz^{e_1}_{d}$. The values in Table 5 confirm this linear relationship: $(1.74 = 1.51 - (-0.23))$. Similar conclusions apply to the other exchange rate shock combinations given in the second, the third, and the fourth row of Table 5.

Figure 4 depicts the development of log discount factors through time. In this setting, all three discount factors are *simultaneously and uniquely* determined. As can be seen from the figure, their behavior closely resembles that for the bilateral country pairs. In fact, all three log discount factors move very closely together, the only difference being assigned to the fluctuations

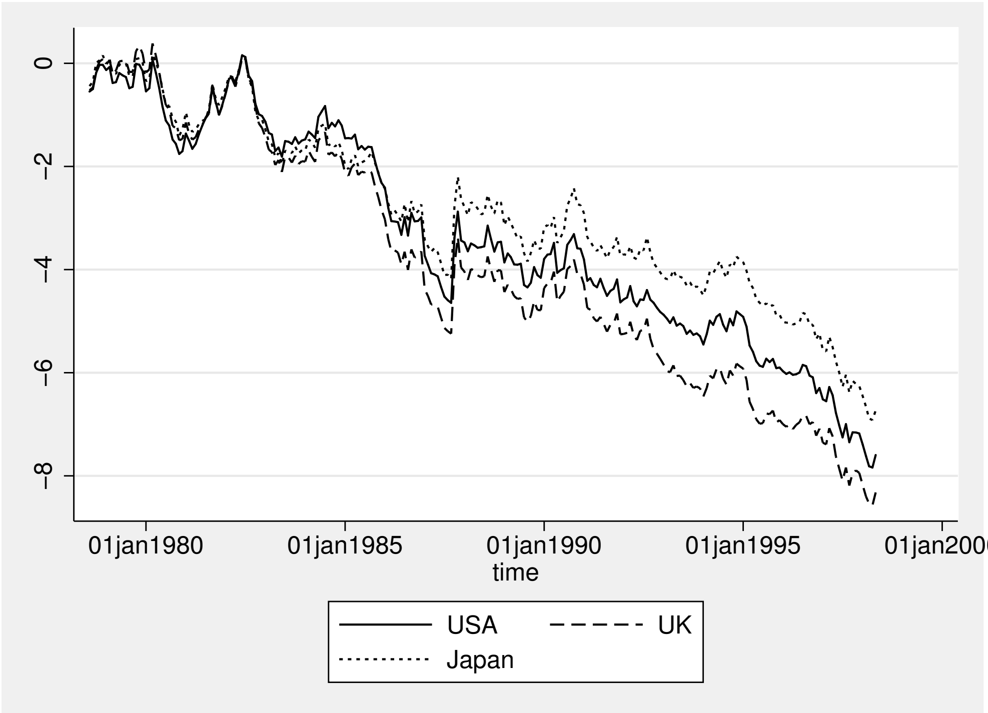

Figure 4: Log Level of Discount Factors

in the real exchange rates.

To get a better visualization, we plot the discount factor growth rates for all three country pairs (bilaterally) in Figure 5. This figure is almost identical to Figure 2, which depicted the correlation of discount factor growth rates in the bilateral setting. As in the previous case, most observations lie on or very close to the 45 degrees lines, suggesting that marginal utility growth rates are almost equalized for each bilateral country pair. This is exactly what the perfect risk-sharing condition implies.

Finally, we complete our visual inspection with a 3-dimensional scatterplot of the discount factor growth rates given in Figure 6. In fact, it represents a combination of all three scatterplots from Figure 5 and visualizes the joint correlation among the discount factor growth rates for all three countries. The figure shows that almost all points (observations) lie

