Indeterminacy and unemployment fluctuations with constant returns to scale in production*

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Abstract

We incorporate imperfectly insured unemployment in the finance constrained economy proposed by Woodford (1986), by introducing unions and unemployment benefits financed by labor taxation. We show that this simple extension of the Woodford model changes drastically its stability conditions and local dynamics around the steady state. In fact, in contrast to related models in the literature, we find that under constant returns to scale in production: (i) indeterminacy *always* prevails in the case of a unitary elasticity of substitution between capital and labor; (ii) flip and Hopf bifurcations occur for empirically credible elasticities of substitution between capital and labor, so that a rich set of dynamics may emerge at "realistic" parameters' values.

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1 Introduction

In this paper, we study the consequences of labor market frictions and imperfect unemployment insurance in an economy where workers are financially constrained. We investigate whether these features affect the emergence of endogenous (sunspot-driven) fluctuations, by analyzing the occurrence of local indeterminacy and local bifurcations.

To do so, we introduce unemployment benefits and unions in the finance constrained economy proposed by Woodford (1986), and extended by Grandmont et al. (1998) to a general production function with constant returns to scale. The crucial assumption of the Woodford model is that "capitalists" discount the future less than "workers", and thus end up owning the whole capital stock. Capitalists then simply live of capital rents, accumulating capital through a traditional, unconstrained, consumption-saving choice, while workers, being submitted to a liquidity constraint, can only consume out of wage earnings. While the Woodford framework is a particularly relevant starting point (since workers do not possess capital, they cannot use it as a collateral and thus face difficulties in financing credit activities), the assumption of financially constrained workers is probably most salient if workers are assumed to face real income uncertainty, due notably to the risk of being unemployed.

We consider therefore a setup in which, as in Lloyd-Braga and Modesto (2004), wages and employment are bargained between unions and firms, and where unemployment emerges as an equilibrium result. However, in contrast to that paper and to many models in the Real Business Cycle literature, we do not assume that there exists a perfect insurance/redistributive mechanism that allows workers to completely insure themselves against the revenue losses they would incur if unemployed.[1] We consider instead an *imperfect unemployment insurance scheme* in which the government guarantees a fixed minimum real income to those unemployed, financed by taxing employed workers. Since unions are able to set real wages (net of taxes) above the real income received when unemployed, unemployment is welfare costly from a worker point of view.

We find that this simple extension of the Woodford model changes drastically its local dynamics around the steady state. By contrast to most related

[1] Obviously, in the presence of such a perfect insurance scheme, unemployment would not be a major problem since optimal diversification of risk would prevent a sharp fall in earnings under these circumstances (Hansen (1985), Rogerson, (1988)).

models in the literature, we find that deterministic and stochastic endogenous fluctuations, driven by self-fulfilling volatile expectations, can emerge in this economy under fairly plausible values for the elasticity of input substitution, without requiring either increasing returns to scale in production or a sufficiently high share of government expenditures. In particular, in the case of a unitary elasticity of substitution between capital and labor, we find that indeterminacy *always* prevails. Furthermore, provided union power is sufficiently strong, flip and Hopf bifurcations are shown to occur for values of the elasticity of input substitution that are relatively close to one and, therefore, in accordance with the empirical literature.

The rest of the paper is organized as follows. In the next section we describe the model and obtain the (deterministic perfect foresight) dynamic equilibrium equations. Section 3 analyzes and discusses the local dynamic properties of the model and the occurrence of local bifurcations. In section 4, we provide an economic interpretation of the indeterminacy mechanism and compare our results with the related literature. Finally, in section 5, we present some concluding remarks.

2 The Model

The economy we consider is composed of 5 types of agents: workers, capitalists, firms, unions and the government. All markets are assumed to be perfectly competitive, with the exception of the labour market where union power will prevent the wage from falling to its walrasian level.

2.1 The agents

Workers There is a continuum of identical infinitely lived workers, each worker supplying inelastically one unit of labor. Preferences of workers are described by the following utility function: $E\sum_{t=0}^{\infty}\gamma^t u(c_t^w)$, where $0<\gamma<1$ is the constant discount factor and c_t^w is consumption in period t.[2] Workers face, as in Woodford (1986), a consumption-saving choice in the presence of liquidity constraints and, in our case, also under income uncertainty.[3]

[2] We assume that u satisfies the usual properties, namely: $u(c_t)$ is a continuous real valued function in $c_t \geq 0$, with $u'(c) > 0$ and $u''(c) \leq 0$ for $c_t > 0$.

[3] Formally, our description of the consumption behavior of workers can be seen as a direct application of the general framework studied in Deaton (1991).

Indeed, in each period t, a worker may be either employed (state e) - receiving in cash, at the beginning of next period, a nominal wage w_t - or unemployed (state u). We assume that the government provides a minimum guaranteed income program, ensuring to all period t unemployed workers a constant real income $b > 0$.[4] As in the case of employed workers, these resources only become available for consumption at the beginning of next period. These transfers are financed by taxing period t employed workers at the beginning of next period (i.e., when their labour income becomes available). Since the government balances its budget, the real lump-sum tax τ_t, paid by each worker employed in period t, will be determined endogenously by the balanced-budget condition. Hence, in period t, a worker will receive a state-dependant revenue $y_t^i \in \{w_{t-1} - p_t\tau_{t-1}, p_t b\}$, conditioned on being in state $i \in \{e, u\}$ in period $t-1$, where p_t is the price of output in period t. We also assume that, when deciding how much to consume in t, the worker does not know yet whether he will be employed or unemployed during that period. However, he can put a probability distribution over the two states, which consists in period t employment (l_t) and unemployment rates $(1 - l_t)$, respectively.

In Appendix A.1 we solve the workers problem and give the conditions under which both employed and unemployed workers always (rationally) choose not to hold capital or money and to spend all their available income on current consumption,[5] so that we have:

$$c_t^w = \frac{y_t^i}{p_t} \qquad i \in \{e, u\} \tag{1}$$

Unions In each period, identical unions bargain with identical firms over wages and employment. We assume that all workers are unionized and that there is one union per firm. Workers are matched exogenously and uniformly with unions and cannot move between firms or unions, so that each union represents the same mass of workers, which we normalize to 1. Assuming that each union wishes to maximize the expected discounted sum of its members' total future consumption, we obtain (see (1)) the following objective function

[4] Note that in most European countries, where such minimum guaranteed income programmes exist, they are indeed indexed to inflation, in order to ensure real purchasing power of the poor.

