A Classification of Infinity Dimensional Walrasian Economies

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Abstract

We consider economies of pure interchange in which the spaces of consumption of the agents, are Banach spaces. The utility functions are not necessarily separable, but, strictly quasi-concave, and Fréchet differentiable. We characterize the set of walrasian equilibria, by the social weight that support each walrasian equilibria and, using techniques of the smooth functional analysis, we show that a suitable large subset of the walrasian equilibrium set, conforms a Banach manifold. In the next sections we focuses on the complement of this set, the set of singular economies, and we analyze the main characteristics of this set.

1 Introduction

We consider an economy where each agent's consumption set is a subset of a Banach lattice. Agents will be indexed by $i \in I = \{1, 2, ...n\}$; and X_+ will denote the positive cone of the Banach space X. We do not assume separability in the utility functions $u_i : X_+ \to R$. Utility functions are in the $C^2(X, R)$ space, i.e. the set of the functions with continuous second Fréchet-derivatives (F-derivatives), and we suppose that they are increasing functions it is to say that, each agent prefer more than less, formally, each first order F-derivative is positive. Where F-derivative define $f'(x)$ in the usual way of the linearize $f(x + h) = f(x) + f'(x)h + o(\|h\|)$. In order to assure the uniqueness of equilibrium allocation we will assume strictly quasi-concave utility functions. In addition, we suppose that for all $x \in X$ the inverse operator $(u_i^{''})^{-1}$ of u_i'' at x, exists. Here $u_i^{''}(x)$ is identified with the quadratic form $(h, k) \to u^{''}(x)hk$. In this work $C^k(X, Y)$ denote the space of $k-times$ continuously F-differentiable operators from X into Y, and $L(X, Y)$ denote the space of linear and continuous operators from X into Y. So, $u' : X_+ \to L(X, R); u'(x) \in L(X, R)$, and $u'' : X_+ \to L(X, L(X, R))$, then, $u''(x) \in L(X, L(X, R))$. By $C^\infty(X, Y)$ we denote the set of functions belonging to $C^k(X, Y)$ for all integer k.

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The consumption space of each agent is the same one, and it will be symbolized by X_+. The cartesian product of these n consumption sets is represented by Ω. So, a bundle set for the i-agent will be symbolized by $x_i \in X_+$ and an allocation will be denoted by $x = (x_1, x_2, ..., x_n) \in \Omega_+$. The i-agent endowments will be symbolized by w_i, and $w = (w_1, w_2, ...w_n)$, symbolize the initial allocation. We assume that $w \in \Omega_+$ where Ω_+ is the positive cone of Ω. The total amounts of available goods will be denoted by $W = \sum_{i=1}^{n} w_i$. We assume that $W \in \Omega_{++}$.

With the purpose to obtain strictly positive equilibria, we will assume that utilities satisfy at least one of the following two, widely used assumptions in economics, see for instance [Aliprantis, C.D; Brown, D.J.; Burkinshaw, O.], conditions:

(i) (Inada condition) $\lim u_i'(x_i) = \infty \; if \; x_i \to \partial(X_+)$, for each $i = 1, 2, ..., n$ and for each utility function, by $\partial(X_+)$, we denote the frontier of the positive cone of X. It assumes that marginal utility is infinite for consumption at zero.

(ii) All strictly positive bundle set is preferable to all bundle set with at least one zero component in one state of the world.

An economy will be represented by

$$\mathcal{E} = \{X, u_i, w_i, I\},$$

where X is the consumption set(in our case X_+) u_i the utility function and w_i the initial endowments of the i-agent, and I the set of the agents, in our case $I = \{1, 2, ..., n\}$.

As examples of economies with the properties above mentioned, consider those where the consumption set is $X_+ = C_{++}(M, R^n)$ and utility functions are $u_i(x) = \int_M U_i(x(t), t)dt$, see [Chichilnisky, G. and Zhou, Y. (1988)] and [Aliprantis, C.D; Brown, D.J.; Burkinshaw, O.].

It is well known that the demand function is a good tool to deal with the equilibrium manifold in economies in which consumption spaces are subset of Hilbert spaces, in particular R^l [Mas-Colell, A. (1985)]. But unfortunately if the consumption spaces are subsets of infinite dimensional spaces (not a Hilbert space), the demand function may not be a differentiable function [Araujo, A. (1987)], or it is not well definite because the price space is very large or the positive cone where prices are definite has empty interior. Despite in many of these cases it is possible, to characterize the equilibria set using the function of excess of utility , see for instance [Accinelli, E. (1996)], and it is possible using this function to introduce in infinite dimensional models differentiable techniques with wide generality. Then it is possible to solve problems defined in spaces of infinite dimension by means of techniques of differential calculus own of the finite case. And in this way to generalize the result obtained by [Chichilnisky, G. and Zhou, Y. (1988)] for smooth infinite dimensional economies to the case with no separable utilities.

In this work, following the Negishi approach, we will characterize the equilibrium set of the economy, as the set of zeroes of the excess utility function $e : int[\Delta] \times \Omega_+ \to R^{n-1}$. So, the equilibrium set will be denoted by

$$\mathcal{E}_q = \{(\lambda, w) \in int[\Delta] \times \Omega_+ : e((\lambda, w) = 0\}$$

Where Δ symbolize the social weight set,

$$\Delta = \left\{ \lambda \in R^n : \sum_{i=1}^{n} \lambda_i = 1, \;\; 0 \leq \lambda_i \leq 1 \; \forall i \right\},$$

and $int[\Delta]$ symbolize the set of $\lambda \in \Delta$, such that $\lambda_i > 0, \forall i$. In the considered hypothesis, the fact that each agent has non-null initial endowments, implies that the result of a process of maximization of the utility functions will be a strictly positive bundle set. Then each relative weight cannot be zero. Because in other case, if for some consumer $j \in [1, 2, ..., n]$ $\lambda_j = 0$ this consumer receive $x_j = 0$. So, without loss of generality, we can consider only cases where $\lambda \in \Delta_{++} = int[\Delta]$. We will use the fact that this set is a Banach submanifold in R^{n-1}. By the symbol $int[\cdot]$ we represent the interior of de set $\cdot$.

In section (3) in order to prove that $\mathcal{E}_q$ restricted to $w \in \Omega_0$, where Ω_0 is an open and dense (residual) subset included in Ω_{++} is a Banach manifold. In this section we assume that the positive cone Ω_+ of the consumption space has non-empty interior. Typically examples of such spaces are $L^\infty(M, R^n)$ where M is any compact manifold, with the supremun norm, see [Chichilnisky, G. and Zhou, Y. (1988)]. So, we show that in this cases, the set of regular economies is large, and its complement is a rare set. This is not a consequence of the Debreu theorems, here it follows from an alternative approach with particular interest in infinite dimensional cases.

Next we will focuses on the complement of regular economies, this kind of economies will be called singular economies. Up till now the characteristics of this kind of economies are no well know. To characterize this subset of economies, we adopt the point of view of the smooth analysis. So, in this section we will consider economies which utility functions are in $C^\infty(X_+, R)$, certainly this is a strong restriction but it is necessary to analyze singularities from the point of view of the smooth analysis.

Singular economies, in contrast with regular economies, characterize the sudden qualitative and unforeseen changes in the economy. More explicitly, all regular economies have locally, the same behavior, this means that in a neighborhood of a regular economy there is not big changes, and all economy in this neighborhood is a regular economy too. If the economy is regular, small changes in the distribution of the endowments do not imply big changes in the behavior of the economy as a system, and the new economy will be a regular economy also (this means that regualar economies are structuraly stable) but, in a neighborhood of a singular economy small changes in the distribution of the endowments usually, imply big changes in the main characteristics of the economy, for instance its number of equilibria, (so singular economies are structuraly instables). Our object in this section will be to analyze this kind of economies.

An economy is singular if the zero is a singular value of the excess utility function of this economy, and as the utilities appear explicitly in the excess utility function, the strong relation between the characteristics of the agent preferences, and the behavior of the economy appear clearly reflected in this function. In spite of to be singular economies from a topological or measure theory point of view a very small set, but it play central role in economics. For instance, the existence of multiplicity of equilibria in an economy is a straightforward result of the existence of singularities in the excess utility function, then its existence depend on characteristics of the utility functions. This relation between the kind of singularities and the characteristics of the utility functions of the economy appear clearly in our approach.

Taking care of the type of singularity, it is possible to introduce in the space of the economies a classification, that introduce a partition of this set in equivalence classes. Economies with the same possible singularities will form an equivalence class and anyone of the members represents them, because they react of way similar to small modifications in its parameters. Regular economies have locally the same behavior, this is not true for singular economies, but the behavior of the economy is locally similar if singularities are of the same class. Small modifications of the parameters do not imply great modifications in the behavior of the economy if this is a regular economy, but in the case of a singular economy, imperceptible changes in the parameters could give rise to situations completely different from the original one, i.e. singular economies are very sensitives to political and social choices.

Despite its importance to understand the economic changes, there are not many works about singular economies. Y. Balasko has several works on singularities, for instance [Balasko, Y. (1988)], [Balasko, Y. (1997a)] and also in [Mas-Colell, A. (1985)] there are characterizations of the singular economies, however the General Equilibrium Theory is indebted with singularities. We hope to make a little collaboration in the long way to pay the debt with this topic.

2 Some of notation and mathematical facts

In this section we recalling some basic mathematical definitions that will be used later. Our main reference for considerations on Functional Analysis is [Zeidler, E. (1993)], in special vol 1 and vol 4.