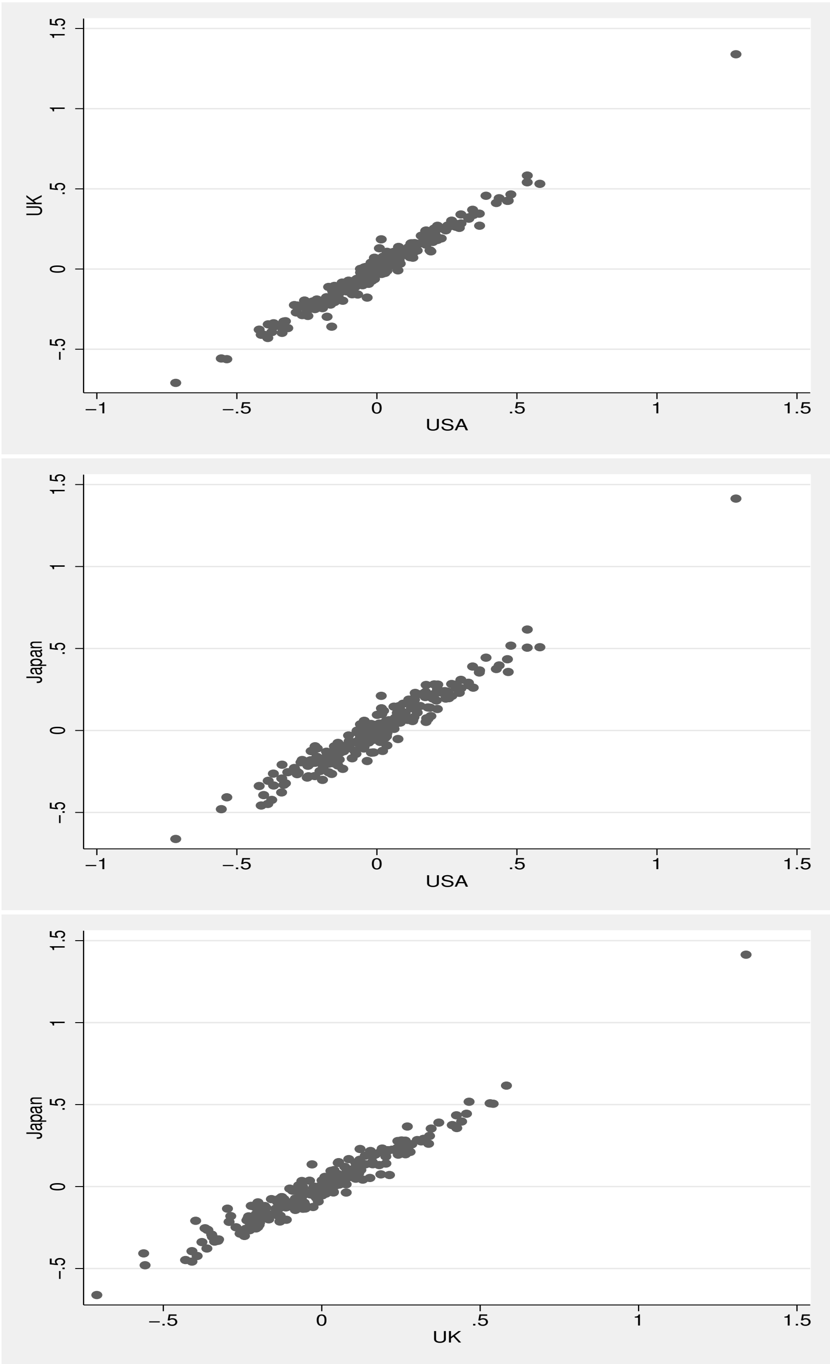

Figure 5: Discount Factor Growth Rates (Trilateral)

