Modeling Regional Interdependencies using a Global Error-Correcting Macroeconometric Model[1]

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Abstract

Financial institutions are ultimately exposed to macroeconomic fluctuations in the global economy. This paper proposes and builds a compact global model capable of generating forecasts for a core set of macroeconomic factors (or variables) across a number of countries. The model explicitly allows for the interdependencies that exist between national and international factors. Individual region-specific vector error-correcting models are estimated, where the domestic variables are related to corresponding foreign variables constructed exclusively to match the international trade pattern of the country under consideration. The individual country models are then linked in a consistent and cohesive manner to generate forecasts for all the variables in the world economy simultaneously. The global model is estimated for 26 countries grouped into 11 regions using quarterly data over 1979Q1-99Q1. The degree of regional interdependencies is investigated via generalized impulse responses where the effects of shocks to a given variable in a given country on the rest of the world are provided. The model is then used to investigate the effects of various global risk scenarios on a bank's loan portfolio.

Keywords: Global interdependencies, global macroeconometric modeling, Credit loss distribution, Risk management, Global Vector Error Correcting Model.

JEL Classification: C32, E17, G20.

1 Introduction

Increased globalization of the world economy has important consequences for the conduct of monetary and financial policies by central bankers and risk management by commercial bankers. In setting interest rates, more than ever before, central bankers need to allow for the inter-relationships that exist between their economy and the rest of the world. In a commercial banking context, the risk analysis of a bank's financial activities needs to take account of domestic economic conditions as well as the economic conditions of countries that directly or indirectly influence the loss distribution of banks' loan portfolios. Thus both constituencies would benefit from working with a global macroeconometric model which is capable of generating forecasts for a core set of macroeconomic factors for a set of regions and countries to which they have risk exposures, and which explicitly allows for interconnections and interdependencies that exist between national and international factors in a coherent and consistent manner.

This paper aims to provide such a global modeling framework by making use of recent advances in the analysis of cointegrating systems. So far applications of the cointegrating approach have been confined to a single country covering only some of the key macroeconomic variables.[1] While in principle it is possible to extend the approach to modeling inter-relationships across different economies, in practice due to data limitations such a strategy will not be feasible. In an unrestricted VAR model covering N regions the number of unknown parameters rises with N, and even if we focus on a few key macroeconomic indicators such as output, inflation, interest rate, and exchange rate there will be $p(kN-1)$ unknown parameters (not counting intercepts or other deterministic/exogeneous variables) to be estimated per each equation, where p is the order of the VAR and k is the number of the endogenous variables per region. For example, in the case of a world economy composed of 10 regions with $p=2$, and $k=5$, there will be at least as many as 98 unknown coefficients to be estimated per equation with the available quarterly time series being of the same order of magnitude for advanced economies and often much less in the case of other regions.

In view of these difficulties global forecasting models are often formed by linking up of the traditional, typically large-scale, macroeconometric models developed originally for the national economies. A prominent example of this approach is Lawrence Klein's Project Link adopted by United Nations. A similar approach, albeit on a smaller scale, has been followed by international agen-

[1] See, for example, King, Plosser, Stock, and Watson (1991), Mellander, Vredin, and Warne (1992), Crowder, Hoffman and Rasche (1999), and Garratt, Lee, Pesaran and Shin (2000, 2001).

cies such as the IMF and OECD. The National Institute's Global Econometric Model (NiGEM) estimates/calibrates a common model structure across OECD countries, China and a number of regional blocks. The country/region specific models in NiGEM are still quite large, each comprised of 60-90 equations with 30 key behavioral relations.[2] Global models with limited geographical coverage have also been developed. For example, Rae and Turner (2001) develop a small forecasting model covering the United States, the Euro area and Japan. These contributions provide significant insights into the important inter-linkages that exist among major world economies and have proved essential in global forecasting. Nevertheless, they are difficult to use for risk management purposes and do not adequately address the important financial inter-linkages that exist amongst the world's major economies.

In this paper we propose a new approach to modeling the global economy which avoids some of these limitations while at the same time providing a consistent and flexible framework for use in a variety of applications such as risk management. We first estimate individual country (or region) specific vector error-correcting models (VECMs) where the domestic macroeconomic variables such as Gross Domestic Product (GDP), the general price level, the level of short-term interest rate, exchange rate, equity prices (when applicable) and money supply, are related to corresponding foreign variables constructed to match the international trade pattern of the country under consideration. For purposes of estimation and inference these country-specific foreign variables can be treated as weakly exogenous (or long-run forcing) for most economies when N is sufficiently large and the idiosyncratic shocks are weakly correlated; a notable exception of course being the U.S. economy. The model for the U.S. can be estimated by treating most of the variables as endogenous. The individual country models are then combined in a consistent and cohesive manner to generate forecasts or impulse response functions for *all* the variables in the world economy simultaneously.

We use the estimated global model as the economic engine for generating conditional loss distributions of a credit portfolio. Business cycle fluctuations can have a major impact on credit portfolio loss distributions. Carey (2002), using resampling techniques, shows that mean losses of a typical portfolio during a recession such as 1990/91 in the U.S. are about the same as losses in the 0.5% tail during an expansion. Bangia et al. (2002), using a regime switching approach, find that capital held by banks over a one-year horizon needs to be 25-30% higher in a recession than in an expansion. The basic idea of our approach

[2] For a recent detailed account see Barrell et al. (2001).

is to make more explicit the linkage between a bank's credit exposures and the underlying international macroeconomic conditions.