[5] This means that we focus on equilibria that are sufficiently close to the steady state equilibrium in which γ is sufficiently low compared to the capitalists discount rate β (see Appendix A.1).

for the representative union:

$$\Omega_t = E_t \left\{ \left[\left(\frac{w_t}{p_{t+1}} - \tau_t \right) l_t + b(1 - l_t) \right] + \gamma \Omega_{t+1} \right\} \tag{2}$$

where l denotes employment at the respective firm.[6]

Government The government guarantees a constant minimal amount of real income to each unemployed worker, b, collecting from each employed individual a given amount τ that balances the budget. Hence we have that:

$$\tau_t = b(1 - l_t)/l_t. \tag{3}$$

The reason for assuming a tax per (employed) worker, instead of considering, as usually done in the literature, a labour income tax, is that the former type of taxation can also be interpreted as an insurance mechanism (*imperfect*, due to the existence of unions, as we shall see) provided by the government. To participate in this programme, each worker pays a real premium τ, receiving in the event of unemployment a real amount b (net of the premium). As usual, the premium must cover the expected value of payments, i.e., $\tau = (b + \tau)(1 - l)$, that we can rewrite as $b(1 - l) = \tau l$. Note also that, since each employed worker supplies one unit of labour, τ is also a tax per unit of labour.

Capitalists As in Woodford (1986), capitalists are identical and maximize $E \sum_{t=0}^{\infty} \beta^t Log c_t^c$, $0 < \beta < 1$, subject to $p_t c_t^c + p_t k_{t+1}^c + m_{t+1}^c = p_t R_t k_t^c + m_t^c$, where c_t^c is consumption in period t, k_t^c and m_t^c are respectively the capital stock and money holdings at the outset of period t, $R_{t+1} = (\rho_t + 1 - \delta)$ is the real gross rate of return on capital, ρ_t is the real rental rate of capital and $0 \leq \delta \leq 1$ is the capital depreciation rate. Under the condition $R_{t+1} > E_t \{p_t/p_{t+1}\}$, the solution to this problem may be written as (see Woodford, 1986):

$$\begin{aligned} c_t^c &= (1 - \beta) R_t k_t^c && (4) \\ k_{t+1}^c &= \beta R_t k_t^c && (5) \\ m_{t+1}^c &= 0 && (6) \end{aligned}$$

[6] As we have normalize the mass of workers per firm to 1, l represents both the employment level and the employment rate in each firm. At a symmetric equilibrium, and as workers are treated anonymously, it also represents the probability of being employed.

Firms Firms are identical and each firm operates under a constant returns to scale (CRS) technology, $Al_t f(x_t)$, where $x \equiv k/l$ is the capital labor ratio and $A > 0$ is a scaling factor.[7] The representative firm wishes to maximize the present value of expected discounted profits, Π_t, but must negotiate wages and employment with the respective union. Also, since period t wages are paid in cash at the beginning of next period, the firm will have to hold, at the end of period t, $m^f_{t+1} \geq w_t l_t$. At each period t, the sequence of decisions is the following. First, the firm pays last period wages out of their money holdings and rents capital, k_t, on the economy-wide capital market, at a given nominal rental rate $p_t \rho_t$. Next, wages, w_t, and employment, l_t, are determined through the bargaining process. Finally, the firm decides the level of money holdings and production takes place.[8] In order to ensure time consistency of the equilibrium, the problem of the firm must be solved backwards, starting with the decision on money holdings. In Appendix A.2 we show that the cash constraint is always binding, i.e. $m^f_{t+1} = w_t l_t$. We proceed now with the wage-employment bargain and then with capital decisions.

2.2 Wage, employment and capital decisions

Wages and employment are determined through an efficient bargaining procedure. This implies that l_t and w_t must solve the generalized Nash bargaining problem:

$$\underset{(w_t,l_t)\in\Re^2_{++}}{Max} \quad \left(\Pi_t - \overline{\Pi}_t\right)^{\alpha} \left(\Omega_t - \overline{\Omega}_t\right)^{(1-\alpha)} \quad s.t. \quad l_t \leq 1 \tag{7}$$

where $0 < \alpha \leq 1$ represents the firm's power in the bargain, and $(\overline{\Pi}_t, \overline{\Omega}_t)$ are the fallback payoffs of each party if no agreement in period t is reached.[9] Using (2), the fallback payoff of a union is given by $\overline{\Omega}_t = b + \gamma\Omega_{t+1}$, so that $\Omega_t - \overline{\Omega}_t = l_t \left(\frac{w_t}{p_{t+1}} - b - \tau_t\right)$. Given the sequence of decisions of firms, it can be shown that $\Pi_t - \overline{\Pi}_t = p_t A l_t f(x_t) - w_t l_t$ (see Appendix A.2).

[7] We also make the following standard assumptions on technology: $f(x)$ is a real, continuous function for $x \geq 0$, positively valued and differentiable as many times as needed for $x > 0$, with $f^{'}(x) > 0$, $f''(x) < 0$, so that $f(x) - f^{'}(x)x > 0$.

[8] As usually done in the literature, we are assuming that workers cannot sign binding wage contracts, so that the wage and employment are determined after the capital stock decision has been made.

[9] If negotiations fail, production does not take place and all workers are unemployed.

We assume that all agents in the economy are "small", in the sense that they take τ_t as given. We also assume that the solution l_t of problem (7) always satisfies $l_t < 1$, so that there is unemployment. Hence, the first order conditions are:

$$(b + \tau_t) E_t \frac{p_{t+1}}{p_t} = A\left[f(x_t) - f^{'}(x_t)x_t\right] \tag{8}$$

$$\frac{w_t}{p_t} = A\left[f(x_t) - \alpha f^{'}(x_t)x_t\right] \equiv \mu(x_t) A\left[f(x_t) - f^{'}(x_t)x_t\right], \tag{9}$$

where $\mu(x_t) \equiv \left[f(x_t) - \alpha f^{'}(x_t)x_t\right] / \left[f(x_t) - f^{'}(x_t)x_t\right]$ is the markup factor.

From (8) we can see that, whatever the union's bargaining power, employment is determined by the intersection of the marginal productivity of labour (MPL) curve, $A\left[f(x_t) - f^{'}(x_t)x_t\right]$, with the real reservation wage schedule, $(b + \tau_t)E_t p_{t+1}/p_t$, see Figure 1. Using also (9), we see that when unions have no power in the bargain, $\alpha = 1$, we recover the perfectly competitive labor market case, where real wages are identical to the marginal productivity of labour and, thereby, to the real reservation wage. By contrast, when $\alpha < 1$, the real wage is set above the MPL (and so above the reservation wage), with a markup $\mu(x)$ which, for a given x, is increasing in the bargaining power of unions $(1 - \alpha)$. Given the absence of perfect redistributive schemes, unemployed individuals are thus clearly worse off. Finally, note also that employment is influenced by expectations of future inflation (shifting the reservation wage locus), which constitutes a potential channel for the emergence of expectations driven fluctuations.