Definition 1 *Let M be a topological space, M is said to be a C^k Banach manifold (B-manifold), if and only if there exists a C^k atlas for M.*

Recall that a C^k atlas for a topological space M is a collection of charts (U_α, ϕ_α) (α ranging in some indexing set) such that the: (i) $\phi : U \to U_\phi$ is a homeomorphism, from an open subset U in M onto U_ϕ an open subset of a Banach space X_ϕ. The map ϕ is called a chart map. (ii) The U_α cover M. (iii) Any two charts $(U, \phi), (V, \psi)$ are C^k compatible. This means that $U \cap V = \emptyset$ or $\phi(\psi)^{-1}$ and $\psi(\phi)^{-1}$ are $C^k, k \geq 0$.

1. Obviously each open set in a Banach space is a C^∞ manifold.

2. If M and N are C^k B-manifold, then $M \times N$ is also a C^k B-manifold.

Definition 2 *Let M be a C^k B-manifold. Then by T_xM we symbolize the* **tangent space** *to M at the point x i.e., the set of all tangent vectors at x.*

As it is well known, much of the theory of the differential calculus in B-space, can be extended to B-manifolds. This can be doing by means of the charts. Certainly if M and N are B-manifolds, and $f : M \to N$ let (U, ϕ) and (V, ψ) be charts in M and N then $\bar{f} = \psi(f((\psi)^{-1})) : X_\phi \to X_\psi$, this is a function between B-spaces, and it is considered as representative of f. So we say that a function $f : (M, \phi) \to (N, \psi)$ is in $C^k(M, N)$ if and only if $\bar{f} : X_\phi \to X_\psi$ is a $C^k(X_\phi, X_\psi)$ function.

Definition 3 *Let $f : Dom(f) \subseteq M \to N$ be a mapping between two B-manifolds M and N here $Dom(f)$ is the domain of f, then there exists a linear continuous map $f'(x) : T_xM \to T_{f(x)}N$ at each point $x \in M$.*

This is an extension to manifolds of the concept of F-derivatives, so $f' : M \to L(TM, TN)$, where $TM = \{(x, T_xM) : x \in M\}$ and $TN = \left\{(f(x), T_{f(x)}N) : x \in M\right\}$, defined by $f'(x) \in L(T_xM, T_{f(x)}N)$. We will call this map the Fréchet derivative (F-derivative) at the point x for the map f. This map is called in some literature the tangent map, and it is symbolized by T_xf or $Df(x)$ or $df(x)$.

Definition 4 *Let M be a C^k B-manifold, $k \geq 0$. A subset S of M is a* **submanifold** *if and only if for each point $x \in S$ there exists and admissible chart (U, ϕ) in M with $x \in U$ such that: (i) The chart space X_ϕ contains a linear, closed subspace Y_ϕ which splits X_ϕ. And (ii) the chart image $\phi(U \cap S)$ is an open set in Y_ϕ.*

It is easy to see that every submanifold S of a C^k B-manifold is itself a C^k B-manifold

Definition 5 *Let $f : M \to N$ be $C^k, k \geq 1$ where M and N are C^k B-manifolds. Then*

- *f is called a* **submersion** *at the point x if and only if $f'(x)$ is onto and the null space*

$$Ker(f'(x)) = \{x \in X : f'(x) = 0\},$$

splits T_xM.

Recall that the linear subspace $Y_1 = Ker(f'(x))$ split T_xM if and only if there exists Y_2, a linear subspace in T_xM, such that T_xM is the topological direct sum of Y_1 and Y_2 i.e. $T_xM = Y_1 \oplus Y_2$.

The function f is called a submersion in the subset $Z \subseteq M$ if and only if f is a submersion at each $x \in Z$.

We will denote the image set of a linear operator $T : X \to Y$ by

$$R(T) = \{y \in Y : \textit{ there exists } x \in X : y = T(x)\},$$

the dimension of $R(T)$ will be denoted by $rank\ T$, and the codimension of $(R(T))$ will be symbolized as $corank\ T = dim\,[X/ker(T)]$, where $X/ker(T)$ is the factor space.

Since $rank\ f'(x) = dimR(f'(x))$ this provide a natural classification of maps between manifolds, according to the behavior of the linearizations.

Definition 6 *The following are well know concepts for maps between B-spaces, here they are carried to maps between B-manifolds. Let $f : M \to N$, be a $C^k, K \geq 1$ M and N are B-manifolds.*

1. *The point $x \in M$ is called a* **regular point** *of f iff f is a submersion at x. Otherwise x is called* **singular point**.
2. *The point $y \in N$ is called a* **regular value** *of f if and only if f^{-1} is empty or consists solely of regular points. Otherwise y is called* **singular value**.
3. *Let M be a B-manifold, it follows that $f : U(x_0) \subset M \to R$ where $U(x_0)$ is an open neighborhood of x_0, has a singular point at x_0 if an only if $f'(x_0) = 0$. Such point will be non-degenerate if and only if the bilinear form $(h, k) \to f''(x_0)hk$ is non-degenerate.*

Definition 7 *A function $f : M \to R$ is called a* **Morse function** *if every singular point is a no degenerate singular point.*

Theorem 1 (Generalized Morse Lemma) *Let M be a Banach manifold, and let $f : U(x_0) \subset M \to R$ be a smooth function, $x_0 \in X$ is a no degenerate singular point of f Then there exists a local diffeomorphism ψ (in a neighborhood $U(x_0)$ of x_0) such that:*

$$f(\psi(y)) = f(x_0) + f^{''}(x_0)y^2/2 \tag{1}$$

is satisfy for all $y \in U(p)$, where $p = \psi(x_0)$, and $U(p) = \psi(U(x_0))$.

The following global result is shown in [Zeidler, E. (1993)]:

Theorem 2 (Preimage Theorem) *Let M and N be B-manifolds. If y is a regular valued of $f : M \to N$, with $1 \leq k \leq \infty$, then the set Z of all solutions of $f(x) = y$ is a C^k submanifold of M.*

Definition 8 *A map $f : M \to N$ is a* **Fredholm map** *if $f'_x : M \to N$ is a linear Fredholm operator, see [Zeidler, E. (1993)].*

A linear map $T : X \to Y$ is called a **Fredholm operator** if and only if is continuous and both numbers the dimension of the $ker(T), dim(Ker(T))$ and the codimension of the rank of $f, codim(R(T))$ are finite. The index of f is defined by: $ind(T) = dim(Ker(T)) - codim(R(T))$.

3 The Negishi approach

The Negishi approach starts considering a social welfare function given by: $U_\lambda : \Omega_+ \to R$ defined as:

$$U_\lambda(x) = \sum_{i=1}^{n} \lambda_i u_i(x_i). \tag{2}$$

where u_i is the utility function of the agent indexed by i, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in int[\Delta]$ (each λ_i represents the social weight of the agent in the market), and Ω_+ is the positive cone in the consumption space $\Omega = X^n$.

As it is well know if $x^* \in \Omega$ solves the maximization problem of $U_{\lambda^*}(x)$ for a given λ^*, subject to be a feasible allocation i.e.,

$$x^* \in \mathcal{F} = \left\{ x \in \Omega_+ : \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} w_i \right\}$$

then x^* is a Pareto optimal allocation [1]. Reciprocally it can be proved that if a feasible allocation x^*, is Pareto optimal, then there exists any $\lambda^* \in \Delta$ such that x^*, maximize U_{λ^*}, see [Accinelli, E. (1996)]. There exist some Pareto optimal allocations where $x_i^* = 0$ for some $i \in I'$ where $I' \subseteq I$. However if each agent has positive no null endowments, these cases are possible if and only if the agents indexed in this subset be out of the market, i.e., if and only if $\lambda_i = 0$ for al $i \in I'$. Then we can restrict ourselves, without loss of generality, to consider only cases where $\lambda \in \Delta_{++}$. Moreover we are interested only in individually rational Pareto optimal allocations, this is the subset of Pareto optimal allocations such that $u_i(x_i) \geq u_i(w_i),\ \forall i$. Clearly if x is an individually rational Pareto optimal allocation, then the corresponding λ is a strictly positive vector.

In this way characterized the set of Pareto optimal allocations, our next step is to choose the elements x^* in the Pareto optimal set such that can be supported by a price p and satisfying $px^* = pw_i$ for all $i = 1, 2, ..., n$ i.e., an equilibrium allocation.

We symbolize by $W = \sum_{i=1}^n w_i \in int[X_+]$ the aggregate endowments of the economy, and by $w \in= \Omega_{++}$ the vector of the initial endowments, $w = (w_1, w_2, ..., w_n)$, such that $w_i > 0, \forall i$. Suppose that the aggregate endowment of the economy is fixed.

We will use the following notation: For any $\lambda \in int\,[\Delta] = \{\lambda \in \Delta : \lambda_i > 0 \ \ \forall i \in I\}$,

$$x(\lambda, W) = argmax \left\{ \sum_{i=1}^n \lambda_i u_i(x_i),\ s.t \ \sum_{i=1}^n x_i = \sum_{i=1}^n w_i \right\}. \tag{3}$$

This element is well defined if the *closedness condition*[2] is verified, see [Accinelli, E. (2002)]. Let $e : int[\Delta] \times \Omega \to R^n$ be the excess utility function, which coordinates are given by:

$$e_i(\lambda, w) = u_i'(x_i(\lambda, W))(x_i(\lambda, W)) - w_i).$$

Here $u_i'(x_i(\lambda, W)) : X \to R$ is the F-differential of the utility $u_i(x_i(\lambda, W))$.

Definition 9 *For fixed utility functions, for each $w \in \Omega_{++}$, we define the set*

$$\mathcal{E}q(w) = \{\lambda \ \in \ int[\Delta] : e_w(\lambda) = 0\},$$

it will be called the set of the **Equilibrium Social Weights***.*

[1]We say that an allocation $x \in \mathcal{F}$ is Pareto optimal if there is no $y \in \mathcal{F}$ such that $u_i(x_i) \geq u_i(y_i)\ \forall i = 1, 2, ...n$; with strict inequality for at least one i.