The plan of the paper is as follows: Section 2 sets out the country/region specific models and establishes the inter-linkages between each of the economies and the rest of the world through trade-based weighting matrices. The different country-specific VECM models are then combined in Section 3, where a complete solution of the global VAR (GVAR) model is provided. Section 4 examines the error-correcting properties of the global model and shows that the number of long-run relationships in the global model can not exceed the sum of the long-run relations of the region specific models. Dynamic and stability properties of the GVAR model are discussed in Section 5. Section 6 derives impulse response functions for the analysis of shocks in one country on the macroeconomic variables in other countries. Section 7 considers the estimation problem of the country-specific models, with the technical details provided in Appendix. Section 8 discusses the practical issues surrounding the construction of regional aggregates. To ensure maximum global coverage while keeping the risk analysis manageable it is often necessary to work at regional levels, and Section 8 also addresses the aggregation bias that this may entail and ways of minimizing such a basis. An empirical illustration of the approach is set out in Section 9, where a GVAR model in seven countries (U.S., U.K., Germany, France, Italy, China and Japan) and four regions (Western Europe, Middle East, South East Asia and Latin America) is estimated and analyzed. This section also reports a number of impulse response functions demonstrating how the model could be used in the analysis of the transmission of stock market and interest rate shocks from one region to the rest of the world economy. In Section 10 we link a firm's return (and default) process to macroeconomic (systematic) variables and then proceed to generate loss distributions *conditional* on the estimated GVAR specification from Section 9, as well as analyzing the impact of economic shocks on loss. Section 11 offers some concluding remarks. The Appendix provides a summary of data sources used, as well as a brief account of the way the regional series were constructed.

2 Country Specific Models

We assume there are $N+1$ countries (or regions) in the global economy, indexed by $i = 0, 1, 2, ..., N$. We adopt country 0 as the reference country. (U.S. seems an obvious choice). For each country/region we assume that the country specific variables are related to the global economy variables measured as country-specific weighted averages of foreign variables plus deterministic variables such as time trends and global (weakly) exogenous variables such as oil prices. For simplicity, here we confine our exposition to a first order dynamic specification which relates the $k_i \times 1$ country-specific factors/variables, $\mathbf{x}_{it}$, to $\mathbf{x}_{it}^*$, a $k_i^* \times 1$ vector of foreign variables specific to country i (to be defined below) and write

$$\begin{aligned} \mathbf{x}_{it} &= \mathbf{a}_{i0} + \mathbf{a}_{i1}t + \mathbf{\Phi}_i \mathbf{x}_{i,t-1} + \mathbf{\Lambda}_{i0}\mathbf{x}_{it}^* + \mathbf{\Lambda}_{i1}\mathbf{x}_{i,t-1}^* + \boldsymbol{\varepsilon}_{it}, \\ t &= 1, 2, ..., T;\ i = 0, 1, 2, ..., N \end{aligned} \tag{2.1}$$

where $\mathbf{\Phi}_i$ is a $k_i \times k_i$ matrix of lagged coefficients, $\mathbf{\Lambda}_{i0}$ and $\mathbf{\Lambda}_{i1}$ are $k_i \times k_i^*$ matrix of coefficients associated with the foreign-specific variables, and $\boldsymbol{\varepsilon}_{it}$ is a $k_i \times 1$ vector of idiosyncratic country-specific shocks. In the special case where $\mathbf{\Lambda}_{i0} = \mathbf{\Lambda}_{i1} = \mathbf{0}$, this model reduces to standard vector autoregressive process of order 1, VAR(1). However, in the presence of foreign-specific variables (2.1) is an augmented VAR model which we denote by VARX*(1,1).[3]

We shall assume that the idiosyncratic shocks, $\boldsymbol{\varepsilon}_{it}$, are serially uncorrelated with a zero mean and a non-singular covariance matrix, $\mathbf{\Sigma}_{ii} = (\sigma_{ii,\ell s})$, where $\sigma_{ii,\ell s} = cov(\varepsilon_{i\ell t}, \varepsilon_{ist})$, or written more compactly

$$\boldsymbol{\varepsilon}_{it} \backsim i.i.d.(\mathbf{0}, \mathbf{\Sigma}_{ii}). \tag{2.2}$$

The assumption that the country-specific variance covariance matrices, $\mathbf{\Sigma}_{ii}$, $i = 0, 1, 2, ..., N$, are time invariant can be relaxed, but for the analysis of quarterly observations this time invariant assumption may not be too restrictive. However, when the focus of the analysis is on contagion or spill-over effects resulting from systemic risk it may be necessary to consider regime switching models where the parameters of the regional models (in particular $\mathbf{\Sigma}_{ii}$) switch between a "normal" and a "crisis" set of values.[4] To accommodate such effects it would be necessary to specify and estimate non-linear switching regional models from which a non-linear global model can be derived, and this is beyond the scope of the present paper.

We also allow the idiosyncratic shocks $\boldsymbol{\varepsilon}_{it}$ to be correlated across regions to a limited degree. The exact nature of this dependence will be clarified later once the linkages between the country-specific foreign variables, $\mathbf{x}_{it}^*$, and the variables in the rest of the world economic system, namely $(\mathbf{x}_{0t}, \mathbf{x}_{1t}, ..., \mathbf{x}_{i-1,t}, \mathbf{x}_{i+1,t}, ..., \mathbf{x}_{Nt})$ are specified.