The firm, anticipating the result of the bargaining process, chooses consequently $k_t > 0$ to maximize the expected discounted flows of future profits, Π_t, or, equivalently, current profits, $(p_t A l_t f(x_t) - p_t \rho_t k_t - w_t l_t)$, see Appendix A.2. Using (9), current profits can be rewritten as $p_t \alpha A f^{'}(x_t) k_t - p_t \rho_t k_t$, where l_t satisfies (8), and we obtain the following first order condition:[10]

$$\alpha A f^{'}(x_t) = \rho_t. \tag{10}$$

2.3 Equilibrium

We now obtain the dynamic equilibrium equations of our model under perfect foresight. Assuming an identical number of capitalists and firms, equilibrium

[10]Note that, because firms operate under constant returns to scale, profits are zero at equilibrium.

in the capital services market requires that $k_{t+1} = k^c_{t+1}$. Using the definition of R and equations (5) and (10), we obtain equation (11) below. Considering, as in Woodford (1986), a constant (per firm) amount of outside money in the economy, m, money market clearing in every period requires that $m = m^f_{t+1} = w_t l_t = m^f_{t+2} = w_{t+1} l_{t+1}$, so that realized inflation is given by $p_{t+1}/p_t = (w_t l_t / p_t) / (w_{t+1} l_{t+1}/p_{t+1})$. Using this last relation, equations (3), (8) and (9), and noticing that under perfect foresight $p_{t+1}/p_t = E_t \{p_{t+1}/p_t\}$, we obtain equation (12) below. Accordingly we have:

Definition 1 *An intertemporal equilibrium with perfect foresight is a sequence* $(k_t,\, l_t) \in \Re^2_{++}$, $t = 0, 1, ...\infty$ *that solves the two-dimensional dynamic system, with* $x_t \equiv k_t/l_t$

$$k_{t+1} = \beta \left[\alpha A f^{'}(x_t) + (1-\delta)\right] k_t \tag{11}$$

$$l_{t+1} A \left[f(x_{t+1}) - \alpha f^{'}(x_{t+1}) x_{t+1}\right] = b \frac{\left[f(x_t) - \alpha f^{'}(x_t) x_t\right]}{\left[f(x_t) - f^{'}(x_t) x_t\right]}. \tag{12}$$

Equations (11) and (12) define implicitly a two dimensional dynamic system[11] that describes the deterministic equilibrium trajectories of employment, a non predetermined variable whose value in period t is influenced by expectations of future inflation (see section 2.2), and capital, a predetermined variable whose value in period t is fixed by past savings of capitalists (see (5)).

3 Local dynamics and bifurcation analysis

To study the local stability properties of our two dimensional dynamic system (11) and (12), around its interior steady state solution (k^*, l^*),[12] we use the

[11] Note that (11) and (12) define locally a two dimensional dynamic system of the form $(k_{t+1}, l_{t+1}) = G(k_t, l_t)$, provided the elasticity of $l_{t+1} A \left[f(x_{t+1}) - \alpha f^{'}(x_{t+1}) x_{t+1}\right]$ with respect to l_{t+1} does not vanish at the steady state under analysis.

[12] Given (11) and (12), a steady state equilibrium $(k^*, l^*) \equiv (x^* l^*, l^*)$ is a solution of the following two equations: $f^{'}(x)\beta\alpha/\theta = 1/A$, where $\theta \equiv 1 - \beta(1-\delta)$, and $b = lA\left[f(x) - f'(x)x\right]$. Under the assumptions made on technology (see footnote 7), $f^{'}(x)$ is a continuous decreasing function, so that there exists at most one solution $x^* > 0$ to the first equation. Given x^*, the second equation also has a unique solution $l^* > 0$. In order to ensure the existence of the unique interior steady state (k^*, l^*), with $l^* < 1$ (so that, at

geometrical method proposed by Grandmont et al. (1998). This method amounts to study how the trace, T, and the determinant, D, of the Jacobian matrix of system (11)-(12), evaluated at the steady steady state, evolve in the space (T, D) when some relevant parameters of the model are made to vary continuously in their admissible range. In Appendix A.3 we show that D and T can be written in terms of $\sigma > 0$, the steady state elasticity of substitution between capital and labor, of $s_L \in (0,1)$, the steady state labor share of output, of the bargaining power of firms, α, and of the parameter $\theta \in (0,1]$, $\theta \equiv 1 - \beta(1-\delta)$. Since empirical values for σ and α are either not precisely estimated or may differ substantially across countries, we shall organize our discussion in terms of these two parameters, considering that θ and s_L take some fixed value in their admissible range.[13] Moreover, to ease the exposition, we assume that $\sigma > 1 - s_L$, which covers all the empirically interesting cases. The main results of this analysis are given in Proposition 1 below (see Appendix A.3 for details).

Proposition 1 *For $\sigma > 1 - s_L$ and defining $\alpha_1 = \frac{2(1-s_L)}{2-s_L}$, $\alpha_2 = \frac{(2+\theta)(1-s_L)}{(2+\theta-2s_L)}$, $\alpha_3 = \frac{4(1-s_L)}{4(1-s_L)+\theta}$, $\sigma_F = \frac{2[(\alpha-1+s_L)-(1-s_L)(1-\alpha)]-(\alpha-1+s_L)s_L(2-\theta)}{2[(\alpha-1+s_L)-(1-s_L)(1-\alpha)]}$ and $\sigma_H = \frac{(1-s_L)}{\alpha}$, the following generically holds:*

(i) *if $1 - s_L < \alpha < \alpha_1$, the steady state is a source for $\sigma < \sigma_H$, undergoes a Hopf bifurcation for $\sigma = \sigma_H < 1$, becomes a sink for $\sigma_H < \sigma < \sigma_F$, undergoes a flip bifurcation for $\sigma = \sigma_F > 1$, and becomes a saddle for $\sigma > \sigma_F$.*

(ii) *if $\alpha_1 < \alpha < \alpha_2$, the steady state is a source for $\sigma < \sigma_H$, undergoes a Hopf bifurcation for $\sigma = \sigma_H < 1$, and becomes a sink for $\sigma > \sigma_H$.*

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the steady state and in its neighborhood, (8) and (9) are indeed the equilibrium conditions on the labor market) we assume that: $(\beta\alpha/\theta)\lim_{x\to 0} f'(x) > 1/A > (\beta\alpha/\theta)\lim_{x\to\infty} f'(x)$, so that the function $F(x) \equiv f'(x)\beta\alpha/\theta$ will cross $1/A$ exactly once at a value $x^* > 0$, and $0 < b < A\left[f(x) - f'(x)x\right]$, so that, given x^*, l^* is necessarily lower than 1.