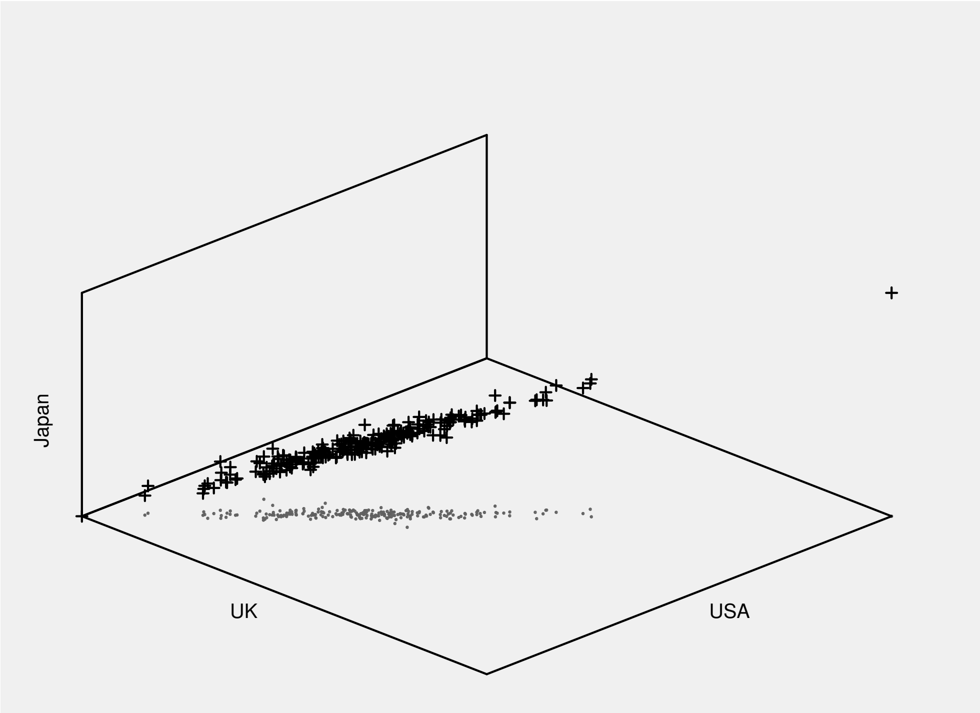

Figure 6: Correlation of Discount Factor Growth Rates (3-D)

along the spatial diagonal, suggesting quasi-equalization of all three discount factor growth rates. Thus, the evidence from this 3-dimensional scatterplot just strengthens the main conclusion: asset markets-based measures imply at least as high international risk-sharing in a trilateral as they do in a bilateral setting.

5 Discussion: Reconciliation with Macroeconomic Evidence

The trilateral framework presented here serves two purposes: first, it eliminates possible criticisms about the inherent incoherence of the bilateral SDF approach, where the evolution of the stochastic discount factor for one country crucially depends on the other country used in the calculation. Second, it offers a simple extension to calculate risk-sharing among a group of (more than two) countries.

Unambiguously, the results from the trilateral framework just strengthen the evidence about a discrepancy between the measures of international risk-sharing derived from asset markets data following the stochastic discount factor approach and those derived with macroeconomic data and specific utility function. One possible reason for these differences is the absence of complete capital markets. In fact, if asset markets account for only a small portion of total macroeconomic risks, then the low values for international risk-sharing implied by macroeconomic data can easily co-exist with the high risk-sharing measures presented here. However, these additional, non-marketable/non-insurable shocks not spanned by assets markets should be very large, negatively correlated across countries, and even more variable than the ones already observed in asset markets. In fact, Brandt et al. (2006) demonstrate that it is extremely difficult to justify the existence of such shocks. Subsequently, shocks must be even larger and more variable to rationalize the results from the trilateral setting presented in this study. Therefore, it is very unlikely that the reconciliation between these two approaches to measuring international risk-sharing would go along these lines.

The arguments above suggest that equation 10 cannot hold if the two different approaches are to be reconciled. In fact, by assuming that equation 10 holds, this approach implicitly "imposed" (almost) perfect risk-sharing. It is important to realize that this whole approach was built upon the assumption that marginal utility growth rates differ only by the changes in the real exchange rate. However, we never tested this condition empirically. In turn, it might be interesting to test not only whether this condition holds as parity (as assumed here), but rather to see whether it has the correct sign (+). If this is not the case, then the reconciliation of the two approaches might go along the lines proposed for dealing with the uncovered interest parity (UIP) anomaly and the Backus-Smith puzzle (consumption-real exchange rate correlation puzzle).

6 Concluding Remarks

In this study we presented an extension of the stochastic discount factor approach to international risk-sharing. At the beginning, we presented the theoretical framework that links the minimum-variance discount factors in two countries with the corresponding real exchange rate. We elaborated on the calculation of the discount factors, the construction of the risk-sharing index and the replication of the results for the bilateral setting given in Brandt et al. (2006). Afterwards, we proposed an extensions of this general approach to a three-country (trilateral) setting.

We conclude that the results of the stochastic discount factor approach to measuring international risk-sharing are quite robust to the number of countries used in the calculation. If anything, the already high bilateral risk-sharing index further increases with the inclusion of additional countries.