[3] Common global stochastic variables, such as oil prices, can also be included in the model. These are not central to our exposition here and will be considered in Section 5.

[4] A comprehensive review of the literature on systemic risk can be found in De Bandt and Hartmann (2000).

Typically $\mathbf{x}_{it}$ will include real output (y_{it}), a general price index (p_{it}) or its rate of change, a real equity price index (q_{it}), the exchange rate (e_{it}, measured in terms of a reference currency, say U.S. dollar), an interest rate (ρ_{it}), and real money balances (m_{it}). To focus ideas we set $\mathbf{x}_{it} = (y_{it}, p_{it}, q_{it}, e_{it}, \rho_{it}, m_{it})'$, with $k_i = 6$.[5] We assume that these variables are observed at quarterly frequencies; $y_{it}, p_{it}, q_{it}, e_{it}$, and m_{it} are measured in natural logarithms and ρ_{it} is an interest rate variable. Output could be measured by real gross domestic (or national) product (GDP); the general price level by the consumer price index (CPI), the real equity price index (when available) could be measured by broad market indices such as theS&P500 index in the U.S., or the FTSE100 index in the U.K., deflated by the CPI, the real money supply by M_0 or M_2 measures of money supply deflated by the CPI, and finally the interest rate variable could be either the nominal interest rate on three months Treasury Bill rate (or its equivalent), or the (ex post) real interest rate defined as the nominal rate minus the rate of inflation.[6] For example, a typical set of endogenous variables for country i ($i \neq 0$), could be:

$$\left.\begin{array}{c} y_{it} = \ln\left(GDP_{it}/CPI_{it}\right), \quad p_{it} = \ln(CPI_{it}), \\ q_{it} = \ln(EQ_{it}/CPI_{it}), \quad m_{it} = \ln\left(M_{it}/CPI_{it}\right), \\ e_{it} = \ln(E_{it}), \quad \rho_{it} = 0.25 * \ln(1 + R_{it}/100), \end{array}\right\} \tag{2.3}$$

where[7]

$$\begin{aligned} GDP_{it} &= \text{Nominal Gross Domestic Product of country } i \\ &\quad\ \text{during period } t, \text{ in domestic currency,} \\ CPI_{it} &= \text{Consumer Price Index in country } i \text{ at time } t, \\ &\quad\ \text{equal to 1.0 in a base year (say 1995),} \\ M_{it} &= \text{Nominal Money Supply in domestic currency,} \\ EQ_{it} &= \text{Nominal Equity Price Index,} \\ E_{it} &= \text{Exchange rate of country } i \text{ at time } t \text{ in terms of } U.S. \text{ dollars,} \\ R_{it} &= \text{Nominal rate of interest per annum, in per cent.} \end{aligned}$$

Notice, that in the case of the base economy $e_{0t} = 0$ and $\mathbf{x}_{0t} = (y_{0t}, p_{0t}, q_{0t}, \rho_{0t}, m_{0t})'$,

[5] However, in practice it may be necessary to consider other transformations of these underlying variables. For example, as can be see from our empirical analysis in Section 9, we argue in favour of using the rate of inflation ($p_{it} - p_{i,t-1}$) instead of the price level (p_{it}) and the "real exchange rate" ($e_{it} - p_{it}$) instead of the nominal exchange rate (e_{it}).

[6] For details of the variables used in our empirical application and their sources see Section 9, and the Data Appendix.

[7] Note that the last transformation specified in (2.3) converts the annual rate of interest, R_{it}, to quarterly interest rate, ρ_{it}, using a logarithmic scale.

with $k_0 = 5$. Also in the case of some of the emerging market economies and the newly constituted economies of the Eastern Europe and Russia where the interest rate and/or the equity price index may not be available over the whole sample period, $\mathbf{x}_{it}$ may be confined to the $y_{it}, p_{it}, e_{it}, m_{it}$, with $k_i = 4$. The foreign variables (indices), denoted by $\mathbf{x}^*_{it}$, is a $k^*_i \times 1$ vector[8] are constructed as weighted averages, with country/region specific weights:

$$\left.\begin{array}{r}\mathbf{x}^*_{it} = (y^*_{it}, p^*_{it}, q^*_{it}, e^*_{it}, \rho^*_{it}, m^*_{it})', \\ y^*_{it} = \sum_{j=0}^N w^y_{ij} y_{jt}, \quad p^*_{it} = \sum_{j=0}^N w^p_{ij} p_{jt}, \\ q^*_{it} = \sum_{j=0}^N w^q_{ij} q_{jt}, \quad e^*_{it} = \sum_{j=1}^N w^e_{ij} e_{jt}, \\ \rho^*_{it} = \sum_{j=0}^N w^\rho_{ij} \rho_{jt}, \quad m^*_{it} = \sum_{j=0}^N w^m_{ij} m_{jt}. \end{array}\right\} \tag{2.4}$$

The weights w^y_{ij}, $w^p_{ij}, w^q_{ij}, w^e_{ij}, w^\rho_{ij}$, and w^m_{ij} for $i, j = 0, 1, ...N$,[9] could be based on trade shares (namely the share of country j in the total trade of country i measured in U.S. dollars) in the case of y^*_{it}, p^*_{it}, e^*_{it} and m^*_{it} and capital flows in the case of equity price indices and interest rates, q^*_{it} and ρ^*_{it}.[10] Notice that

$$w^y_{ii} = w^p_{ii} = w^q_{ii} = w^\rho_{ii} = w^m_{ii} = w^e_{ii} = 0, \text{ for all } i.$$