[13] We think that it is more interesting to study the dynamics in terms of the labour share of output $s_L = \left[f(x) - \alpha f'(x)x\right]/f(x)$, which is an economic meaningful 'parameter' for which there are empirical estimations, than in terms of the technological 'parameter', $f'(x)x/f(x)$. Of course, doing so implies that when we consider different configurations for α, while keeping fixed the value of s_L, we implicitly assume that the elasticity of $f(x)$ adjusts, so that s_L can indeed remain constant. Moreover, since we keep $0 < s_L < 1$ fixed, the assumptions made on technology (see footnote 7) imply that $\alpha > 1 - s_L$.

(iii) *if $\alpha_2 < \alpha < \alpha_3$, the steady state is a saddle if $\sigma < \sigma_F$, undergoes a flip bifurcation for $\sigma = \sigma_F < 1$, becomes a source for $\sigma_F < \sigma < \sigma_H$, undergoes a Hopf bifurcation for $\sigma = \sigma_H < 1$, and becomes a sink for $\sigma > \sigma_H$.*

(iv) *if $\alpha_3 < \alpha \leq 1$, the steady state is a saddle if $\sigma < \sigma_F$, undergoes a flip bifurcation for $\sigma = \sigma_F < 1$, and becomes a sink for $\sigma > \sigma_F$.*

Proof. See Appendix A.3. ■

From direct inspection of Proposition 1, it is easy to see that when $\sigma = 1$ the steady state is always sink. Since a Cobb-Douglas technology is often taken as a benchmark case in the literature, we highlight this result in the following corollary:

Corollary 1 *The Cobb-Douglas case*
For $\sigma = 1$ the steady state is a sink.

The above findings on local dynamics and bifurcations are depicted in Figure 2, where we have plotted in the (α, σ) plane the bifurcation values (σ_H and σ_F) that divide the plane into different regions in which the steady state is either a sink, a source or a saddle.

Indeterminacy A well known feature of dynamic models is that, when the steady state is locally indeterminate, there exist infinitely many nondegenerate stochastic equilibria driven by self fulfilling expectations (sunspots equilibria) that stay arbitrarily close to the steady state, as shown for instance in Grandmont et al. (1998). In the context of our model, where only capital is a predetermined variable, the steady state is locally indeterminate when it is a sink.

In that respect, one main striking feature highlighted by Figure 2 is that, in this economy with constant returns to scale in production, *indeterminacy occurs for a wide range of values for the elasticity of substitution σ*, including the Cobb-Douglas case. Moreover, *in the* latter *case of $\sigma = 1$*, as emphasized in Corollary 1, *the steady state is always indeterminate, independently of the values of the other parameters.* These two results are important since empirical studies point to values of σ that may differ among countries, but are not very far from one (Hamermesh (1993), Duffy and Papageorgiou, 2000). As these empirical values typically belong to the range of values for which the steady state is a sink, indeterminacy truly appears as a pervasive phenomenon in our economy.

In this dimension, our results differ substantially from related models in the literature that have also addressed the indeterminacy issue within the Woodford (1986) framework. For example, it is well-known that when the standard Woodford model is extended to a general production function (Grandmont et al., 1998), indeterminacy can only occur with constant returns to scale in production if capital and labor are highly complementary, implying a value for σ that is not supported by the available empirical evidence. As shown in Cazzavillan et al. (1998), indeterminacy is compatible with larger values for σ if increasing returns to scale are assumed. But, with a unitary elasticity of substitution, the required degree of increasing returns (around 30% for a quarterly calibration, see Barinci and Chéron, 2001) is also at odds with the recent empirical estimates provided by Burnside et al. (1995) and Basu and Fernald (1997). Moreover, in Lloyd-Braga and Modesto (2004), where the Cazzavillan et al. (1998) framework is extended to account for wage and employment bargaining between unions and firms, but within a framework with no taxes or unemployment benefits, indeterminacy still requires the same amount of increasing returns, when a unitary elasticity of substitution (and the same quarterly calibration) is considered.

Bifurcations Another well known feature of dynamic models is that both deterministic and stochastic endogenous fluctuations may also emerge through the occurrence of bifurcations. When a *Hopf* bifurcation occurs, deterministic cycles – periodic or quasi periodic orbits – surrounding the steady state in the state space emerge, and when a *flip* bifurcation occurs, deterministic cycles of period two appear. Moreover, theses cycles *even appear when the steady state is locally determinate*, provided *Hopf* (flip) bifurcations are supercritical. In this case, as shown in Grandmont et al. (1998), there are also infinitely many bounded stochastic equilibria driven by extrinsic uncertainty, remaining in a compact set that contains in its interior the stable cycle.

However, for such situations to be considered seriously as a possible explanation of actual business cycles, the relevant issue is not only whether bifurcations are possible, but mostly if they occur for *empirically plausible* values of the parameters. For example, in Grandmont et al. (1998) and in many related papers, bifurcations - although possible - are in a certain way a mere theoretical phenomenon, since they only emerge for very low elasticities of substitution between factors or strong increasing returns.

By contrast, our model suggests that such situations may easily occur when unions are sufficiently strong. In fact, it is easy to see from Proposition 1 that when $\alpha = 1$ (the competitive labour market case), there is no Hopf bifurcation and, using the Cooley and Prescott (1995) calibration, the flip bifurcation occurs at $\sigma_F = 0.407$, a value of the capital-labor elasticity of substitution which is too low to be empirically credible. On the other hand, when union power is high (low α), flip and Hopf bifurcations appear for values of σ that are not very far from one, and that become *arbitrarily* close to one as α tends to $1 - s_L$ (see figure 2). For example, when $\alpha = 0.5$ (the value which is usually considered in the labor economics literature), flip and Hopf bifurcations occur for elasticities of substitution between capital and labor given by $\sigma_F = 1.59$ and $\sigma_H = 0.8$, respectively. Interestingly, both values fall within the range of estimated values in the empirical literature (Duffy and Papageorgiou, 2000).

4 The indeterminacy mechanism

In this section, we first provide an economic interpretation explaining why indeterminacy easily occurs in this economy, and then compare our indeterminacy mechanism with related ones in the literature.