Finally, we give a note of caution on the interpretation of the results in this study. The stochastic discount factor approach to international risk-sharing is derived under the assumption that equation 10 always holds. Moreover, the replication of the results for the bilateral setting, but also the extension to a trilateral setting were performed retaining the assumption that equation 10 prices all assets at any point in time. However, if this is not the case, i.e. if the economies are far-away from what is implied by the first principles, then this approach cannot give valid measures of international risk-sharing in the first place.

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A Excess Returns in Bilateral Setting

A.1 Domestic - USA

Domestic stock

$$\frac{dS^d}{S^d} - \frac{dB^d}{B^d} = (\theta^d - r^d)dt + dz^d \tag{34}$$

Foreign bond

$$\frac{d(eB^f)}{eB^f} - \frac{dB^d}{B^d} = \frac{de}{e} + r^f dt - r^d dt = (\theta^e + r^f - f^d)dt + dz^e \tag{35}$$

Foreign stock

$$\begin{aligned}
\frac{d(eS^f)}{eS^f} - \frac{d(eB^f)}{eB^f} &= \frac{dS^f}{S^f} + \frac{de}{e}\frac{dS^f}{S^f} - \frac{dB^f}{B^f} - \frac{de}{e}\frac{dB^f}{B^f} \\
&= \Big(1 + \frac{de}{e}\Big)\Big(\frac{dS^f}{S^f} - \frac{dB^f}{B^f}\Big) \\
&= (1 + \theta^e dt + dz^e)(\theta^f dt + dz^f - r^f dt) \\
&= \theta^f dt + dz^f - r^f dt \\
&+ \theta^e dt\theta^f dt + \theta^e dtdz^f - \theta^e dtr^f dt + dz^e\theta^f dt + dz^e dz^f - dz^e r^f dt \\
&= (\theta^f - r^f)dt + dz^e dz^f + dz^f \\
&= (\theta^f - r^f + \Sigma^{ef})dt + dz^f \qquad (36)
\end{aligned}$$

A.2 Foreign

Domestic bond

$$\begin{aligned}
\frac{d\Big(\frac{B^d}{e}\Big)}{\Big(\frac{B^d}{e}\Big)} - \frac{dB^f}{B_f} &= \Big(\frac{dB^d}{B^d} - \frac{de}{e} + \frac{de_1^2}{e_1^2} - \frac{de}{e}\frac{dB^d}{B^d}\Big) - \frac{dB^f}{B_f} \\
&= r^d dt - \theta^e dt - dz^e + \Sigma^{ee}dt - \theta^d dtr^d dt - r^f dt \\
&= (r^d - r^f - \theta^e + \Sigma^{ee})dt - dz^e \\
&= -[(\theta^e + r^f - r^d - \Sigma^{ee})dt + dz^e] \qquad (37)
\end{aligned}$$

Domestic Stock

$$\frac{d\Big(\frac{S^d}{e}\Big)}{\frac{S^d}{e}} - \frac{d\Big(\frac{B^d}{e}\Big)}{\frac{B^d}{e}} = \Big(\frac{dS^d}{S^d} - \frac{de}{e} + \frac{de_1^2}{e_1^2} - \frac{de}{e}\frac{dS^d}{S^d}\Big) - \Big(\frac{dB^d}{B^d} - \frac{de}{e} + \frac{de_1^2}{e_1^2} - \frac{de}{e}\frac{dB^d}{B^d}\Big)$$

$$
\begin{aligned}
&= \frac{dS^d}{S^d} - \frac{dB^d}{B^d} - \frac{de}{e}\Big(\frac{dS^d}{S^d} - \frac{dB^d}{B^d}\Big) \\
&= (1 - \frac{de}{e})\Big(\frac{dS^d}{S^d} - \frac{dB^d}{B^d}\Big) \\
&= (1 - \theta^e dt - dz^e)(\theta^d dt + dz^d - r^d dt) \\
&= \theta^d dt + dz^d - r^d dt - \theta^e dt \theta^d dt - \theta^e dt dz^d + \theta^e dt r^d dt - dz^e \theta^d dt + dz^e dz^d - dz^e r^d dt \\
&= (\theta^d - r^d - \Sigma^{ed})dt + dz^d \qquad (38)
\end{aligned}
$$