[2]Recall that the closedness condition is verified, if and only if the utility set, $U = \{(u_1(x_1), u_2(x_2), ..., u_n(x_n)) : (x_1, x_2, ..., x_n) \in \mathcal{F}\}$ is closed, [Mas-Colell, A. Zame, W. (1991).]. Every economy with order interval $[0, W]$ weakly compact, verify the closedness condition. However an exchange economy can satisfy closedness condition without order interval $[0, W]$ being weakly compact. The weakly compactness of order interval is, a sufficient condition for the existence of a rational Pareto optimal allocation. See [Aliprantis, C.D; Brown, D.J.; Burkinshaw, O.]

In [Accinelli, E. (1996)] is show that the equilibrium social weights is a non-empty set.

Theorem 3 *Let $\lambda \in \mathcal{E}q(w)$, and let $x^*(\lambda)$ be a feasible allocation, solution of the maximization problem of W_λ and let $\gamma(\lambda)$ be the corresponding vector of Lagrange multipliers. Then, the pair $(x^*(\lambda), \gamma(\lambda))$ is a walrasian equilibrium and reciprocally, if (p, x) is a walrasian equilibrium then, there exists $\bar{\lambda} \in \mathcal{E}q$ such that x maximize $W_{\bar{\lambda}}$ restricted to the feasible allocations set, and p will be the corresponding vector of Lagrange multipliers i.e., $p = \gamma(\bar{\lambda})$.*

The proof can be see in [Accinelli, E. (1996)].

4 The equilibrium set as a Banach manifold

The allocation $x^* \in \Omega_{++}$ solve (3), if and only if there exists $\bar{\gamma} \in X^*$ (here X^* symbolize the dual space of X, i.e: the set of the linear continuous functionals,) such that the following identities are verified: see [Luenberger, D. (1969)]:

$$\begin{aligned} &\lambda_i u_i'(x_i^*) - \bar{\gamma} = 0 \\ &\textstyle\sum_{i=1}^n x_i^* - \sum_{i=1}^n w_i = \theta, \end{aligned} \tag{4}$$

Both terms in the first equation of (4) are linear operator, consequently the second member symbolize de null operator. In the second equation by θ we represent de null element of X.

Then for an arbitrary $h \in X$ it follows that:

$$\begin{aligned} &\lambda_i u_i'(x_i^*)h - \bar{\gamma}h = 0 \\ &\textstyle\sum_{i=1}^n x_i^* - W = \theta, \end{aligned} \tag{5}$$

These equalities represent the first order conditions for the maximization problem. Observe that from our hypothesis there are necessary and sufficient conditions for a solution of this problem. Let us define γ^* as the real number given by $\bar{\gamma}h$. Then if for a given, $(\tilde{\lambda}, \tilde{W}) \in int[\Delta] \times X_{++}$ there exist $x^* \in X_{++}$ and a real number γ^* solving the equations (5) then, x^* is a solution for the maximization problem (3) with $\lambda = \tilde{\lambda}$ and $W = \tilde{W}$.

Using the implicit function theorem (see appendix), we can show that there exist functions $f : \mathcal{U}_{\tilde{\lambda}} \times U_{\tilde{W}} \to \Omega_{++}$ defined by $f(\lambda, W) = x^*$ and $g : \mathcal{U}_{\tilde{\lambda}} \times U_{\tilde{W}} \to R$ defined by $g(\lambda, W) = \gamma^*$. Where $\mathcal{U}_{\tilde{\lambda}} \subseteq int[\Delta]$ is an open neighborhood of $\tilde{\lambda}$ and $U_{\tilde{W}} \subseteq X_{++}$ is an open neighborhood of $\tilde{W}$. We recall that $int[\Delta]$ and X_{++} are B-manifolds .

Moreover from this theorem, and from the preliminary hypothesis on utility functions, it follows that this functions are C^k. Let us now redefine this function as $x^*(\lambda, W)$ and $\gamma^*(\lambda, W)$.

From this notation, we will denote by $x^*_{i,\lambda_j}(\lambda, W)$ and $x_{i,w_j}(\lambda, W)$ de partial F-derivatives with respect to the variable λ_j and w_j respectively, $j \in \{1, 2, ..., n\}$. The derivatives with respect to w_j follows using the chain rule [3].

The following are well know properties of the excess utility function:

(1) $\lambda e(\lambda, w) = 0$.

(2) $e(\alpha\lambda, w) = e(\lambda, w), \forall \alpha > 0$.

See for instance [Accinelli, E. (1996)].

From item (1) it follows that the rank of the jacobian matrix $J_\lambda e(\cdot, w)$ [4] of the excess utility function $e(\cdot, w) : int[\Delta] \to R^n$ is at most equal to $n - 1$. And as from item (2) we know that if $e_i(\lambda, w) = 0 \ \forall i = 1, 2, ..., n-1$, then $e_n(\lambda, w) = 0$, we will consider the restricted function $\bar{e} : int[\Delta] \times \Omega_+ \to R^{n-1}$ obtained from the excess utility function removing one of its coordinates, for instance e_n.

The following theorem holds:

Theorem 4 *If the positive cone of the consumption space, has a non-empty interior* [5] *then, there exists an open and dense subset $\Omega_0 \subseteq \Omega_\epsilon$ such that*

$$\mathcal{E}q/\Omega_0 = \{(\lambda, w) \in int[\Delta] \times \Omega_0 : e(\lambda, w) = 0\}$$

is a Banach manifold.

Proof: To prove this theorem, we will prove the following assertions:

(i) There exists a residual set $\Omega_0 \subseteq \Omega$ such that, the mapping $\bar{e} : int[\Delta] \times \Omega_0 \to R^{n-1}$ is C^1, [6] and zero is a regular value of e i.e. for all $(\lambda, w) \in int[\Delta] \times \Omega_0$, such that $e(\lambda, w) = 0$ the mapping $\bar{e}$ is a submersion.

(ii) For each parameter $w \in \Omega_0$, the mapping $\bar{e}(\cdot, w) : int[\Delta] \to R^{n-1}$ is Fredholm of index zero.

- Now, from [Zeidler, E. (1993)] section (4.19), the theorem follows.

[3] $x^*_{i,w_j}(\lambda, w) = \frac{\partial x^*(\lambda, W)}{\partial W} \frac{\partial W}{\partial w_j}$.

[4] As $int[\Delta]$ is a B-manifold whose chart map is $X_\phi = R^{(n-1)}$ we can consider the concept of $F - derivative$ of this map, here we represent $e'(\cdot, w)$ by means of the symbol $J_\lambda e(\cdot, w)$.

[5] Basically these space are $C^k(X, R)$, and L_∞.

[6] We use here the concept of generalized local F derivative. As $int[\Delta]$ and Ω_0 are Banach manifolds, then $int[\Delta] \times \Omega_0$ is a Banach manifold (where the cart space is $X_\phi = R^{n-1} \times \Omega$)then, from the definition of manifold there is a natural way to define the derivative of $\bar{e}$. See [Zeidler, E. (1993)] vol 1.

From this theorem, it follows that:

Corolary 1 *For each* $w \in \Omega_0$ *the equation* $e(\lambda, w) = 0, \ \lambda \in int[\Delta]$ *has at most finitely many solutions* λ *of* $e_w(\lambda) = 0$.

Proof of the corollary: The convergence of $\bar{e}(\lambda_n, w_n) \to 0$ as $n \to \infty$ and convergence of $\{w_n\}$ implies the existence of a convergent subsequence of $\{\lambda_n\}$ in $int[\Delta]$. Note that under the assumptions of our model and as $w_i > 0 \ \forall i$, if $\lambda_n \to \bar{\lambda} \in Fr(\Delta_{++})$ then there exists some i such that $\bar{\lambda}_i = 0$ then $x_i(\lambda_n) \to 0$ and $u_i'(x((\lambda_n) \to \infty$ when $\lambda_n \to \bar{\lambda}$. So $\|e_i(\lambda_n)\| \to \infty$.

- The oddness of this solutions is proved in [Accinelli, E. (1996)].

Proof of the step (i): Consider the mapping from $int[\Delta] \times \Omega_+ \to R^{n-1}$ defined by the formula:

$$\lambda, w \to \bar{e}(\lambda, w),$$

where $\bar{e}(\lambda, w) \in R^{n-1}$ defined by $n-1$ first coordinates of the vector $e(\lambda, w)$.

We will prove that 0 is a regular value of the restricted excess utility function restricted to an open and dense subset, $\Omega_0 \subset \Omega_+$. It is to say that the restricted excess utility function $\bar{e}/\Omega_0$ is a submersion at each point $(\lambda, w) \in int[\Delta] \times \Omega_0$, i.e, $\bar{e}'(\lambda, w) : R^{n-1} \times \Omega \to R^{n-1}$ is surjective and the null space $Ker(e'(\lambda, w))$ splits $R^{n-1} \times \Omega$, for each $(\lambda, w) \in \mathcal{E}_q/\Omega_0$.

We begin showing that the linear tangent mapping is always onto, or equivalently that the rank of the linear map $\bar{e}'$ is equal to $n-1$ in each $(\lambda, w) \in \mathcal{E}_q/\Omega_0$, where by $\mathcal{E}_q/\Omega_0$ we represent the set of $(\lambda, w) \in \mathcal{E}_q$ with $w \in \Omega_0$. We will prove that the affirmation is true in the residual set Ω_0.