Economic intuition Comparing our framework to that of Grandmont et al. (1998), two new ingredients are considered: (i) the presence of unions and wage/employment bargaining, and (ii) unemployment insurance financed by taxation. From the discussion in section 3, one might infer that it is not union power, but the insurance scheme provided by the government, that constitutes the main channel through which indeterminacy occurs. We now explain why this is indeed the case.

As it is frequent in this type of literature (e.g. Benhabib and Farmer,1994), most things can be understood by referring to the equilibrium conditions on the labour market, comparing in particular the slopes of the relevant "labor supply" and "labor demand" curves. In that respect taxation is important because it renders the reservation wage schedule – which was horizontal at the partial equilibrium, see Figure 1 – *negatively sloped* at equilibrium, with a constant elasticity of -1.[14] A necessary condition for indeterminacy is that

[14] Indeed, using the equilibrium condition (3), the reservation wage expression, given in

this equilibrium reservation wage schedule (ERW) be steeper than the MPL curve, which is also negatively sloped. This requirement is satisfied if the elasticity of the MPL curve, $-(1-s_L)/\alpha\sigma$, is higher than -1, i.e. when $\sigma > \sigma_H \equiv (1-s_L)/\alpha \in (1-s_L, 1)$, see Figure 2.

An intuitive economic explanation of why this condition on the slopes is required for local indeterminacy can be given as follows. When this condition is satisfied, if, departing from the steady state, there is an increase in expected inflation, the upward shift in the ERW curve will imply an increase in current employment. Under our assumption $\sigma > 1 - s_L$, this increase in employment increases the current real wage bill. It also increases the rental cost of capital (through the decrease in the capital-labor ratio), so that the future capital stock will also be higher than its steady state level. This increase in future capital will in turn shift the future MPL curve upwards, which, provided that the future ERW schedule does not shift too much due to further changes in expected inflation, will decrease future employment. If the steady state is locally indeterminate, two things should be observed: (i) a reversal in the increase in capital stock, and (ii) a realized value of inflation that fulfills the original increase in expectations. The first condition is easily satisfied as the future increase in capital and the future decrease in employment both tend to decrease the future rental rate of capital. Note that this reversal in the future rental rate of capital would not appear if the slopes condition was not met, since in this case both current and future employment would go in the same direction in response to a change in expected inflation. In what concerns the second condition, observe that realized inflation can be written as the ratio of the current and future wage bills, i.e. $\hat{p}_{t+1} \equiv p_{t+1}/p_t = (w_t l_t/p_t) \,/\, (w_{t+1} l_{t+1}/p_{t+1})$ (see section 2.3). We have already seen that the current wage bill increases. Furthermore, the decrease in future employment will tend to moderate the future wage bill, making therefore possible an increase in realized inflation consistent with initial expectations.

Of course, as we have mentioned, for this reasoning to be correct, the ERW curve should not shift too much in the future due to further increases in expected inflation. In the Cobb-Douglas case, which satisfies the slopes condition ($\sigma = 1 > \sigma_H$), we can prove that this is indeed the case. This is because, in this case, the markup factor of wages over the reservation wage, $\mu(x)$, becomes constant. Therefore, the value of realized inflation,

section 2.2, $(b+\tau)E_t(p_{t+1}/p_t)$ can be written as $(b/l)\,E_t(p_{t+1}/p_t)$, so that its elasticity at the steady state (where $E_t(p_{t+1}/p_t) = p_{t+1}/p_t = 1$) is -1.

$\hat{p}_{t+1} = (w_t l_t/p_t) / (w_{t+1} l_{t+1}/p_{t+1}) = \mu(x_t) E_t(\hat{p}_{t+1})/\mu(x_{t+1}) E_{t+1}(\hat{p}_{t+2})$ (see section 2.2), simplifies to $\hat{p}_{t+1} = E_t(\hat{p}_{t+1})/E_{t+1}(\hat{p}_{t+2})$. Perfect foresight then requires $\hat{p}_{t+1} = E_t(\hat{p}_{t+1})$ and, therefore, $E_{t+1}(\hat{p}_{t+2}) = 1$. Thus, in the particular case of a Cobb-Douglas technology, the future ERW schedule *does not shift at all.* This is the reason why indeterminacy occurs in this configuration *for any value of the other parameters.* On the contrary, when σ is different from 1, markup variability implies a change in the ratio $\mu(x_t)/\mu(x_{t+1})$, so that consistency between expected and realized inflation requires further changes in future expected inflation (leading to a shift in the future ERW curve). As the elasticity of the markup $\mu(x_t)$ depends on union power, further conditions on α are then needed for indeterminacy (see Figure 2).

Related literature Since government policy is the main mechanism responsible for indeterminacy, our model fits in the line of research that explores the role of different balanced-budget policy rules on the stability properties of the equilibrium, as for instance Schmitt-Grohé and Uribe (1997) and Pintus (2004), where full employment economies with CRS technologies are considered. Indeed, both in our framework and in these papers, the indeterminacy mechanism operates through the impact of government policy on the labour market equilibrium. There remains, however, a major difference between our indeterminacy mechanism/results and those obtained in these former papers. Both Pintus (2004), considering a Woodford model, and Schmitt-Grohé and Uribe (1997), considering a Ramsey model with a Cobb-Douglas technology, find that, with fixed public spending, indeterminacy requires a lower bound on public spending as a share of GDP. With the standard quarterly calibration of Cooley and Prescott (1985), and assuming $\sigma = 1$, this lower bound is around 23%. But Pintus (2004) also shows that a model with fixed public spending is in fact isomorphic to a model with increasing returns to scale and without government. Under this interpretation, indeterminacy therefore only prevails if the "fixed cost" imposed to the economy by the constant level of public spending is sufficiently high, a mechanism which, he concludes, is similar to imposing external increasing returns to scale in the first place.[15] In our framework, on the contrary, the emergence of indeterminacy is independent of the values assumed for the policy variables.