It is worth noting that the exchange rate variable, e^*_{it}, defined for country i is not the same as the more familiar concept of the 'effective exchange rate' as defined below. To see this denote the exchange rate of country i in terms of the currency of country j by E_{ijt}. Then

$$\ln(E_{ijt}) = \ln(E_{it}/E_{jt}) = e_{it} - e_{jt}. \tag{2.5}$$

Let the trade share of country i with respect to country j be w_{ij} and write the (log) effective exchange rate of country i as (recall that $e_{0t} = 0$):[11]

$$\tilde{e}_{it} = \sum_{j=0}^N w_{ij}(e_{it} - e_{jt}) = (\sum_{j=0}^N w_{ij})e_{it} - \sum_{j=1}^N w_{ij} e_{jt}.$$

But noting that $\sum_{j=0}^N w_{ij} = 1$, then $\tilde{e}_{it} = e_{it} - \sum_{j=1}^N w_{ij} e_{jt}$, and hence $e^*_{it} = e_{it} - \tilde{e}_{it}$. Only in the case of the base country where $e_{0t} = 0$, do the two concepts (apart from a sign convention) coincide, namely we have $e^*_{0t} = -\tilde{e}_{0t}$.

[8] In our application, $k^*_i = 5$ or 6. See Section 9.

[9] In practice, it may also be desirable to allow for these weights to vary over time in order to capture secular movements in the geographical patterns of trade and capital flows. However, too frequent changes in the weights could introduce an undesirable degree of randomness into the analysis. This is the classic index number problem to which a totally satisfactory answer does not exist. In our empirical analysis we use fixed trade weights but base their computation on averages of trade flows over a three year period.

[10] See Glick and Rose (1999) who discuss the importance of trade links in the analysis of contagion.

[11] w^T_{ij} can be measured as the total trade between country i and country j divided by the total trade of country i with *all* its trading partners.

It is also worth noting that in the case of countries or regions that attempt to maintain (approximately) a fixed effective exchange rate by pegging their currency to a basket of currencies, there will be a close correlation between e_{it} and e_{it}^*. Hence for purposes of econometric analysis it may not be advisable to include e_{it}^* as an exogenous variable in $\mathbf{x}_{it}^*$, considering that e_{it} is already included amongst the endogenous variables. The inclusion of e_{it} in the model ought to be sufficient to accommodate the possible effects of exchange rate variations on the domestic economy. For the base economy, however, under our set-up e_{0t}^* will be determined by the models for the rest of the world via equation (2.1), for $i = 1, 2, .., N$. Hence, for internal consistency e_{0t}^* must be treated as an exogenous variable in the model for the base economy. Otherwise, there will be two sets of equations explaining e_{0t}^*; one equation derived by combining the exchange rate equations from the models for the regions $i = 1, 2, .., N$, and a second equation obtained directly from the model of country $i = 0$, if e_{0t}^* is included in that model as endogenous.

In general, the GVAR model allows for interactions amongst the different economies through three separate but inter-related channels:

1. Contemporaneous dependence of $\mathbf{x}_{it}$ on $\mathbf{x}_{it}^*$ and on its lagged values.

2. Dependence of the country-specific variables on common global exogenous variables such as oil prices. (see Section 5).

3. Non-zero contemporaneous dependence of shocks in country i on the shocks in country j, measured via the cross country covariances, Σ_{ij}

$$\Sigma_{ij} = Cov(\boldsymbol{\varepsilon}_{it}, \boldsymbol{\varepsilon}_{jt}) = E(\boldsymbol{\varepsilon}_{it}\boldsymbol{\varepsilon}_{jt}'), \text{ for } i \neq j. \quad (2.6)$$

   where $\boldsymbol{\varepsilon}_{it}$ is defined by (2.1). A typical element of Σ_{ij} will be denoted by $\sigma_{ij,\ell s} = cov(\varepsilon_{i\ell t}, \varepsilon_{jst})$ which is the covariance of the ℓ^{th} variable in country i with the sth variable in country j.

The $N+1$ country-specific models, (2.1), together with the relations linking the (weakly) exogenous variables of the country-specific models to the variables in the rest of the global model, (2.4), provide a complete system. As emphasized in the introduction, due to data limitations even for moderate values of N, a full system estimation of the global model may not be feasible. To avoid this difficulty we propose to estimate the parameters of the country-specific models separately, treating the foreign-specific variables as weakly exogenously on the grounds that most economies (possibly with the exception of the U.S.) are small relative to the size of the world economy. This is the standard assumption in the small-open-economy macroeconomic literature, pioneered by Fleming (1962) and Mundell (1963) and developed further by Dornbusch (1976), where it is routinely assumed that "world" interest rate, output and prices are exogenously given. Whether such exogeneity assumptions hold in practice depends on the relative sizes of the countries/regions in the global model and the degree of cross-country dependence of the idiosyncratic shocks, $\boldsymbol{\varepsilon}_{it}$, as captured by the cross-covariances Σ_{ij}. Sufficient conditions under which foreign-specific variables can be viewed as weakly exogenous are discussed in Section 7. Empirical evidence on weak exogeneity of these variables is provided in Section 9.5.