To see this consider a a vector $h \in X_{++}$ and a little change in the endowments given by $w(\eta) = (w + \eta)$, where $\eta \in \Omega$ such that $w + \eta \in \Omega_{++}$ where $\eta = (\eta_1, \eta_2, ..., \eta_n)$ $\eta_i = v_i h, \ v_i \in R, \ i = 1, 2, ..., (n-1)$ and $v_n = -\sum_{i=1}^{n-1} v_i$. The vector η will be thought as a state-independent parameter for redistributions of initial endowments. We define $\bar{\eta} = (\eta_1, ..., \eta_{n-1})$. Observe that $\sum_{i=1}^{n} w_i(\eta) = \sum_{i=1}^{n} w_i = W$. So, the allocations that solve (3) for the economies $\mathcal{E}(v)$ and $\mathcal{E}$ are the same.

The excess utility function for the economy $\mathcal{E}(\eta) = \{u_i, w(\eta)_i, I\}$ will be:

$$e(\lambda, w(\eta)) = (e_1(\lambda, w_1 + v_1 h), ..., e_n(\lambda, w_n + v_n h)), \tag{6}$$

where

$$e_i(\lambda, w_i + v_i h) = u_i'(x_i^*(\lambda, W))[x_i(\lambda, W) - w_i - v_i h].$$

Observe that the function $e_i(\lambda, w_i + v_i h)$ depends only on the real variable v_i, and we write $e_i(\lambda, w_i+v_i h) = \tilde{e}_i(v_i)$. So $e(\lambda, w+\eta)$ is a function on the n real variables $v = (v_1, ..., v_n)$ and let us define $\bar{v} = (v_1, ..., v_{n-1})$. So we can write the following identity $e(\lambda, w + \eta) = \tilde{e}(v_1, ..., v_n) = \tilde{e}(v)$, observe that $\tilde{e} : R^{(n-1)} \to R^n$.

The derivative of $\tilde{e_i}$ with respect to v_i evaluated at $(\lambda, w(\eta))$ is given by:

$$\frac{\partial e_i(\lambda, w_i + v_i h)}{\partial v_i} = \frac{\partial \tilde{e}_i(v_i)}{\partial v_i} = -u_i'(x_i(\lambda, W))h.$$

Then it follows that:

$$\frac{\partial e(\lambda, w(\eta))}{\partial v_i} = \frac{\partial \tilde{e}(v)}{\partial v_i} = (0, ..., 0, \frac{\partial \tilde{e}_i(v_i)}{\partial v_i}, 0..., 0) = (0, ..., 0, -u_i'(x_i(\lambda, W))h, 0, ..., 0)$$

Let $\bar{e} : R^{n-1} \to R^{n-1}$ be the function defined by the $n-1$ first coordinates of $\tilde{e}$, i.e:

$$\bar{e}(\lambda, w + \eta) = (e_1(\lambda, v_1 h), ..., e_{n-1}(\lambda, v_{n-1} h) = (\tilde{e_1}(v_1), ..., \tilde{e}_{n-1}(v_{n-1})).$$

Then:

$$\frac{\partial \bar{e}}{\partial \bar{v}}(\lambda, w(v)) = -\begin{bmatrix} u_1'(x_1^*)h & 0 & \dots & 0 \\ 0 & u_2'(x_2^*)h & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & u_{n-1}'(x_{n-1}^*)h \end{bmatrix} \in L\left(R^{(n-1)}, R^{(n-1)}\right). \qquad (7)$$

The rank of this matrix is equal to $n-1$, as the rank of a matrix is locally invariant, then for all w there exists an arbitrarily close vector $w(\eta)$: such that the rank of $e(\lambda, w(\eta))$ is equal to $n-1$ this prove the denseness of Ω_0.

Let $\Delta_w = \{\lambda \in int[\Delta] : u_i(x(\lambda) \geq u_i(w_i)\}$ be the set of the *rational social weights*. Then for give $\epsilon > 0$ there exists $\delta > 0$ such that if $|e_i(\lambda, w(\eta) - e_i(\lambda, w)| \leq \|u_i'\|\|h\| < \epsilon$ for $h : \|h\| < \delta$, where $\|u_i'\| = sup|u_i'(x(\lambda, W)|, \lambda \in \Delta_w$, i.e. the excess utility function of the perturbed economy is in a neighborhood of the excess utility function of the original one.

To prove that zero is a regular value for e we need to prove that $Ker(e')$ splits $R^{n-1} \times \Omega$. In our case, as the image of $e = R^{n-1}$, the quotient space $(R^{n-1} \times \Omega)/Ker(e')$ has finite dimension, then $codim[Ker(e')] < \infty$ and the splitting property is automatically satisfied, see [Zeidler, E. (1993)].

Proof of the step (ii) We will prove that, $\bar{e}(\cdot, w) : int[\Delta] \to R^{n-1}$ is a Fredholm operator of index zero. This map will be a Fredholm operator if is F-differentiable and if $J_\lambda \bar{e}(\cdot, w) : int[\Delta] \to L(R^{n-1}, R^{n-1})$ is a linear Fredholm operator for each $\lambda \in int[\Delta]$. The index of $J_\lambda \bar{e}(\cdot, w)$ at λ is

$$ind(J_\lambda \bar{e}(\lambda, w)) = dim(Ker(J_\lambda \bar{e}(\lambda, w))) + codim(R(J_\lambda \bar{e}(\lambda, w))).$$

The operator, $(J_\lambda e(\lambda, w))$ is, for each $w \in \Omega_0$ a finite linear operator from $R^{n-1} \rightarrow R^{n-1}$ and then, for each $\lambda \in int[\Delta]$ is a Fredholm map of index zero.

The economies $\mathcal{E} = \{w_i, u_i, I\}$ where $w \in \Omega_0$ will be called **Regular Economies**.

In [Mas-Colell, A. (1990)] is proved that the set of regular economies is an open and dense set in the space of the economies, and to obtain this result it is not necessary to assume the non-emptiness of the interior of the positive cone of the consumption space. It is sufficient to allow for the possibility that w is not positive. In this work, we need this assumption to characterize the equilibria set as a Banach manifold.

In the next sections we attempt to show some of the main characteristics of the complementary set of Ω_0, this is the set of singular economies.

5 Singular economies and their properties

In this section utility functions are fixed and we describe each economy by its excess utility function $e : int[\Delta] \times \Omega \rightarrow R^{n-1}$. The equilibria of an economy are described by the state variables $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n), \in \mathcal{E}q(w)$ these equilibrium states change when the parameters $w \in \Omega$ change, these parameters are called external or control parameters. Given w the set of λ such that $e(\lambda, w) = e_w(\lambda) = 0$ determine the state of the system, i.e. the equilibrium in which the system rest. The parameters w describe the dependence of the system on external forces, the action of these forces cause changes in the states of the economy. Generically these changes are no so big, and the new state is similar to the previous one, this is because generically economies are regular. Nevertheless in some cases, a sudden transition resulting from a continuous parameter change, can be shown. This kind of changes is referred to as a **catastrophe**. A catastrophe can take place only in a neighborhood of a singular economy.

A state, or equilibrium $\lambda \in \mathcal{E}q(w)$ such that the corank of the jacobian matrix $J_\lambda e_w$ is positive, will be called **singular or critical equilibrium.** Singular economies will be classified in two big classes:

Definition 10 *The set of singular economies such that:*

1. *for all $\bar{\lambda} \in \mathcal{E}q(w)$ the $corank J_\lambda e_w(\bar{\lambda}) \leq 1$ and with strict inequality for at least one $\lambda \in \mathcal{E}q(w)$ This is the set of* **no degenerate singular economies.** *And the states of equilibria corresponding will be called* **critical no degenerate equilibria.**

2. *And the set of all remain singular economies, it will will be called the set of* **degenerate singular economies.** *An equilibrium $\bar{\lambda} \in \mathcal{E}q'(w)$ where $corank J_\lambda e_w(\bar{\lambda}) > 1$, will be called*

a **degenerate critical equilibrium.**

The corank of $J_\lambda e_w(\bar{\lambda})$ *is given by:*

$$corank\left[J_\lambda e_w(\bar{\lambda})\right] = (n-1) - dim\left[J_\lambda e_w(\bar{\lambda})\right].$$

In this way we can say that the corank is a measure for the degree of the degeneration of the equilibria.

To clarify these considerations and to justify the introduction to the Catastrophe Theory in economics, let us now consider the following two examples:

Example 1 *Let E(W)* $= \{R^2_+, u_i, w_i; i = 1,2\}$ *be the set of interchange economies which total endowment* $W = (W_1, W_2)$ *are fixed. This means that:*

$$W_j = w_{1j} + w_{2j},\ j = 1,2;\ \ (*)$$

where w_{ij} *is the initial endowment of agent i in the commodity j. Initial endowment may be redistributed but the total endowment can not be modified, so the components of W are constants.*

The equilibrium set will be symbolized by:

$$\mathcal{V}_W = \{(\lambda, w) \in int[\Delta] \times \Omega, : e(\lambda, w) = 0,\ w_{1j} + w_{2j} = W_j; j = 1,2\}$$

An equilibrium is a pair (λ, w) such that $e_1(\lambda, w) = 0,\ e_2(\lambda, w) = 0$. As in this example the total supply is fixed, to characterize the equilibrium, we can consider, without loss of generality the initial endowments of the only one agent, for instance the agent indexed by 1. And from the fact that social weight are in the sphere of radius 1, it is enough to consider only one component of λ. So, a pair (λ, w) will be an equilibrium if and only if, $e_1(\lambda_1; w_{11}, w_{12}) = 0$.

Suppose that the excess utility function of the agent 1 is given by:

$$e_1(\lambda_1, w_{11}, w_{12}) = 3W_1\lambda_1 - 3w_{11}(\lambda_1)^{\frac{1}{3}} + w_{12}. \tag{8}$$

In terms of catastrophe theory λ_1 is the state variable and w_1 are the control parameters.