What explains the difference? In all models, the (steady state) labour

[15] This formal equivalence between fixed public spending and increasing returns is also made in a recent contribution by Seegmuller (2004), where it is shown that models with markup variability or taste for variety can also be analyzed in a similar manner.

market equilibrium condition may be expressed in terms of the intersection between the marginal productivity of labour (MPL) curve and the equilibrium reservation wage (ERW) schedule. However, the elasticity of the ERW curve strongly depends on the way government spending is financed. In Schmitt-Grohé and Uribe (1997) and Pintus (2004), government spending (G) is financed trough proportional (ad valorem) taxes on labour income. After some manipulation, their labour market equilibrium condition may be written as $MPL = a/(1 - pG/wl)$,[16] where a denotes the constant desutility of labor. In our model, government spending is instead financed through a tax per unit of labour, implying the following equilibrium condition on the labour market: $MPL = b/l$. Therefore, the elasticity of the ERW curve is a fixed constant (-1) in our economy, while it clearly depends on policy parameters in the framework considered by Schmitt-Grohé and Uribe (1997) and Pintus (2004). Hence, it is clear that the difference in the taxation scheme, which leads to different implications for the elasticities of the supply side of the labour market, is the main explanation for why indeterminacy may occur in our economy without further conditions on the size of government spending as a share of GDP.[17]

5 Concluding Remarks

We considered an economy with constant returns to scale in production, where finance constrained unionized workers face income uncertainty due to the risk of imperfectly insured unemployment. We have shown that indeterminacy is a very pervasive feature in this economy, occurring in particular for any parameters' values when the technology is Cobb-Douglas. In addition,

[16] Indeed, when the labor supply is infinitely elastic (as in Schmitt-Grohé and Uribe, 1997), the ERW curve is given by $a + \tau$, where τ denotes the amount of taxes per labor unit. In their case, with ad valorem taxes, $\tau = t_w w/p$, where t_w is the tax rate. Hence, noting that, under perfect competition in the labour market, $w/p = MPL$, and taking into account the budget equilibrium condition, the labour market equilibrium condition $MPL = a + \tau$ becomes $MPL = a/(1 - pG/wl)$.

[17] Note that the assumption that government spending consists of transfers to the unemployed, $b(1 - l)$, is not important for our indeterminacy results. Indeed, if instead government spending was a constant flow of purchases of goods, G, financed by a tax per unit of labour, the ERW would still exhibit an elasticity of -1. In this case, the steady state reservation wage would be given by τ (since $b = 0$), so that, taking the budget constraint $G = \tau l$ into account, the ERW schedule would become G/l.

flip and Hopf bifurcations also emerge for plausible elasticities of substitution between capital and labour when unions bargaining power is sufficiently strong. The mechanism driving these results is the economic policy of the government and the way taxes are raised to finance unemployment benefits.

What is suggested by this analysis? First, a rich set of dynamics, including periodic and irregular, deterministic and stochastic cycles, may easily emerge in this economy. Second, countries with different elasticities of substitution or union bargaining power may have considerably different stability properties of the equilibrium. Our model may therefore explain why European countries, which are very similar in many dimensions but may slightly differ in terms of union power or input substitution, may experience drastically different patterns of unemployment fluctuations. An exploration of these implications in a simulated version of the model would be a natural extension of the present paper.

A Appendix

A.1 Binding liquidity constraints with income uncertainty

Workers receive at the beginning of each period t a state-contigent revenue $y_t^i \in \{w_{t-1} - p_t\tau_{t-1}, p_t b\}$ for $i \in \{e, u\}$, and wish to maximize their expected lifetime utility $E\sum_{t=0}^{\infty}\gamma^t u(c_t)$ with respect to $\{c_t, m_{t+1}, k_{t+1}\}_{t=0}^{\infty}$, under the budget constraint $m_{t+1} + p_t k_{t+1} = m_t + y_t^i + p_t R_t k_t - p_t c_t$, the borrowing constraint $m_{t+1} \geq 0$, and $k_{t+1} \geq 0$ for all t. Denoting by λ_t^i, υ_t^i and η_t^i the Lagrange multipliers associated respectively with these three constraints, the first order conditions for this problem are given by:

$$u'\left(c_t^i\right) = p_t \lambda_t^i \tag{13}$$

$$\lambda_t^i - \upsilon_t^i = \gamma E_t \left\{ l_t \lambda_{t+1}^e + (1 - l_t)\lambda_{t+1}^u \right\} \tag{14}$$

$$p_t \lambda_t^i - \eta_t^i = \gamma E_t \left\{ p_{t+1} R_{t+1} \left[l_t \lambda_{t+1}^e + (1 - l_t)\lambda_{t+1}^u \right] \right\} \tag{15}$$

We are looking for the conditions under which a consumer will choose not to hold capital or money under all possible states (employed or unemployed). This means that we are looking for the sequences of revenues and probability distributions over employment and unemployment that are consistent with $\upsilon_t^i > 0$, so that $m_{t+1} = 0$, and $\eta_t^i > 0$, so that $k_{t+1} = 0$, for all $t = 0, ..., \infty$

and all $i \in \{e, u\}$. This implies that the following inequalities

$$u'\left(c_t^i\right) > \gamma E_t \left\{ \frac{p_t}{p_{t+1}} \left[l_t u'\left(c_{t+1}^e\right) + (1-l_t) u'\left(c_{t+1}^u\right) \right] \right\} \quad (16)$$

$$u'\left(c_t^i\right) > \gamma E_t \left\{ R_{t+1} \left[l_t u'\left(c_{t+1}^e\right) + (1-l_t) u'\left(c_{t+1}^u\right) \right] \right\} \quad (17)$$

must hold for all $i \in \{e, u\}$, and where, for $t = 0, ..., \infty$, we have $c_t^e = (w_{t-1} - p_t \tau_{t-1}) / p_t$ and $c_t^u = b$.[18] Since, as in Woodford (1986), we assume that $R_{t+1} > E_t \{p_t / p_{t+1}\}$ (so that capitalists do not hold money) condition (17) is more restrictive than (16). Condition (17) is in particular verified at the steady state, where (see (5)) $R = 1/\beta > 1$, β being the discount factor of capitalists, if $u'(w/p - \tau) > (\gamma/\beta) [lu'(w/p - \tau) + (1-l)u'(b)]$ and $u'(b) > (\gamma/\beta) [lu'(w/p - \tau) + (1-l)u'(b)]$. Due to concavity of u, only the first of these two condition is actually required, as long as $w/p - \tau \geq b$ (a condition that is implied by the wage bargaining process). In summary, at the steady state and in its neighborhood, capitalists do not hold money and workers do not hold money or capital (their consumption being identical, in every period t, to y_t^i) iff $\gamma < \beta \{u'(w/p - \tau) / [lu'(w/p - \tau) + (1-l)u'(b)]\}$. Of course, since the expression between curled brackets is lower than 1, this last condition can only be verified if $\gamma < \beta$.