Foreign Stock

$$\frac{dS^f}{S^f} - \frac{dB^f}{B^f} = (\theta^f - r^f)dt + dz^f \qquad (39)$$

A.3 Expected Excess Returns

Domestic investor

$$\mu^d = \begin{bmatrix} \theta^d - r^d \\ \theta^e + r^f - f^d \\ \theta^f - r^f + \Sigma^{ef} \end{bmatrix}$$

Foreign investor

$$\mu^f = \begin{bmatrix} \theta^d - r^d - \Sigma^{ed} \\ -(\theta^e + r^f - r^d - \Sigma^{ee}) \\ \theta^f - r^f \end{bmatrix}$$

B Excess Returns in Trilateral Setting

B.1 Domestic - USA

B.1.1 Domestic Stock

$$\frac{dS^d}{S^d} - \frac{dB^d}{B^d} = (\theta^d - r^d)dt + dz^d \tag{40}$$

B.1.2 Foreign 1 Bond

$$\begin{aligned}
\frac{d\Big(\frac{B^d}{e_1}\Big)}{\Big(\frac{B^d}{e_1}\Big)} - \frac{dB^{f_1}}{B_{f_1}} &= \Big(\frac{dB^d}{B^d} - \frac{de_1}{e_1} + \frac{de_1^2}{e_1^2} - \frac{de_1}{e_1}\frac{dB^d}{B^d}\Big) - \frac{dB^{f_1}}{B_{f_1}} \\
&= r^d dt - \theta^{e_1}dt - dz^{e_1} + \Sigma^{e_1e_1}dt - \theta^d dt r^d dt - r^{f_1}dt \\
&= (r^d - r^{f_1} - \theta^{e_1} + \Sigma^{e_1e_1})dt - dz^{e_1} \\
&= -[(\theta^{e_1} + r^{f_1} - r^d - \Sigma^{e_1e_1})dt + dz^{e_1}]
\end{aligned} \tag{41}$$

B.1.3 Foreign 2 Bond

$$\frac{d(e_2B^{f_2})}{e_2B^{f_2}} - \frac{dB^d}{B^d} = \frac{de_2}{e_2} + r^{f_2}dt - r^d dt = (\theta^{e_2} + r^{f_2} - r^d)dt + dz^{e_2} \tag{42}$$

B.1.4 Foreign 1 Stock

$$\begin{aligned}
\frac{d(e_1S^{f_1})}{e_1S^{f_1}} - \frac{d(e_1B^{f_1})}{e_1B^{f_1}} &= \frac{dS^{f_1}}{S^{f_1}} + \frac{de_1}{e_1}\frac{dS^{f_1}}{S^{f_1}} - \frac{dB^{f_1}}{B^{f_1}} - \frac{de_1}{e_1}\frac{dB^{f_1}}{B^{f_1}} \\
&= \Big(1 + \frac{de_1}{e_1}\Big)\Big(\frac{dS^{f_1}}{S^{f_1}} - \frac{dB^{f_1}}{B^{f_1}}\Big) \\
&= (1 + \theta^{e_1}dt + dz^{e_1})(\theta^{f_1}dt + dz^{f_1} - r^{f_1}dt) \\
&= \theta^{f_1}dt + dz^{f_1} - r^{f_1}dt \\
&+ \theta^{e_1}dt\theta^{f_1}dt + \theta^{e_1}dtdz^{f_1} - \theta^{e_1}dtr^{f_1}dt + dz^{e_1}\theta^{f_1}dt + dz^{e_1}dz^{f_1} - dz^{e_1}r^{f_1}dt
\end{aligned}$$

$$= (\theta^{f_1} - r^{f_1})dt + dz^{e_1}dz^{f_1} + dz^{f_1}$$

$$= (\theta^{f_1} - r^{f_1} + \Sigma^{e_1 f_1})dt + dz^{f_1} \tag{43}$$

B.1.5 Foreign 2 Stock

$$\frac{d(e_2 S^{f_2})}{e_2 S^{f_2}} - \frac{d(e_2 B^{f_2})}{e_2 B^{f_2}} = \frac{dS^{f_2}}{S^{f_2}} + \frac{de_2}{e_2}\frac{dS^{f_2}}{S^{f_2}} - \frac{dB^{f_2}}{B^{f_2}} - \frac{de_2}{e_2}\frac{dB^{f_2}}{B^{f_2}}$$