3 Solution of the GVAR Model

Due to the contemporaneous dependence of the domestic variables, $\mathbf{x}_{it}$, on the foreign variables, $\mathbf{x}^*_{it}$, the country-specific VAR models (2.1) need to be solved simultaneously for all the domestic variables, $\mathbf{x}_{it}$, $i = 0, 1, ..., N$. The solution can then be used for a variety of purposes, such as forecasting, impulse response analysis, and risk management.

For construction of the GVAR model from the country-specific models we first define the $(k_i + k^*_i) \times 1$ vector

$$\mathbf{z}_{it} = \begin{pmatrix} \mathbf{x}_{it} \\ \mathbf{x}^*_{it} \end{pmatrix}, \tag{3.1}$$

and rewrite (2.1) as

$$\mathbf{A}_i \mathbf{z}_{it} = \mathbf{a}_{i0} + \mathbf{a}_{i1} t + \mathbf{B}_i \mathbf{z}_{i,t-1} + \boldsymbol{\varepsilon}_{it}, \tag{3.2}$$

where

$$\mathbf{A}_i = (\mathbf{I}_{k_i}, -\mathbf{\Lambda}_{i0}), \quad \mathbf{B}_i = (\mathbf{\Phi}_i, \mathbf{\Lambda}_{i1}). \tag{3.3}$$

The dimensions of $\mathbf{A}_i$ and $\mathbf{B}_i$ are $k_i \times (k_i + k^*_i)$ and $\mathbf{A}_i$ has a full row rank, namely $Rank(\mathbf{A}_i) = k_i$.

Collect all the country-specific variables together in the $k \times 1$ global vector $\mathbf{x}_t = (\mathbf{x}'_{0t}, \mathbf{x}'_{1t}, ..., \mathbf{x}'_{Nt})'$ where $k = \sum_{i=0}^{N} k_i$ is the total number of the endogenous variables in the global model. Recall that $\mathbf{x}_{0t} = (y_{0t}, p_{0t}, q_{0t}, \rho_{0t}, m_{0t})'$ and $\mathbf{x}_{it} = (y_{it}, p_{it}, q_{it}, e_{it}, \rho_{it}, m_{it})'$ for $i = 1, 2, ..., N$. Our analysis is invariant to the way the endogenous variables are stacked in $\mathbf{x}_{it}$, and the ordering of the countries in $\mathbf{x}_t$.

It is now easily seen that the country specific variables can all be written in terms of $\mathbf{x}_t$:

$$\mathbf{z}_{it} = \mathbf{W}_i \mathbf{x}_t, \quad i = 0, 1, 2, ..., N, \tag{3.4}$$

where $\mathbf{W}_i$ is a $(k_i+k_i^*)\times k$ matrix of fixed (known) constants defined in terms of the country specific weights w_{ij}^y, $w_{ij}^p, w_{ij}^q, w_{ij}^e, w_{ij}^\rho$, and w_{ij}^m. $\mathbf{W}_i$ can be viewed as the 'link' matrix that allows the country-specific models to be written in terms of the global variable vector, $\mathbf{x}_t$.

Using (3.4) in (3.2) we have:

$$\mathbf{A}_i\mathbf{W}_i\mathbf{x}_t = \mathbf{a}_{i0} + \mathbf{a}_{i1}t + \mathbf{B}_i\mathbf{W}_i\mathbf{x}_{t-1} + \boldsymbol{\varepsilon}_{it},$$

where $\mathbf{A}_i\mathbf{W}_i$ and $\mathbf{B}_i\mathbf{W}_i$ are both $k_i \times k$ dimensional matrices. Stacking these equations now yields:

$$\mathbf{G}\mathbf{x}_t = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{H}\mathbf{x}_{t-1} + \boldsymbol{\varepsilon}_t, \tag{3.5}$$

where

$$\mathbf{a}_0 = \begin{pmatrix} \mathbf{a}_{00} \\ \mathbf{a}_{10} \\ \vdots \\ \mathbf{a}_{N0} \end{pmatrix}, \ \mathbf{a}_1 = \begin{pmatrix} \mathbf{a}_{01} \\ \mathbf{a}_{11} \\ \vdots \\ \mathbf{a}_{N1} \end{pmatrix}, \ \boldsymbol{\varepsilon}_t = \begin{pmatrix} \boldsymbol{\varepsilon}_{0t} \\ \boldsymbol{\varepsilon}_{1t} \\ \vdots \\ \boldsymbol{\varepsilon}_{Nt} \end{pmatrix}, \tag{3.6}$$

$$\mathbf{G} = \begin{pmatrix} \mathbf{A}_0\mathbf{W}_0 \\ \mathbf{A}_1\mathbf{W}_1 \\ \vdots \\ \mathbf{A}_N\mathbf{W}_N \end{pmatrix}, \ \mathbf{H} = \begin{pmatrix} \mathbf{B}_0\mathbf{W}_0 \\ \mathbf{B}_1\mathbf{W}_1 \\ \vdots \\ \mathbf{B}_N\mathbf{W}_N \end{pmatrix}. \tag{3.7}$$

It is easily seen that $\mathbf{G}$ is a $k \times k$ dimensional matrix and in general will be of full rank, and hence non-singular. Then the GVAR model in all the variables can be written as

$$\mathbf{x}_t = \mathbf{G}^{-1}\mathbf{a}_0 + \mathbf{G}^{-1}\mathbf{a}_1 t + \mathbf{G}^{-1}\mathbf{H}\mathbf{x}_{t-1} + \mathbf{G}^{-1}\boldsymbol{\varepsilon}_t,$$

which may also be solved recursively forward to obtain the future values of $\mathbf{x}_t$. See Section 5 below for further details.