The social equilibria of this economy will be given by the set of pairs (λ, w) such that its components $(\lambda_1, w_{11}, w_{12})$ solve the equation $e_1(\lambda_1, w_{11}, w_{12}) = 0$ and by the corresponding $(\lambda_2, w_{21}, w_{22})$ obtained from the former. The set

$$C_F = \{(\lambda_1, w_{11}, w_{12}) \in \mathcal{V}_W : detJ_{\lambda_1}e_1(\lambda_1, w_{11}, w_{12}) = 0\},$$

is the **Catastrophe surface.**

The economies whose endowments are in this surface are the singular economies. In our case this surface is defined by:

$$C_F = \left\{ (\lambda_1, w_{11}, w_{12}) \in \mathcal{V}_W : \frac{\partial e}{\partial \lambda_1} = 3W_1 - w_{11}\lambda^{-\frac{2}{3}} = 0 \right\}.$$

Explicitly:

$$C_F = \left\{ \left(\frac{w_{11}}{3W_1} \right)^{\frac{3}{2}}, \ w_{11}, \ \frac{2w_{11}^{\frac{3}{2}}}{(3W_1)^{\frac{1}{2}}} \right\}.$$

The projection of this set in the space of parameters will be called the **Bifurcation set.** In our case:

$$B_F = \left\{ w_{11}, \ \frac{2w_{11}^{\frac{3}{2}}}{(3W_1)^{\frac{1}{2}}} \right\}.$$

This set is represented in the space of parameters, w_{11}, w_{12} by a parabola. By varying the parameters continuously, and crossing this parabola, something unusual happens: *the number of possible states of equilibria associated with the initial endowments w change:* increases or decreases by two.

The number of equilibria is given by the sign of δ where:

$$\delta = 27 \left(\frac{w_{11}}{W_1} \right)^2 - 4 \left(\frac{w_{12}}{W_1} \right)^3$$

so if:

- $\delta < 0$ associate with w, there exist three regular equilibria.
- $\delta > 0$ there is one regular equilibrium associate with w.
- $\delta = 0, \ w_{11}w_{22} \neq 0$ there exists one critical (or singular) equilibrium and one regular equilibrium.

The additional consideration taken from [Balasko, Y. (1997b)]: *the set of regular economies with a unique equilibrium is arc connected in the two agents case,* help us to obtain a good geometric representation of economies. Therefore, the set of economies where $\delta > 0$ is an arc-connected set.

The hessian matrix of the considerate excess utility function (the matrix defined by the second order derivatives of e_w at λ) is singular, this means, as we will see later, that the critical equilibrium is degenerate. So economies with endowments which satisfy $\delta = 0$ are degenerate singular economies.

Example 2 *Consider the economy* $E = \left\{R^2_+, u_{\alpha,i}, w_i, i = 1, 2\right\}$ *whose utility functions are:*

$$\begin{aligned} u_{\alpha,1} = & \quad x_{11} - \tfrac{1}{\alpha} x_{12}^{-\alpha} \\ u_{\alpha,2} = & \quad -\tfrac{1}{\alpha} x_{21}^{-\alpha} + x_{22}, \end{aligned}$$

and endowments $W = w_1 + w_2$.

Following the Negishi approach we begin solving the optimization problem:

$$max W_\lambda(x) = \lambda_1 u_1(x_1) + \lambda_2 u_2(x_2),$$

restricted to the factible set:. $\mathcal{F} = \left\{x \in R^4_+ : \sum_{i=1}^2 x_i \leq \sum_{i=1}^2 w_i\right\}$

Denoting $\lambda_1 = \lambda$ it follows that $\lambda_2 = 1 - \lambda$. Then we write the excess utility function:

$$e_{uw} = \begin{cases} \left(\frac{1-\lambda}{\lambda}\right)^{\frac{\alpha}{1+\alpha}} - \left(\frac{1-\lambda}{\lambda}\right)^{\frac{1}{1+\alpha}} - w_{12}\left(\frac{1-\lambda}{\lambda}\right) + w_{21} \\ \left(\frac{1-\lambda}{\lambda}\right)^{\frac{-\alpha}{1+\alpha}} - \left(\frac{1-\lambda}{\lambda}\right)^{\frac{-1}{1+\alpha}} - w_{21}\left(\frac{1-\lambda}{\lambda}\right)^{-1} + w_{12} \end{cases}$$

The catastrophe surface is given by:

$$C_F = \left\{(\lambda, w_{11}, w_{12}) \in \mathcal{V}_W : w_{12} = \frac{\alpha}{1+\alpha} h^{\frac{1}{1+\alpha}} - \frac{1}{1+\alpha} h^{\frac{\alpha}{1+\alpha}}\right\}$$

where $h = \frac{\lambda}{1-\lambda}$.

Then economies E, which endowments are given by $(w_{11}, w_{12}, w_{21}, w_{22})$ verifying

$$W = w_1 + w_2$$

and

$$w_{12} = \frac{\alpha}{1+\alpha} h^{\frac{1}{1+\alpha}} - \frac{1}{1+\alpha} h^{\frac{\alpha}{1+\alpha}}$$

are singular. Solving $e_u(\lambda, w) = 0$ it is easy to see that in all neighborhood of this economies there exist economies with one equilibrium and economies with three equilibria.

6 Catastrophe theory and economic theory

The catastrophe theory can be applied with wide generality in quasiestatical models, (models which equilibria states are modified only by cause of external forces) in which little changes in its parameters cause sudden changes. When the system is a rest in a position of equilibrium the state variables, (λ in our case,) determine the state of the system. The parameters, (initial endowments of the economy) describe the dependence of the system on external forces. The action of these forces usually give raise to sudden jump from an equilibrium position to another, these sudden transitions, when originate from continuous modifications in parameters are referred as catastrophes. In General Equilibrium models this kind of transition only can be observed in a neighborhood of a singular economy.

Catastrophe theory shows that it is possible to analyze this kind of transition by means of few canonical forms. The behavior of economies which utility functions give place to the same kind of singularities is locally similar, then it is possible to classify the economies according to the the stereotype in correspondence with its singularities. So, the aims of this work is to get a survey of the possible qualitative structures in economies.

We start this section considering the most elemental case of economies with two agents. In this case the equilibrium states can be characterized by only one of the components of the excess utility function, for instance $e_i : int[\Delta] \times \Omega_+ \to R$ where i may be equal to 1 or equal to 2, that is a real function. In this case the main theorem to study singularities is the Generalized Morse theorem [Zeidler, E. (1993)]. This theorem states that locally around a no degenerate singular economy all excess utility function can be transformed to a simple standard form by changing coordinates. There are exactly 3 such forms and these are quadratic forms. To each function corresponds exactly one of these canonical forms.

Later more general cases will be considered.

6.1 Two agents economies

If $e_w : U_{\lambda_0} \subseteq int[\Delta] \to R^{n-1}$ is a C^k submersion at λ_0 then there exists a local c^k diffeomorphism ϕ with $\phi(\lambda_0) = 0$, and $\phi'(\lambda_0) = I$, such that the following normal local form holds: $e(\phi(\lambda_0)) = e'(\lambda_0)\lambda$, see [Zeidler, E. (1993)] vol 4. The question is when e_w is not a submersion at λ_0 is the a coordinate transformation ϕ and a local normal form? *For an economy with two agents, the Morse lemma is the answer.*

Let $\mathcal{E} = \{u_i, w_i; i = 1, 2\}$ be an interchange economy with two agents and l commodities. The property 2 of definition 4, allow us to characterize the economy by one component of its excess utility function as a function of the initial endowments, and property 1 of the same definition, allow us consider only one of the two social weight. Let $e_i : (0,1) \times \Omega_+ \to R$ be the excess utility function of the agent indexed by i. The function is defined by $(\lambda_i, w) \to e_i(\lambda_i, w)$.

Then we can classify this kind of economies by looking for the Taylor expansion of e_i. If $g : R \to R$ is a smooth function such that $g(\bar{x}) = g'(\bar{x}) = ... = g^{(k)}(\bar{x}) = 0$ then there exista a smooth function l such that $g(x) = (x - \bar{x})l(x)$, and $l(\bar{\lambda}) = 0$.

But if $g^{(k)}(\bar{x}) \neq 0$. Then there exists a smooth local change of coordinates under which g takes the form x^k, (k *odd*); $\pm x^k$, (k *even*). See [Poston, T.; Stewart, l. (1978)].

The characterization of no degenerates singular points in terms of the *hessian matrix* (see section (2))is a comfortable condition to characterize singular economies:

Remark 1 *A two agents interchange economy w, is a degenerate singular economy if and only if the hessian matrix of e_{wi} :*

$$\partial e_{wi} = \left\{ \frac{\partial e_{wi}(\lambda)}{\partial \lambda_h \lambda_k} \right\}, \ h, k = 1, 2..., n.$$

is singular for at least one $\lambda \in \mathcal{E}q(w)$.

The significance of Morse's Lemma is in reducing the family of all smooth functions vanishing at the origin ($f(p) = 0$) in R^n with the origin as a no degenerate singular point, to just $n + 1$ simple stereotypes.