$$= \Big(1 + \frac{de_2}{e_2}\Big)\Big(\frac{dS^{f_2}}{S^{f_2}} - \frac{dB^{f_2}}{B^{f_2}}\Big)$$

$$= (1 + \theta^{e_2}dt + dz^{e_2})(\theta^{f_2}dt + dz^{f_2} - r^{f_2}dt)$$

$$= \theta^{f_2}dt + dz^{f_2} - r^{f_2}dt$$

$$+ \theta^{e_2}dt\theta^{f_2}dt + \theta^{e_2}dtdz^{f_2} - \theta^{e_2}dtr^{f_2}dt + dz^{e_2}\theta^{f_2}dt + dz^{e_2}dz^{f_2} - dz^{e_2}r^{f_2}dt$$

$$= (\theta^{f_2} - r^{f_2})dt + dz^{e_2}dz^{f_2} + dz^{f_2}$$

$$= (\theta^{f_2} - r^{f_2} + \Sigma^{e_2 f_2})dt + dz^{f_2} \tag{44}$$

B.2 Foreign 1 - UK

B.2.1 Domestic Stock

$$\frac{d\Big(\frac{S^d}{e_1}\Big)}{\frac{S^d}{e_1}} - \frac{d\Big(\frac{B^d}{e_1}\Big)}{\frac{B^d}{e_1}} = \Big(\frac{dS^d}{S^d} - \frac{de_1}{e_1} + \frac{de_1^2}{e_1^2} - \frac{de_1}{e_1}\frac{dS^d}{S^d}\Big) - \Big(\frac{dB^d}{B^d} - \frac{de_1}{e_1} + \frac{de_1^2}{e_1^2} - \frac{de_1}{e_1}\frac{dB^d}{B^d}\Big)$$

$$= \frac{dS^d}{S^d} - \frac{dB^d}{B^d} - \frac{de_1}{e_1}\Big(\frac{dS^d}{S^d} - \frac{dB^d}{B^d}\Big)$$

$$= (1 - \frac{de_1}{e_1})\Big(\frac{dS^d}{S^d} - \frac{dB^d}{B^d}\Big)$$

$$= (1 - \theta^{e_1}dt - dz^{e_1})(\theta^d dt + dz^d - r^d dt)$$

$$= \theta^d dt + dz^d - r^d dt - \theta^{e_1}dt\theta^d dt - \theta^{e_1}dtdz^d + \theta^{e_1}dtr^d dt - dz^{e_1}\theta^d dt + dz^{e_1}dz^d - dz^{e_1}r^d dt$$

$$= (\theta^d - r^d - \Sigma^{e_1 d})dt + dz^d \tag{45}$$

B.2.2 Domestic Bond

$$\frac{d(e_1 B^{f_1})}{e_1 B^{f_1}} - \frac{dB^d}{B^d} = \frac{de_1}{e_1} + r^{f_1}dt - r^d dt = (\theta^{e_1} + r^{f_1} - r^d)dt + dz^{e_1} \tag{46}$$

B.2.3 Foreign 2 Bond

$$\begin{aligned}
\frac{d\Big(\frac{B^{f_2}}{e_3}\Big)}{\Big(\frac{B^{f_2}}{e_3}\Big)} - \frac{dB^{f_1}}{B_{f_1}} &= \Big(\frac{dB^{f_2}}{B^{f_2}} - \frac{de_3}{e_3} + \frac{de_3^2}{e_3^2} - \frac{de_3}{e_3}\frac{dB^{f_2}}{B^{f_2}}\Big) - \frac{dB^{f_1}}{B_{f_1}} \\
&= r^{f_2}dt - \theta^{e_3}dt - dz^{e_3} + \Sigma^{e_3 e_3}dt - \theta^{f_2}dt r^{f_2}dt - r^{f_1}dt \\
&= (r^{f_2} - r^{f_1} - \theta^{e_3} + \Sigma^{e_3 e_3})dt - dz^{e_3} \\
&= -[(\theta^{e_3} + r^{f_1} - r^{f_2} - \Sigma^{e_3 e_3})dt + dz^{e_3}]
\end{aligned} \tag{47}$$

B.2.4 Foreign 1 Stock

$$\frac{dS^{f_1}}{S^{f_1}} - \frac{dB^{f_1}}{B^{f_1}} = (\theta^{f_1} - r^{f_1})dt + dz^{f_1} \tag{48}$$