It is worth illustrating the above solution technique by means of a simple example. Consider a global model composed of three regions in three variables, say output, prices and exchange rates (all in logs). Then

$$\mathbf{x}_t = \begin{pmatrix} \mathbf{x}_{0t} \\ \mathbf{x}_{1t} \\ \mathbf{x}_{2t} \end{pmatrix} = \begin{pmatrix} y_{0t} \\ p_{0t} \\ y_{1t} \\ p_{1t} \\ e_{1t} \\ y_{2t} \\ p_{2t} \\ e_{2t} \end{pmatrix}, \ z_{0t} = \begin{pmatrix} y_{0t} \\ p_{0t} \\ y_{0t}^* \\ p_{0t}^* \\ e_{0t}^* \end{pmatrix}, \quad z_{it} = \begin{pmatrix} y_{it} \\ p_{it} \\ e_{it} \\ y_{it}^* \\ p_{it}^* \end{pmatrix}, \ i = 1, 2.$$

Using the trade shares, simply denoted by w_{ij}, to construct the foreign variables and noting that $e_{0t}^* = w_{01}e_{1t} + w_{02}e_{2t}$, then the link matrices for these three regions are

$$\mathbf{W}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w_{01} & 0 & 0 & w_{02} & 0 & 0 \\ 0 & 0 & 0 & w_{01} & 0 & 0 & w_{02} & 0 \\ 0 & 0 & 0 & 0 & w_{01} & 0 & 0 & w_{02} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & w_{01}\mathbf{I}_3 & w_{02}\mathbf{I}_3 \end{pmatrix},$$

$$\mathbf{W}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ w_{10} & 0 & 0 & 0 & 0 & w_{12} & 0 & 0 \\ 0 & w_{10} & 0 & 0 & 0 & 0 & w_{12} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ w_{10}\mathbf{I}_2 & \mathbf{0} & w_{12}\mathbf{I}_2 & \mathbf{0} \end{pmatrix},$$

$$\mathbf{W}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ w_{20} & 0 & w_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & w_{20} & 0 & w_{21} & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 \\ w_{10}\mathbf{I}_2 & w_{12}\mathbf{I}_2 & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Notice that the country-specific weights are non-negative and satisfy the adding up restrictions $w_{01} + w_{02} = 1$, $w_{10} + w_{12} = 1$, $w_{20} + w_{21} = 1$. Furthermore, in the case where trade shares are non-zero it is easily seen that the link matrices are of full row ranks, a property that will be of importance when we come to consider the error-correction properties of the global model in the following section. Finally,

$$\mathbf{A}_0 = (\mathbf{I}_2, -\mathbf{\Lambda}_{00}), \ \mathbf{A}_1 = (\mathbf{I}_3, -\mathbf{\Lambda}_{10}), \ \mathbf{A}_2 = (\mathbf{I}_3, -\mathbf{\Lambda}_{20}),$$

where $\mathbf{I}_s$ is an identity matrix of order s. Using the above $\mathbf{W}_i$ and $\mathbf{A}_i$ matrices the $\mathbf{G}$ matrix defined by (3.7) can now be readily constructed. In this example $\mathbf{G}$ is 8×8 and must be non-singular if the global model is to be complete.[12]

4 Error-Correcting and Trending Properties of the Global Model

It would be interesting to relate the error correcting and trending properties of the country-specific models to those of the associated global model. The error

[12] A model is said to be complete if it is possible to uniquely solve for all its endogenous variables.

correction representation of (2.1) is given by

$$\begin{aligned}\Delta\mathbf{x}_{it} &= \mathbf{a}_{i0}+\mathbf{a}_{i1}t-(\mathbf{I}_{k_i}-\mathbf{\Phi}_i)\mathbf{x}_{i,t-1}+(\mathbf{\Lambda}_{i0}+\mathbf{\Lambda}_{i1})\mathbf{x}^*_{i,t-1} \\ &\quad +\mathbf{\Lambda}_{i0}\Delta\mathbf{x}^*_{it}+\boldsymbol{\varepsilon}_{it},\ i=0,1,....,N. \end{aligned} \tag{4.1}$$

and using (3.1)

$$\Delta\mathbf{x}_{it}=\mathbf{a}_{i0}+\mathbf{a}_{i1}t-(\mathbf{A}_i-\mathbf{B}_i)\mathbf{z}_{i,t-1}+\mathbf{\Lambda}_{i0}\Delta\mathbf{x}^*_{it}+\boldsymbol{\varepsilon}_{it}, \tag{4.2}$$

where as before $\mathbf{z}_{it}=(\mathbf{x}'_{it},\mathbf{x}^{*\prime}_{it})'$, and $\mathbf{A}_i$ and $\mathbf{B}_i$ are already defined by (3.3). The error-correction properties of the model for country/region i are summarized in the $k_i\times(k_i+k_i^*)$ matrix

$$\mathbf{\Pi}_i=\mathbf{A}_i-\mathbf{B}_i. \tag{4.3}$$

In particular, the rank of $\mathbf{\Pi}_i$, say $r_i\leq k_i$, specifies the number of "long-run" relationships that exists amongst the domestic and the country-specific foreign variables, namely $\mathbf{x}_{it}$ and $\mathbf{x}^*_{it}$. Therefore, we have

$$\mathbf{A}_i-\mathbf{B}_i=\boldsymbol{\alpha}_i\ \boldsymbol{\beta}'_i, \tag{4.4}$$

where $\boldsymbol{\alpha}_i$ is the $k_i\times r_i$ loading matrix of full column rank, and $\boldsymbol{\beta}_i$ is the $(k_i+k_i^*)\times r_i$ matrix of cointegrating vectors, also of full column rank.