A.2 The firms problem

A firm wishes to maximize the present value of expected profits, given by

$$\Pi_t = m_t^f + p_t A l_t f(x_t) - p_t \rho_t k_t - w_{t-1} l_{t-1} - m_{t+1}^f + \varphi E_t \Pi_{t+1} \quad (18)$$

where $0 < \varphi < 1$ is the constant discount factor and m_t^f is money held by firms at the beginning of period t. Since wages must be paid in cash, we have that $m_{t+1}^f \geq w_t l_t$. Given the sequence of events, we have to solve the firm problem backwards, starting with the money holdings decision. This means that, at this stage, firms choose the level of money holdings that maximize (18) subject to $m_{t+1}^f \geq w_t l_t$, with m_t^f given and for given values of k_t, w_t and l_t. Denoting by λ_t the lagrange multiplier associated with the constraint,

[18] For the proof to be correct, further conditions on the initial amounts of capital and cash in hands of the worker at period 0, $(k_0, m_0 + y_0)$, have to be introduced to ensure that constraints (16) and (17) are satisfied in the first period. For simplicity and consistency of notation, we simply assume that $k_0 = 0$, $m_0 = 0$, and that y_0 takes either the steady state values $w - p\tau$ or pb with probability l and $(1-l)$ respectively.

the first order condition for this problem is $\lambda_t = 1 - \varphi$. We therefore see straightforwardly that, for any $\varphi \neq 1$, the cash in advance constraint will be binding: $m^f_{t+1} = w_t l_t$. Therefore, at the second and first stages, the firms' objective becomes $\Pi_t = (p_t A l_t f(x_t) - p_t \rho_t k_t - w_t l_t) + \varphi E_t \Pi_{t+1}$. It is then easy to see that the fallback payoff of firms is $\overline{\Pi}_t = -p_t \rho_t k_t + \varphi E_t \Pi_{t+1,}$ so that $\Pi_t - \overline{\Pi}_t = p_t A l_t f(x_t) - w_t l_t$.

A.3 Proof of Proposition 1

Proof. Note first that the trace, T, and the determinant, D, of J, the Jacobian matrix of the system (11)-(12) evaluated at the steady state, which correspond respectively to the product and sum of the two roots (eigenvalues) of the associated characteristic polynomial $Q(\lambda) \equiv \lambda^2 - \lambda T + D$, can be written as:

$$D = -\frac{(1-s_L)(1-\alpha)(\sigma-1)}{(\alpha-1+s_L)(\sigma-1+s_L)} \tag{19}$$

$$T = 1 + D - \frac{\theta s_L}{(\sigma - 1 + s_L)} \tag{20}$$

where $\theta \equiv 1 - \beta(1-\delta) \in (0,1)$, $s_L = [f(x) - \alpha f'(x)x]/f(x) \in (0,1)$, and $\sigma = -f'(x)\left[f(x) - f'(x)x\right]/f(x)f''(x)x > 0$. We proceed now by applying the geometrical method proposed by Grandmont et al. (1998). In Figure 3, we have represented three lines relevant for this purpose: the *line* AC, $D = T - 1$, where a local eigenvalue is equal to 1; the *line* AB, $D = -T - 1$, where one eigenvalue is equal to -1; and the *segment* BC, defined by $D = 1$ and $|T| < 2$, where two eigenvalues are complex conjugates of modulus 1. When T and D fall in in the interior of triangle ABC, both eigenvalues have modulus lower than one, and the steady state is a *sink*, i.e., is locally stable. In all other cases, the steady state is locally unstable. It is a *saddle* when $|T| > |D+1|$ (one eigenvalue with modulus higher than one and one eigenvalue with modulus lower than one), and a *source* in the remaining regions (both eigenvalues with modulus higher than one). Consider that θ and s_L take some fixed value in their admissible range. We first fix $\alpha \in (1 - s_L, 1)$ and analyze how T and D evolve as σ, the bifurcation parameter, is made to vary continuously within its domain. From (19) and (20), it is easy to show that this locus of points (T_σ, D_σ) is defined by the following linear expression, the Δ line:

$$\begin{aligned} D &= \Delta(T) \equiv -\frac{(1-s_L)(1-\alpha)\theta}{[(\alpha-1+s_L)\theta-(1-s_L)(1-\alpha)]}+\Delta'(T-1) \\ & \quad \textit{where} \\ \Delta' &= -\frac{(1-s_L)(1-\alpha)}{[(\alpha-1+s_L)\theta-(1-s_L)(1-\alpha)]} \end{aligned}$$

Since we assume that $\sigma > 1-s_L$ only the part of the Δ line corresponding to these values of σ is relevant. Using (19), we see that as σ increases from $1-s_L$ to $+\infty$, D decreases from $D_{1-s_L} = +\infty$ to $D_\infty = \frac{(1-\alpha)(1-s_L)}{(\alpha-1+s_L)}$. Using (20), we can also note that $D_\infty = T_\infty - 1$. Hence, the relevant part of the Δ line is just a half-line Δ whose origin (T_∞, D_∞), for $\sigma = +\infty$, lies on the line AC, and as σ decreases to $1-s_L$, points upwards to the right or to the left, depending on whether its slope is positive or negative (see Figure 3). We shall now study how this half line Δ shifts in the space (T, D) with α. As α changes from 1 to $1-s_L$, the origin of the half line Δ moves downwards along line AC, taking the values $(T_\infty, D_\infty) = (1, 0)$ for $\alpha = 1$ and $(T_\infty, D_\infty) = (-\infty, -\infty)$ when α tends to $1-s_L$. Also, the slope of the half line Δ decreases from zero to -∞ as α decreases from 1 to some critical value, and then decreases gradually from $+\infty$ tending to 1 as α increases further towards $1-s_L$. Note finally that the half line Δ crosses point $P = (1-\theta, 0)$, for any value of α, when $\sigma = 1$. All this implies that the half line Δ, lying on the left of line AC, rotates in the clockwise direction around point P as α decreases from 1 to $1-s_L$, being horizontal for $\alpha = 1$, becoming vertical for some α included between $1-s_L$ and 1, its slope tending to 1 as α tends to $1-s_L$. Using figure 3, it is then straightforward to see that several critical values for α have to be considered: α_3, the value of α such that the half line Δ_{α_3} crosses point B, α_2 the value of α for which the slope of the half line Δ_{α_2} becomes -1, and α_1 the value of α such that the half line Δ_{α_1} crosses point A. Indeed, using Figure 3, one can then easily check that for $\alpha_3 < \alpha < 1$, as σ continuously increases from $(1-s_L)$ to $+\infty$, the steady-state is first a saddle and changes to a sink trough a flip bifurcation at $\sigma = \sigma_F < 1$ (σ_F being the value of σ at which the half line Δ crosses line AB). When $\alpha_2 < \alpha < \alpha_3$, the steady state is first a saddle, undergoes a flip bifurcation for $\sigma = \sigma_F$, becomes a source for $\sigma_F < \sigma < \sigma_H$, undergoes a Hopf bifurcation at $\sigma = \sigma_H < 1$ (σ_H being the value of σ at which the half line Δ crosses segment BC) and then turns to a sink. For $\alpha_1 < \alpha < \alpha_2$, the steady state is first a source, undergoes a Hopf bifurcation when $\sigma = \sigma_H < 1$, and then becomes a sink.