B.2.5 Foreign 2 Stock

$$\begin{aligned}
&\frac{d\Big(\frac{S^{f_2}}{e_3}\Big)}{\frac{S^{f_2}}{e_3}} - \frac{d\Big(\frac{B^{f_2}}{e_3}\Big)}{\frac{B^{f_2}}{e_3}} = \Big(\frac{dS^{f_2}}{S^{f_2}} - \frac{de_3}{e_3} + \frac{de_3^2}{e_3^2} - \frac{de_3}{e_3}\frac{dS^{f_2}}{S^{f_2}}\Big) - \Big(\frac{dB^{f_2}}{B^{f_2}} - \frac{de_3}{e_3} + \frac{de_3^2}{e_3^2} - \frac{de_3}{e_3}\frac{dB^{f_2}}{B^{f_2}}\Big) \\
&= \frac{dS^{f_2}}{S^{f_2}} - \frac{dB^{f_2}}{B^{f_2}} - \frac{de_3}{e_3}\Big(\frac{dS^{f_2}}{S^{f_2}} - \frac{dB^{f_2}}{B^{f_2}}\Big) \\
&= (1 - \frac{de_3}{e_3})\Big(\frac{dS^{f_2}}{S^{f_2}} - \frac{dB^{f_2}}{B^{f_2}}\Big) \\
&= (1 - \theta^{e_3}dt - dz^{e_3})(\theta^{f_2}dt + dz^{f_2} - r^{f_2}dt) \\
&= \theta^{f_2}dt + dz^{f_2} - r^{f_2}dt - \theta^{e_3}dt\theta^{f_2}dt - \theta^{e_3}dt dz^{f_2} + \theta^{e_3}dt r^{f_2}dt \\
&- dz^{e_3}\theta^{f_2}dt + dz^{e_3}dz^{f_2} - dz^{e_3}r^{f_2}dt \\
&= (\theta^{f_2} - r^{f_2} - \Sigma^{e_3 f_2})dt + dz^{f_2}
\end{aligned} \tag{49}$$

B.3 Foreign 2 - Japan

B.3.1 Domestic Stock

$$\begin{aligned}
\frac{d\left(\frac{S^d}{e_2}\right)}{\frac{S^d}{e_2}} - \frac{d\left(\frac{B^d}{e_2}\right)}{\frac{B^d}{e_2}} &= \Big(\frac{dS^d}{S^d} - \frac{de_2}{e_2} + \frac{de_2^2}{e_2^2} - \frac{de_2}{e_2}\frac{dS^d}{S^d}\Big) - \Big(\frac{dB^d}{B^d} - \frac{de_2}{e_2} + \frac{de_2^2}{e_2^2} - \frac{de_2}{e_2}\frac{dB^d}{B^d}\Big) \\
&= \frac{dS^d}{S^d} - \frac{dB^d}{B^d} - \frac{de_2}{e_2}\Big(\frac{dS^d}{S^d} - \frac{dB^d}{B^d}\Big) \\
&= (1 - \frac{de_2}{e_2})\Big(\frac{dS^d}{S^d} - \frac{dB^d}{B^d}\Big) \\
&= (1 - \theta^{e_2}dt - dz^{e_2})(\theta^d dt + dz^d - r^d dt) \\
&= \theta^d dt + dz^d - r^d dt - \theta^{e_2}dt\theta^d dt - \theta^{e_2}dtdz^d + \theta^{e_2}dtr^d dt - dz^{e_2}\theta^d dt + dz^{e_2}dz^d - dz^{e_2}r^d dt \\
&= (\theta^d - r^d - \Sigma^{e_2 d})dt + dz^d \qquad (50)
\end{aligned}$$