In the case where $\mathbf{\Pi}_i$ is rank deficient and the linear trend coefficients, $\mathbf{a}_{i1}$, are unrestricted the linear trend in the error correction model transforms into a quadratic trends in $\mathbf{x}_{it}$, which is clearly undesirable. It would be more appropriate to retain the same deterministic trend properties for the elements of $\mathbf{x}_{it}$ under different rank restrictions on $\mathbf{\Pi}_i$. As shown, for example, by Pesaran, Shin and Smith (2000), this can be achieved by restricting the trend coefficients so that

$$\mathbf{a}_{i1}=(\mathbf{A}_i-\mathbf{B}_i)\boldsymbol{\kappa}_i, \tag{4.5}$$

where $\boldsymbol{\kappa}_i$ is a $(k_i+k_i^*)\times 1$ vector of fixed constants. This specification imposes k_i-r_i restrictions on the trend coefficients.

Consider now the global model, given by (3.5), which has the following error-correction form

$$\mathbf{G\Delta x}_t=\mathbf{a}_0+\mathbf{a}_1t-(\mathbf{G}-\mathbf{H})\mathbf{x}_{t-1}+\boldsymbol{\varepsilon}_t. \tag{4.6}$$

The number of long-run relationships in the global model is similarly determined by the rank of $\mathbf{G}-\mathbf{H}$. Using (3.7) and (4.4) we first note that

$$\mathbf{G}-\mathbf{H}=\begin{pmatrix}(\mathbf{A}_0-\mathbf{B}_0)\mathbf{W}_0\\(\mathbf{A}_1-\mathbf{B}_1)\mathbf{W}_1\\ \vdots \\ (\mathbf{A}_N-\mathbf{B}_N)\mathbf{W}_N\end{pmatrix}=\begin{pmatrix}\boldsymbol{\alpha}_0\boldsymbol{\beta}'_0\mathbf{W}_0\\ \boldsymbol{\alpha}_1\boldsymbol{\beta}'_1\mathbf{W}_1\\ \vdots \\ \boldsymbol{\alpha}_N\boldsymbol{\beta}'_N\mathbf{W}_N\end{pmatrix},$$

which can be written equivalently as

$$\mathbf{G} - \mathbf{H} = \tilde{\boldsymbol{\alpha}}\, \tilde{\boldsymbol{\beta}}',$$

where $\tilde{\boldsymbol{\alpha}}$ is the $k \times r$ block diagonal matrix of the global loading coefficients

$$\tilde{\boldsymbol{\alpha}} = \begin{pmatrix} \boldsymbol{\alpha}_0 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\alpha}_1 & & \mathbf{0} \\ \vdots & & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{\alpha}_N \end{pmatrix}, \tag{4.7}$$

$$\tilde{\boldsymbol{\beta}} = \begin{pmatrix} \mathbf{W}_0'\boldsymbol{\beta}_0, & \mathbf{W}_1'\boldsymbol{\beta}_1, & \dots & ,\mathbf{W}_N'\boldsymbol{\beta}_N \end{pmatrix}, \tag{4.8}$$

$r = \sum_{i=0}^{N} r_i$, and $k = \sum_{i=0}^{N} k_i$. It is clear that $Rank(\tilde{\boldsymbol{\alpha}}) = \sum_{i=0}^{N} Rank(\boldsymbol{\alpha}_i) = r$.

Consider now the global $k \times r$ cointegrating matrix $\tilde{\boldsymbol{\beta}}$. Each of the blocks in $\tilde{\boldsymbol{\beta}}$, namely $\mathbf{W}_i'\boldsymbol{\beta}_i$, are of dimension $k \times r_i$ with rank at most equal to r_i. Therefore, the rank of $\tilde{\boldsymbol{\beta}}$ will be at most equal to r. Namely, *the number of the long-run relationships in the global model cannot exceed the sum of the numbers of long-run relations that exist in the country/region specific models.*[13]

The deterministic trend properties of the GVAR model is also related to those of the underlying country-specific models. As with the country-specific models, to ensure that rank restrictions on $\mathbf{G} - \mathbf{H}$ do not lead to quadratic trends in the variables of the global model the vector of trend coefficients, $\mathbf{a}_1$, must satisfy the restrictions

$$\mathbf{a}_1 = (\mathbf{G} - \mathbf{H})\, \boldsymbol{\gamma},$$

where $\boldsymbol{\gamma}$ is a $k \times 1$ vector of fixed constants. Therefore, for the deterministic trend properties of the variables to be the same in the global model as in the underlying country-specific models, using (4.5), we must have

$$(\mathbf{G} - \mathbf{H})\, \boldsymbol{\gamma} = \begin{pmatrix} (\mathbf{A}_0 - \mathbf{B}_0)\boldsymbol{\kappa}_0 \\ (\mathbf{A}_1 - \mathbf{B}_1)\boldsymbol{\kappa}_1 \\ \vdots \\ (\mathbf{A}_N - \mathbf{B}_N)\boldsymbol{\kappa}_N \end{pmatrix}.$$

This condition is satisfied if

$$\boldsymbol{\kappa}_i = \mathbf{W}_i \boldsymbol{\gamma}, \text{ for } i = 0, 1, ..., N.$$

[13] It is worth noting that this result is conditional on the choice of the link matrices, $\mathbf{W}_i$, and in principle it would be possible to obtain a different number of cointegrating relations in the global model for different choices of the link matrices.