Applying this theorem in economic setting it follows that, in a neighborhood $U_{\bar{\lambda}}$ of a social equilibrium $\bar{\lambda}$ of a no degenerate singular economy $\bar{w}$, the excess utilities functions $e_{\bar{w}}$ will behave in similar way for every non degenerate $\bar{w}$ with independence of utilities. Moreover, if given the utility function, there are only no degenerates singular economies, then by smooth coordinate transformation it is possible to reduce the family of all excess utility function to just 3 simple stereotypes, namely:

$$e_{\bar{w}i}(\psi(\lambda)) = \pm\lambda_1^2 \pm \lambda_2^2.$$

The following two theorems, help us to know some characteristics of the no degenerates singular economies

Theorem 5 *Let $f : X \to R$ be a smooth function with a no degenerate singular point p. Then there exists a neighborhood V of p in X such that no other singular point of f are in V, i. e., no degenerate singular points are isolates.*

So, no degenerate singular points are isolates, and if we consider endowments in a finite subset of Ω there are finite number of they. Moreover, generically in Ω, there exists only one λ such that $e_w(\lambda) = 0$ is a critical no degenerate social equilibrium. This follows as a conclusion of the next theorem:

Theorem 6 *Let X be a smooth manifold. The set of Morse functions all of whose singular values are distinct (i.e., if p and q are distinct singular points of f in X, then $f(p) \neq f(q)$) form a residual set in $C^\infty(X, R)$.*

Figure 1: Two goods two agents economies

This means that generically, if the economy $\mathcal{E} = \{u_i, w_i, I\}$ is singular no degenerate, then there exists only one critical equilibrium $\lambda \in \mathcal{E}q(w)$.

Remark 2 (About singularities and oddness in the number of equilibria) *In terms of the economic theory this means that, generically a singular no degenerate economy w, with 2 agents has only one critical equilibrium. The oddness of the number of equilibria force that in a neighborhood of the singular economy there are economies $\bar{w}$ with only one $\lambda \in \mathcal{E}q(\bar{w})$ and economies $\bar{\bar{w}}$ with three distinct $\lambda \in \mathcal{E}q(\bar{\bar{w}})$*

If we add the hypothesis of 2 commodities, the oddness and the arc-connectedness properties of the regular economies with one equilibrium before mentioned, allows us to show the picture as generically representative of the behavior of this kind of economies.

More in general, the splitting theorem, see [Poston, T.; Stewart, l. (1978)] allow us to classify degenerates singular two agents economies. This theorem say that if a smooth function $F : R^n \to R$ has codimension $n - r$ (i.e the corank of the hessian matrix is $n - r$) then there exists a change of coordinates, that allow us to write F in this new coordinates in the form:

$$F(u_1, u_2, ..., u_n) = \pm u_1^2 \pm u_2^2 ... \pm u_r^2 + f(u_{r+1} + ... + u_n).$$

This means that the excess utility function, in a neighborhood of a critical degenerate equilibrium with corank 1, after a change of coordinates has the form: $e(\lambda_1, \lambda_2) = \pm\lambda_1^2 + g(\lambda_2)$.

Finally, the economic interpretation of the above considerations is that:

1) Regular economies have a similar behavior around an equilibrium.

2) Two agent economies can be classified from the Taylor expansion of the one of the two coordinates of its excess utility function.

3) The excess utility function of all no degenerate singular economy with two agents, have a similar behavior in a neighborhood of a no degenerate critical equilibrium. And this behavior is characterized by a second order polynomial.

4) Singular degenerate two agent economies can be classified using the splitting theorem.

Our objective is to classify the economies in more general cases too, then the following question is of major importance for a qualitative understanding of many economical (in general scientific) phenomena:

When does the Taylor expansion up to some order k

$$j_x^k f(u) = f(x) + f(x)u + ... + f^k(x)u^k/k!$$

provide enough information to understand the local behavior of a function f at x?

We say that a function f is k-determined if and only if it follows from $j^k f = j^k g$ that such that f and g are *locally right-equivalent* i.e., there exists a diffeomorphism ϕ such that

$$\phi(0) = 0, \ g(u) = f(\phi(u)) + constant.$$

Example 3 *Consider $f : U(0) \subset R^n \to R$*

- *a)Let $f'(0) \neq 0$, i.e., not all the terms vanish in the Taylor expansion of f at zero. Then F and H, with $H(u) = \xi_1$ are locally right equivalent. Consequently, f is 1-determined. And the local normal form is given by: $f(\phi(x)) = f(0) + f'(0)x$.*

- *b) If $f'(0) = 0$, and the matrix $f''(0)$ for the second-order partial derivatives of f at* 0 *is invertible, then f is 2-determined in R^n. And the local normal form is given by: $f(\phi(x)) = f(0) + f''(0)x^2$.*

- *c) Recall that in general a function cannot be determined by its Taylor polynomial in an arbitrary point x. For instance the functions $f : R^2 \to R$, $f(x,y) = x^2$, and $g : R^2 \to R$, $g(x,y) = x^2 - y^{2l}$, have the same $k-$polynomial at $0 \in R^2$ when $l > k/2$ holds, but if if $\phi = (\phi_1, \phi_2)$ is any local diffeomorphism at $0 \in R^2$ then:*

$$f(\phi(0,y)) = (\phi_1(0,y))^2 \neq -y^{2l} = g(0,y)$$

is true for nonzero $y \in R$. Thus, there is no number k such that f is k-determined.

In economics terms this question takes the following expression: *it would be possible, to characterize the behavior of an economy from the Taylor expansion up to some order k, of its excess utility function ?*

- The above example, prove that all regular two-agent economy is one-determined. Then the local normal form for the excess utility function is: $e(\phi(\lambda_0)) = e'(\lambda_0)\lambda$.

- As we shown above, using the Morse lemma no degenerate two-agent economies are 2- determined. And the local normal form for the excess utility function is: $e(\phi(\lambda_0)) = e''(\lambda_0)\lambda^2$.

7 Starting a classification: The S_r classification

We begin this section with an important question of the catastrophe theory, the k-determination of $C^k(X,Y)$ functions. After discussing this point in abstract, we will relate it to the possible changes in the qualitative behavior of an economy in a neighborhood of herself [7]. We will consider economies with an arbitrary but finite number of consumers and then we focus our attention on two kind of singularities: the folds and the cusps. Finally we will connect the kind of the singularities that it can appear in a particular economy with the number of agent and goods that this economy has.

The question previously formulated, is to say, the question of the $k-$determination of a function:

When a function f is determined in a neighborhood of a point x by some of its Taylor polynomials at x in the sense that, every other function having the same Taylor polynomial coincide with f in a neighborhood of x up to a diffeomorphism? It is a fundamental question to the catastrophe theory.

Let us introduce another characterization of functions with an esential similar behavior.

We will say that **a map** $f \in C^k(X,Y)$, **is k-equivalent at a point** $x_0 \in X$ to a map $g \in C^k(U,V)$ at a point u_0 if and only if there exist local C^k diffeomorphisms at u_0 and $f(x_0)$ respectively with $\phi(u_0) = x_0$ and $\psi(f(x_0) = g(u_0)$. In this case f and g need only be defined in a neighborhood of x_0 and u_0. Where X, Y, U, and V are Banach-manifolds. There is no obvious relationship between this two kind of equivalence, despite these two concepts are strongly related.

Example 4 *1. Let $f : U(p) \subset X \to Y$ be $C^k(X,Y), k \geq 1$ and X and Y Banach manifolds and let $g = j^1_k(f)$ i.e.,*

$$g(u) = f(x) + f'(x)u$$

If f is submersion or inmersion at x then f is $k-$equivalent to g at 0.

2. If $X = R^n$ and $Y = R^m$ and f is a submersion at x then f is k-equivalent at x to g at 0. Moreover, if $\text{rank} f'(x) = r$, then f is k-equivalent at x to $h : X \to Y$ with

$$h(x_1, ..., x_n) = (x_1, ..., x_r, 0...0)$$

Definition 11 *We will say that the economy $\mathcal{E} = \{u_i, w_i, i \in I\}$ is k-equivalent at a $\lambda^0 \in \mathcal{E}q(w)$ to the economy $\mathcal{E}' = \{u_i, w'_i, i \in I\}$ at $\lambda^1 \in \mathcal{E}q(w)$ if and only if its respective excess utility functions e_w and e'_w are $k-$equivalent functions at λ^0 and λ^1 .*

[7] We say that the economy, $\mathcal{E} = \{X_i, u_i, w_i, I\}$ is in a neighborhood of the economy $\mathcal{E}' = \{X_i, u_i, w'_i, I\}$ if w is in a neighborhood $U_{w'} \subset \Omega_+$ of w'.

This definition and (2) in example 4, show that the excess utility function of all regular economy is $1-determined$.

Thus, if the polynomial of Taylor of first order of two functions of excess of utility in its respective points of equilibrium,λ^0 and λ^1 agree (to less of diffeomorphisms) then the economies that this function represent display the same behavior, locally speaking.

Now let us to relate the k-equivalence concept with k-determination. *A function* $f : U(x) \subset X \to R^m$ *is* $k-$ *determined if and only if for each function* $g : U(x) \subset X \to R^m$ *with the same Taylor polinomial of degree* k, $j_p^k f$, *there exists a local* C^∞ *diffeomorphism* $\phi \in R^n$ *such that* $g(\phi(u)) = j_x^k f(u)$ *in a neighborhood of* x.

Roughly speaking, a function f will be k-determined if all function which differ from f only in terms of order higher than k behave qualitatively like the $k-th$ Taylor polinomial of f. This means that the Taylor expansion up to order k completely determines f and its perturbations with terms of order higher than k.

So, if the excess utility function of a given economy e_w is k-determined, then all economy which excess utility function have the same Taylor polinomial up to order k, show the same qualitative behavior than the former, i.e., for some k, $j^k e_w$ express the essential behavior of the economy in a neighborhood of each of its equilibria.

We will look at what is called the $k-jet$ of that function at $p \in Dom(f)$, and then we will show some characteristic of the set of singularities of each clase of functions identified in this way.

Definition 12 Jet Bundles: *Let* X *and* Y *be* n *and* m *dimensional, smooth manifolds and* $f,\ g : X \to Y,\ f(x) = g(x) = y$ *be smooth functions. Consider the following equivalence relation:* $f \sim_k g$ *will mean that the* $k-th$ *Taylor expansion of* f *coincides with the* $k-th$ *expansion of* g *at* x. *The equivalence class of* f *at* x *under this relation is called the* **k-jet of f at** x, *and will be denoted by* $J^k(f)_x$.