B.3.2 Domestic Bond

$$\begin{aligned}
\frac{d\left(\frac{B^d}{e_2}\right)}{\left(\frac{B^d}{e_2}\right)} - \frac{dB^{f_2}}{B_{f_2}} &= \Big(\frac{dB^d}{B^d} - \frac{de_2}{e_2} + \frac{de_2^2}{e_2^2} - \frac{de_2}{e_2}\frac{dB^d}{B^d}\Big) - \frac{dB^{f_2}}{B_{f_2}} \\
&= r^d dt - \theta^{e_2}dt - dz^{e_2} + \Sigma^{e_2 e_2}dt - \theta^d dtr^d dt - r^{f_2}dt \\
&= (r^d - r^{f_2} - \theta^{e_2} + \Sigma^{e_2 e_2})dt - dz^{e_2} \\
&= -[(\theta^{e_2} + r^{f_2} - r^d - \Sigma^{e_2 e_2})dt + dz^{e_2}] \qquad (51)
\end{aligned}$$

B.3.3 Foreign 1 Bond

$$\frac{d(e_3 B^{f_1})}{e_3 B^{f_1}} - \frac{dB^{f_2}}{B^{f_2}} = \frac{de_3}{e_3} + r^{f_1}dt - r^{f_2}dt = (\theta^{e_3} + r^{f_1} - f^{f_2})dt + dz^{e_3} \qquad (52)$$

B.3.4 Foreign 1 Stock

$$\frac{d(e_3 S^{f_1})}{e_3 S^{f_1}} - \frac{d(e_3 B^{f_1})}{e_3 B^{f_1}} = \frac{dS^{f_1}}{S^{f_1}} + \frac{de_3}{e_3}\frac{dS^{f_1}}{S^{f_1}} - \frac{dB^{f_1}}{B^{f_1}} - \frac{de_3}{e_3}\frac{dB^{f_1}}{B^{f_1}}$$

$$
\begin{aligned}
&= \Big(1 + \frac{de_3}{e_3}\Big)\Big(\frac{dS^{f_1}}{S^{f_1}} - \frac{dB^{f_1}}{B^{f_1}}\Big) \\
&= (1 + \theta^{e_3}dt + dz^{e_3})(\theta^{f_1}dt + dz^{f_1} - r^{f_1}dt) \\
&= \theta^{f_1}dt + dz^{f_1} - r^{f_1}dt \\
&+ \theta^{e_3}dt\theta^{f_1}dt + \theta^{e_3}dtdz^{f_1} - \theta^{e_3}dtr^{f_1}dt + dz^{e_3}\theta^{f_1}dt + dz^{e_3}dz^{f_1} - dz^{e_3}r^{f_1}dt \\
&= (\theta^{f_1} - r^{f_1})dt + dz^{e_3}dz^{f_1} + dz^{f_1} \\
&= (\theta^{f_1} - r^{f_1} + \Sigma^{e_3 f_1})dt + dz^{f_1} \qquad (53)
\end{aligned}
$$

B.3.5 Foreign 2 Stock

$$
\frac{dS^{f_2}}{S^{f_2}} - \frac{dB^{f_2}}{B^{f_2}} = (\theta^{f_2} - r^{f_2})dt + dz^{f_2} \qquad (54)
$$

C Expected Excess Returns

C.1 Domestic - USA

$$\mu^d = \begin{bmatrix} \theta^d - r^d \\ -(\theta^{e_1} + r^{f_1} - r^d - \Sigma^{e_1 e_1}) \\ \theta^{e_2} + r^{f_2} - r^d \\ \theta^{f_1} - r^{f_1} + \Sigma^{e_1 f_1} \\ \theta^{f_2} - r^{f_2} + \Sigma^{e_2 f_2} \end{bmatrix}$$

C.2 Foreign 1 - UK

$$\mu^{f_1} = \begin{bmatrix} \theta^d - r^d - \Sigma^{e_1 d} \\ \theta^{e_1} + r^{f_1} - r^d \\ -(\theta^{e_3} + r^{f_1} - r^{f_2} - \Sigma^{e_3 e_3}) \\ \theta^{f_1} - r^{f_1} \\ \theta^{f_2} - r^{f_2} - \Sigma^{e_3 f_2} \end{bmatrix}$$

C.3 Foreign 2 - Japan

$$\mu^{f_2} = \begin{bmatrix} \theta^d - r^d - \Sigma^{e_2 d} \\ -(\theta^{e_2} + r^{f_2} - r^d - \Sigma^{e_2 e_2}) \\ \theta^{e_3} + r^{f_1} - f^{f_2} \\ \theta^{f_1} - r^{f_1} + \Sigma^{e_3 f_1} \\ \theta^{f_2} - r^{f_2} \end{bmatrix}$$