Finally, for $1 - s_L < \alpha < \alpha_1$, the steady state is first a source, undergoes a Hopf bifurcation for $\sigma = \sigma_H < 1$, becomes a sink for $\sigma_H < \sigma < \sigma_F$, a flip bifurcation occurs for $\sigma = \sigma_F > 1$, and then turns into a saddle. Combining all these results Proposition 1 follows. ■

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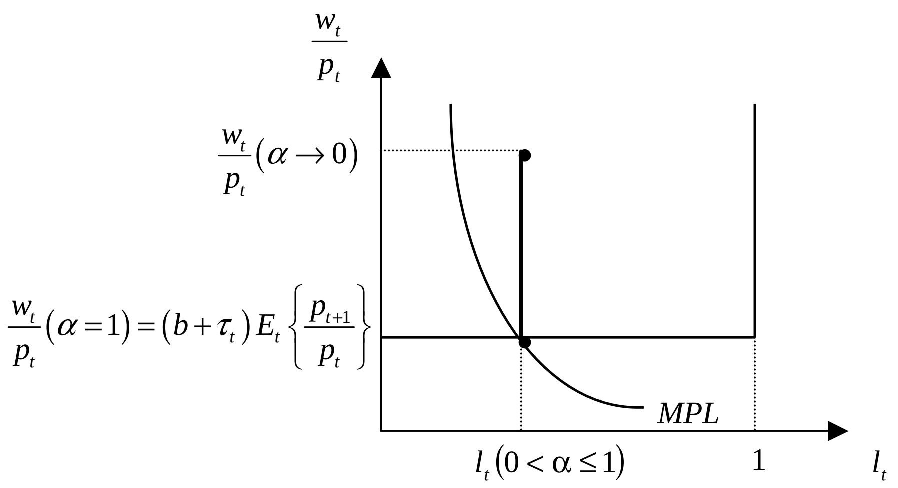

Figure 1: Labor market at temporary partial equilibrium

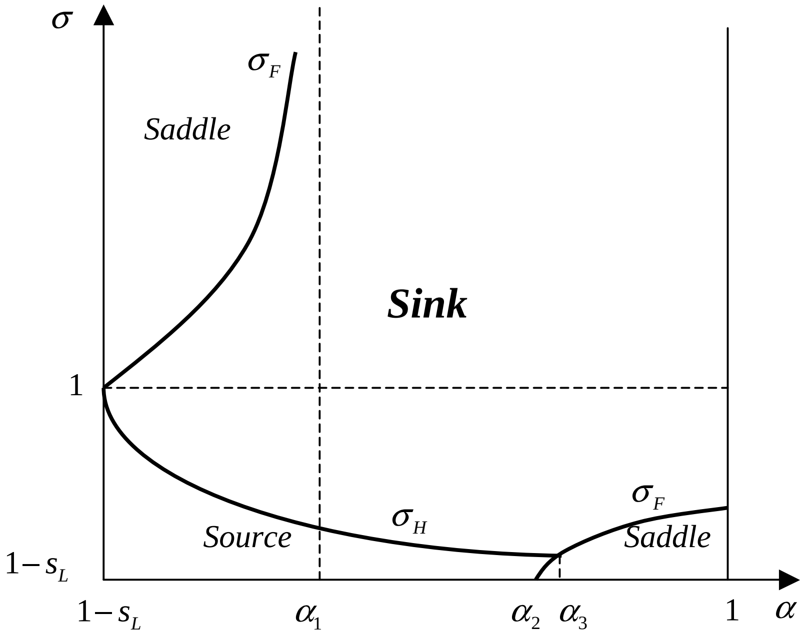

Figure 2: The local dynamics regimes in the (α,σ) plane

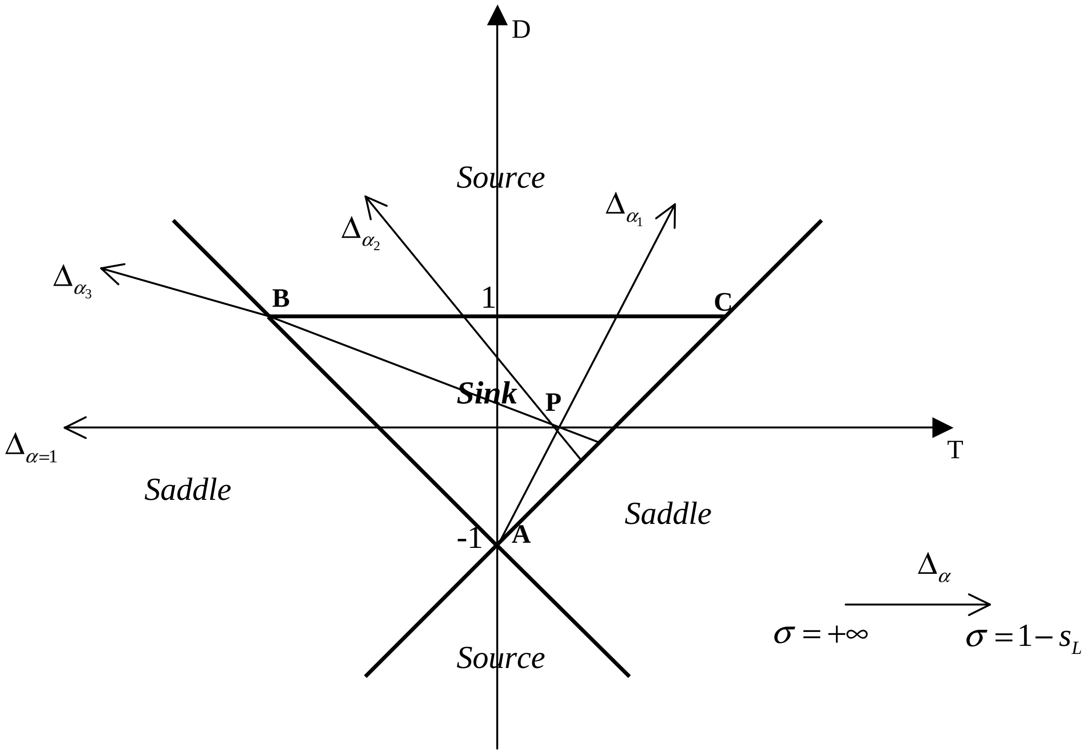

Figure 3: The local dynamics regimes in the (T, D) plane