- By the symbol $D^h f$ we represent the set of partial derivatives such that: $\frac{\partial^{|h|} f}{\partial x_1^{h_1} \ldots \partial x_n^{h_n}}$ where $|h| = \sum_{j=1}^n h_j,\ h_i \geq 0\ \ i = 1, 2, \ldots, n$.
- Let $J^k(X,Y)_{x,y}$ denote the set of equivalence classes under $\sim_k$ at p of mapping $f : X \to Y$ where $f(x) = y$.
- An element $\sigma \in\ J^k(X,Y) = \cup_{(x,y) \in X \times Y} J^k(X,Y)_{x,y}$, is called **k-jet** where $f(x) = y$.
- let $J^k(X,Y) = \cup_{(x,y) \in X \times Y} J^k(X,Y)_{x,y}$ (disjoint union). Then $J^k(X,Y)$ is the set of all k-jet with source X and target Y.

Theorem 7 *Let X and Y be smooth manifolds with $n = dimX$ and $m = dimY$. Then, $J^k(X,Y)$ is a smooth manifold with:*

$$dimJ^k(X,Y) = m + n + dim(B^k_{n,m}),$$

where $B^k_{n,m}$ is the space of formed by the direct sum of polynomial in $n-$variables with degree $\leq k$.

The object of our analysis is the excess utility function, and the social equilibria. Obviously its critical values can be other than zero, but our interest is focused at the origin, because only this value have an economical means: The preimagen of zero by e is the set of the social equilibria, i.e. $e^{-1}(0) = \mathcal{E}q$. So we will consider the functions $e_w : \int[\Delta] \rightarrow R^{n-1}$.

Then we are interested in consider the class $J^k(X,Y)_{(\lambda,w),0}$ that is, the $k-$jet σ with source $\lambda) \in int[\Delta]$ and target $0 \in Y = R^{n-1}$.

Remark 3 (Notation) *To avoid future possible mistakes arose from the notation, from now on we will represent the jacobian matrix of a mapping f at p by the symbol: $(\partial f)_x$.*

Let $\sigma \in J^1(X,Y)$; then σ defines a unique linear mapping of $T_xX \rightarrow T_yY$, where x is the source of σ and y is the target of σ. Let f be a representative of σ in $C^\infty(X,Y)$. Then $(\partial f)_x$ is that linear mapping. Define $rank(\sigma) = rank(\partial f)_p$ and $corank(\sigma) = \mu - rank\sigma$, where $\mu = min(dimX, dimY)$. Let

$$S_r = \left\{\sigma \in J^1(X,Y) : corank(\sigma) = r\right\}.$$

This is the subset of the equivalence classes under $\sim_1$ in $C^\infty(X,Y)$ such that the $corank(\partial f)_p = r$ where p is the source of σ. The subset S_r is a submanifold of $J^1(X,Y)$ with

$$codim\ S_r = (n - \mu + r)(m - \mu + r),$$

see [Golubistki, M. and Guillemin,V.(1973)].

As we said above our interest is the class $\sigma \in J^1(int[\Delta], R^{n-1})$ with source λ and target $0 \in Y = R^{n-1}$. It follows that: $dimX = (n-1)$ and $dimY = n-1$, then $\mu = n-1$. So, $codimS_r = r^2$.

The set of singularities of $f : X \rightarrow Y$ where the rank of it jacobian matrix drops by r i.e., the set $x \in X$ where $rank(\partial f)_x = min(dimX, dimY) - r$ is represented by the symbol: $S_r(f) = (j^1 f)^{-1}(S_r)$. Then $S_r(f)$ will be, generically, a manifold of the same codimension that (S_r), [Golubistki, M. and Guillemin,V.(1973)].

As $codimS_r(f) = dimX - dimS_r(f) \geq 0$ there is a relation between the kind of singularities possible for each $f \in C^\infty(X,Y)$ and the dimension of the manifold.

Applying this concepts to economics, $S_r(e_w)$ is the set of critical points of e_w where the jacobian matrix of e_w drops rank by r. This set is a manifold and the set of critical social equilibria is the subset of $(\lambda, w) : \lambda \in S_r(e_w)$ and $e(\lambda, w) = 0$. For instance, $S_1(e_w)$ is the set of no degenerate critical social equilibria that is, the set of pairs $(\lambda, w) \in int[\Delta] \times \Omega$ such that $e(\lambda, w) = 0$.

It follows that, there exists a relation between the number of agents and the form of possible singularities. In others words, the excess utility function could have only some types of singularities, and these will be determined by the number of consumers in the economy.

Then, $codimS_r(f) > |dimX - dimY|$, then $dimS_r(f) < dimY$. Applying this observation to economics, where: $X = int[\Delta]$, $Y = R^{n-1}$ and f is the excess utility function e_w, it follows that: if n is the number of consumers of the economy then, $dimS_r(e_w) < n - 1$. In cases where $n = 2$ we obtain that singular economies are generically isolates points in Ω.

It is important to remind that the topology used in theorems about transversality of maps in $C^\infty(X, Y)$ is the Withney topology, this is a very strong topology, therefore if a proposition is satisfy generically in a topological space with the Whitney topology, is indeed satisfy in quite large sense and is a strong result.

The next example clarify these considerations:

Example 5 *If the economy have n consumers then it follows that that $dimX = dimY = n - 1$ and $codimS_r = r^2$ so, e_w could have only singularities of kind S_r such that $r^2 < n - 1$. Note that if $n = 2$ we obtain that critical social equilibria are isolate points.*

Now we will show **some characteristic of S_1 singularities:**

7.1 The Fold and the Cusp in economics

Definition 13 (Submersions with Folds) *Let X and Y be a smooth manifolds with $dimX \geq dimY$ Let $f : X \to Y$ be a smooth mapping, such that $J^1 f$ is transversal to S_1. Then a point $p \in S_1(f)$ is called fold point if:*

$$T_x S_1(f) + Ker(\partial f)_x = T_x X.$$

Definition 14 *We say that a map is one generic if $J^1 f$ is transversal to S_1. This is a generic situation. [Golubistki, M. and Guillemin, V.(1973)].*

Where S_1 is the submanifold of $J^1(X, Y)$ of jets of corank 1, then $S_1(f) = (j^1 f)^{-1}(S_1)$ is a submanifold of X with $codimS_1(f) = codim(S_1) = k + 1$ where $k = dimX - dimY$. Note that is $x \in S_1(f)$ then $dimKer(\partial f)_x = k + 1$. That is, the tangent space to $S_1(f)$ and the kernel of $(\partial f)_x$ have complementary dimensions.

Therefore, $codimS_1(e) = 1$ it follows that if $(\lambda, w) \in S_1(e_w)$ then, $dimKer(\partial e : w)_{\lambda)} = 1$.

The next theorem characterize the local behavior of a submersion with folds near a fold, similar to the Morse theorem for real function. (Recall that if X and Y are manifolds, and $f : X \to Y$ is differentiable mapping, with rank$(\partial f)_p$ the maximum possible, is a submersion if $dimX \geq dimY$.)

Theorem 8 *Let $f : X \to Y$ be a submersion with folds and let p be in $S_1(f)$. Then there exist coordinates $x_1, x_2, ..., x_n$ centered at x_0 and $y_1, y_2, ...y_n$ centered at $f(p)$ so that in these coordinates f is given by:*

$$(x_1, x_2, ..., x_n) \to (x_1, x_2, ..., x_{m-1}, x_m^2 \pm ... \pm x_n^2)$$

This theorem is proved in [Golubistki, M. and Guillemin,V.(1973)].

Taking a particularly simple example of 2-manifolds (manifolds with dimension equal 2), we see the reason for the nomenclature fold point. In this case the normal form is given by: $(x_1, x_2) \to (x_1, x_2^2)$. This transformation is obtained by means of the following steps:

- (1))Map the (x_1, x_2) map onto the parabolic cylinder, (x_1, x_2, x_2^2),
- (2))then, project onto the (x_1, x_3) plane.

Example 6 *3-agent economies:*

Let X and Y be 2-manifolds and let $f : X \to Y$ be a one generic mapping. By our computation $codimS_1(f) = 1$ in X, and S_2 does not occur, since it would to have codimension 4. Let p be a point in $S_1(f)$ and $q = f(p)$. One of the following two situations can occur:

(a) $T_pS_1(f) \oplus Ker(\partial f)_p = T_pX$.

(b) $T_pS_1(f) = Ker(\partial f)_p$

Remark 4 *Whitney proved that if f belongs to $C^\infty(X, Y)$ generically the only singularities are folds and simple cusp.*

Note that only if the interchange economy has 3 agents and fixed initial endowment the excess utility function is a mapping between 2-manifolds, $e_w : int[\Delta] \to R^2$.

Let $\bar{\lambda} = (\bar{\lambda_1}, \bar{\lambda_2}, \bar{\lambda_3}) \in \Delta$ be a singular social equilibrium for the economy w.

i) In the first case (item (a)) applying 8 one can choose a system of coordinates (λ_1, λ_2) centered at $(\bar{\lambda 1}, \bar{\lambda 2}) \in S_1(e_w)$ and (e_1, e_2) centered at $e_w(\bar{\lambda}) = 0$ such that e_w is a fold: $(\lambda_1, \lambda_2) \to (\lambda_1, \lambda_2^2)$.

ii) If (b) holds the situation is considerable more complicated. Generically singularities, in this case, are simple cusps. In this case one can find coordinates $(\bar{\lambda_1}, \bar{\lambda_2})$ centered at $e(\bar{\lambda})$ such that:

$$(\bar{\lambda}_1, \bar{\lambda}_2) \rightarrow (\bar{\lambda}_1, \bar{\lambda}_1 \bar{\lambda}_2 + \bar{\lambda}_2^3).$$

In a neighborhood of a cusp or a fold there exist regular economies with different number of equilibrium. Recall that, at the moment of through a singularity the changes in the number of equilibria appear.

7.2 The $S_{r,s}$ singularities

Let $f : X \rightarrow Y$ be one generic. We will denote by $S_{r,s}(f)$ the set of points where the map $f : S_r(f) \rightarrow Y$ drops by rank s. Analogous to the S_r it si possible to build:

$$S_{r,s} \subset \{\sigma \in J^2(X, Y) : corank(\sigma) = r\}.$$

Note that $x \in S_{r,s}(f)$ if and only if $x \in S_r(f)$ and the kernel of $(\partial f)_x$ intersects the tangent space to $S_r(f)$ in a subspace a s dimensional subspace. From $dimS_r(e) < n - 1$ it follows that in cases of economies where $n = 2$ the singularities are $S_{1,0}(e)$ folds, or $S_{1,1}(e)$ cusps.

Using the Transversality Theorem in [Golubistki, M. and Guillemin,V.(1973)] is proved that $j^2 f$ is generically transversal to $S_{r,s}$ and then the sets $S_{r,s}$ are submanifolds in $J^2(X, Y)$ and like in the case of $S_r(f)$,

$$S_{r,s}(f) = (j^2(f))^{-1}(S_{r,s}).$$

In the cited work the dimension of $S_{r,s}(f)$ is computed.

Generically $S_{r,s}(f)$ are submanifolds in X whose dimensions are given by:

$$dimS_{r,s}(f) = dimX - r^2 - \mu r - (codimS_{r,s}(f) \; in \; S_r(f)) \,. \tag{9}$$

where

$$codimS_{r,s} = \frac{m}{2}(k+1) - \frac{m}{2}(k-s)(k-s+1) - s(k-s), \tag{10}$$

where

- $m = dimY - dimX + k$
- $k = r + max(dimX - dimY, 0)$ and
- $\mu = min \{dimX, dimY\}$.

[Golubistki, M. and Guillemin,V.(1973)]

In this way we see that the set of possible singularities in economics are strongly relate with the number of agents and commodities, then it follows that some kind of singularities appear only if the number of agents are big enough.

7.3 Singularities and its relations with the dimension of the economy

Applying to economies with a finite number of agents and commodities, we obtain that:

- $k = m = r$
- where n is the number of agents, and r is the codimension of (∂e_w) evaluated at the singularity.

So, generically, we obtain substituting in (10) that:

$$codim S_{r,s}(e) = \frac{r}{2}(r+1) - \frac{r}{2}(r-s)(r-s+1) - s(r-s).$$

In particular for $S_{1,1}$ it follows that $codim S_{11} = 1$ and $dim S_{11} = n - 2$ holds.

Substituting $r = 1$ and $s = 2$ in the formula above it follows that generically, singularities like $S_{1,2}(e)$ only could appear if the number of consumer is $n > 3$, because $n > 3$ is a necessary condition to be $dim S_{1,2}(e) \geq 0$. So for economies with three agents, are only possible singularities of the type $S_{1,0}$ and $S_{1,1}$, folds and cusps, as we already saw it in example (6).

8 Conclusions

The introduction of the function excess of utility allows us to work with economies of infinite dimension in analogous form to which we do in the finite case and on the other hand, to improve our understanding of the way in which equilibria depend upon economic parameters (initial endowments or utility functions) and shows the strong relation existing between preferences and the behavior of the economic system.

This function reflects the weight of consumers in the markets, and show the changes in their relative weights in equilibrium, when the initial endowments change. Near a regular economy these changes are smooth and there is not qualitative changes, but around a singularity sudden and big changes occur. The economic weight of the agents change drastically, overthrowing the existent order. The uncertainty in the behavior of the economy is a direct result of the existence of singular economies. If there were not singularities, economics would be a science with perfect prediction. In a neighborhood of a singular economy, the central planer need to be extremely careful. Because if he acts according with its experience, can do small changes in endowment hopping that things follows not much different than in the previous situation, but this hope can not occur, if the economy is a singular one the perturbed economy will be to much different than the previous one, and it is no possible to can back by means of small changes.

Nevertheless, most part of the literature in economics have focused on regular economies whose equilibria change smoothly according to the changes in the endowments. The study of the discontinuous behavior requires to consider singularities, this led us to the catastrophe theory. This theory refers to drastic changes, however to be sudden, abrupt and unexpected the catastrophe theory show that these changes have a similar substratum that allows us to do a classification according its geometric representation. So, the study of singularities require catastrophe theory and the theory of mapping and their singularities, in this way one might have an approximation to understanding the forms of the unexpected changes in economics. The economies can be characterized by their singularities, and these are those that really characterize the essential of their behavior. Economies with the same type of singularities will present the same possibilities of changes.

A final consideration: The excess utility function allows us to extend the analysis of singularities for economies with finite dimensional consumption spaces, to infinite dimensional economies. Showing in this way that also in these cases, the catastrophe theory may be a gate to begin to understand the behavior of an economical system with infinitely many goods in a neighborhood of a singularity.

9 Appendix

In this section we consider the implicit function theorem in Banach manifolds. The concept of chart map alow us to carry much of the differential theory useful in Banach space to Banach Manifold. The charts establish a local diffeomorphism between a B-manifold and a B-space. And the correspondences between manifolds are translated, by means of these chart maps in local correspondences between B-spaces. So if ϕ and ψ are charts for M and N, and $f : M \to N$ then $\bar{f} = \psi(f((\phi)^{-1}) : X_\phi \to X_\psi$ where X_ϕ and X_ψ are B-spaces (the chart spaces for (M, ϕ), and (N, ψ) respectively). Recall that $f' : T_xM \to T_{f(x)}N$, where T_xM and $T_{f(x)}N$ are respectively, the tangent spaces for M at the point x and for N at the point $f(x)$.

Theorem: Let E, F, G Banach manifolds, let $A \subset E \times F$ be a nonempty open set. Let $f : A \to G$ continuously differentiable in A. Let (u_0, v_0) be a point in A such that $f(u_0, v_0) = 0$ and let the derivative with respect to the second variable, $D_2 f(u_0, v_0) : F \to G$ be an homeomorphism on G. Then there exists a neighborhood U of u_0 in E and a neighborhood W of (u_0, v_0) in A and a differentiable map $g : U \to F$ such that for all $u \in U; v = g(u)$ verify $f(u, g(u)) = 0$. This map is a C^k map if $f \in C^k$. See [Zeidler, E. (1993)].

For a given $k \in X$ let us introduce the following n real variables $\alpha = (\alpha_1, ..., \alpha_n)$. We introduce the following notation: $x^* + \alpha k = (x_1^* + \alpha_1 k, ..., x_n^* + \alpha_n k)$. Let us now consider the functions $f_i : (0,1) \times R \times R \to R; i = 1, 2, ..., n$ and $f_{n+1} : R^n \times X_+ \to X$ consider also the function $f : int[\Delta] \times X_{++} \times R^n \times R \to R^n \times X_+$ given by:

$$f(\lambda, W, \alpha, \gamma) = (f_1(\lambda_1, \alpha_1, \gamma), ..., f_n(\lambda_n, \alpha_n, \gamma), f_{n+1}(\alpha, W)$$

defined as:

$$\begin{aligned} f_i(\lambda, \alpha_i, \gamma^*) &= \lambda_i u_i'(x_i^* + \alpha_i k)h - \gamma^* \\ f_{n+1}(\alpha, W) &= \textstyle\sum_{i=1}^n (x_i^* + \alpha_i k) - W \end{aligned} \tag{11}$$

We know that for $f(\tilde{\lambda}, \tilde{W}, 0, \gamma^*) = 0$. Let us introduce the notation $m = (\lambda, W)$ and $v = (x, \gamma)$. Let $\mathcal{M}_{\tilde{m}} = \mathcal{U}_{\tilde{\lambda}} \times \mathcal{U}_{\tilde{W}}$ be a neighborhood of $\tilde{m} = (\tilde{\lambda}, \tilde{W})$. Analogously, let $\mathcal{V}_{v^*} = \mathcal{V}_0 \times \mathcal{V}_{\gamma^*}$ be a neighborhood of $v^* = (0, \gamma^*)$. The jacobian matrix of this system with respect to the variables α and γ is given by $D_2 f(\tilde{u}, v^*)k = f_v'(\tilde{u}, v^*)k$ i.e.:

$$\begin{bmatrix} \lambda_1 u_1''(x_1^*)hk & 0 & 0 & \dots & 0 & 1 \\ 0 & \lambda_2 u_2''(x_2^*)hk & 0 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n u_n''(x_n^*)hk & 1 \\ k & k & k & \dots & k & 0 \end{bmatrix}$$

from our hypothesis on utility functions, this is a bijective transformation from $R^n \times X$ onto $R^n \times X$, so we can apply the implicit function theorem to obtain $x^*(\lambda, W) = x^* + \alpha(\lambda, W)k$, so for all $(\lambda, W) \in \mathcal{M}_m$ and $h \in X$ the following identities are verified:

$$\lambda_i u_i'(x_i^*(\lambda, W))h - \gamma^*(\lambda, W) = 0, \quad i = 1, 2, ..., n.$$

$$\textstyle\sum_{i=1}^n x_i^*(\lambda, W) - W = 0$$

then, for each $(\lambda, W) \in \mathcal{M}_m$ the feasible allocation $x^*(\lambda, W)$ solves the problem (3).

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