These impose additional cross-country restrictions on the trend coefficients. While in principal it should be possible to test these restrictions, their simultaneous imposition will be infeasible when N is large compared to the available time series data, T.

5 Dynamic Properties, Stability Conditions and Forecasts of the GVAR Model

In this section we shall consider the dynamic properties of a slightly generalized version of the global model that allows for "common global variables" such as oil prices. Such an augmented VARX* model is given by[14]

$$\mathbf{x}_{it} = \mathbf{a}_{i0} + \mathbf{a}_{i1}t + \mathbf{\Phi}_i \mathbf{x}_{i,t-1} + \mathbf{\Lambda}_{i0}\mathbf{x}^*_{it} + \mathbf{\Lambda}_{i1}\mathbf{x}^*_{i,t-1} + \mathbf{\Psi}_{i0}\mathbf{d}_t + \mathbf{\Psi}_{i1}\mathbf{d}_{t-1} + \varepsilon_{it}, \quad (5.1)$$

for $t = 1, 2, ..., T$, and $i = 0, 1, 2, ..., N$, where $\mathbf{d}_t$ is an $s \times 1$ vector of common global variables assumed to be weakly exogenous to the global economy. The global model associated with these country specific models is now given by

$$\mathbf{G}\mathbf{x}_t = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{H}\mathbf{x}_{t-1} + \mathbf{\Psi}_0\mathbf{d}_t + \mathbf{\Psi}_1\mathbf{d}_{t-1} + \varepsilon_t,$$

where $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{G}$, $\mathbf{H}$ and ε_t are as already defined by (3.6) and (3.7), and

$$\mathbf{\Psi}_0 = \begin{pmatrix} \mathbf{\Psi}_{00} \\ \mathbf{\Psi}_{10} \\ \vdots \\ \mathbf{\Psi}_{N0} \end{pmatrix}, \quad \mathbf{\Psi}_1 = \begin{pmatrix} \mathbf{\Psi}_{01} \\ \mathbf{\Psi}_{11} \\ \vdots \\ \mathbf{\Psi}_{N1} \end{pmatrix}. \quad (5.2)$$

Assuming $\mathbf{G}$ is non-singular we now have the following reduced-form global model

$$\mathbf{x}_t = \mathbf{b}_0 + \mathbf{b}_1 t + \digamma \mathbf{x}_{t-1} + \mathbf{\Upsilon}_0\mathbf{d}_t + \mathbf{\Upsilon}_1\mathbf{d}_{t-1} + \mathbf{u}_t, \quad (5.3)$$

for $t = 1, 2, ..., T, T+1, ..., T+n$, where

$$\begin{aligned} \mathbf{b}_i &= \mathbf{G}^{-1}\mathbf{a}_i,\ i = 0, 1,\ \digamma = \mathbf{G}^{-1}\mathbf{H}, \quad (5.4) \\ \mathbf{\Upsilon}_0 &= \mathbf{G}^{-1}\mathbf{\Psi}_0,\ \mathbf{\Upsilon}_1 = \mathbf{G}^{-1}\mathbf{\Psi}_1, \text{ and } \mathbf{u}_t = \mathbf{G}^{-1}\varepsilon_t. \end{aligned}$$

Suppose now that the global economy is observed over the period $t = 1, 2, ..., T$, and we wish to forecast $\mathbf{x}_t$ over the future periods $t = T+1, T+$

[14] The distinction between foreign variables, $\mathbf{x}^*_{it}$, and the global exogenous variables, $\mathbf{d}_t$, is relevant for the analysis of the dynamic properties of the global model, but is not of material consequence for estimation of the country-specific models. For the latter purpose $\mathbf{x}^*_{it}$ and $\mathbf{d}_t$ can be combined and treated jointly as weakly exogenous.

$2, ..., T+n$, where n is the forecast horizon. To simplify the exposition we assume that the exogenous variables $\mathbf{d}_t$ for $t = T+1, T+2, ...$ are given.[15] Solving the difference equation (5.3) forward we obtain:

$$\begin{aligned} \mathbf{x}_{T+n} &= \digamma^n \mathbf{x}_T + \sum_{\tau=0}^{n-1} \digamma^\tau \left[\mathbf{b}_0 + \mathbf{b}_1 (T+n-\tau) \right] + \\ & \sum_{\tau=0}^{n-1} \digamma^\tau \left[\boldsymbol{\Upsilon}_0 \mathbf{d}_{T+n-\tau} + \boldsymbol{\Upsilon}_1 \mathbf{d}_{T+n-\tau-1} \right] + \sum_{\tau=0}^{n-1} \digamma^\tau \mathbf{u}_{T+n-\tau}. \end{aligned} \tag{5.5}$$

This solution has four distinct components. The first component, $\digamma^n \mathbf{x}_T$, measures the effect of initial values, $\mathbf{x}_T$, on the future state of the system. The second component captures the deterministic trends embodied in the underlying VAR model. The third component measures the effect of the global exogenous variables, $\mathbf{d}_t$, on the model's endogenous variables, $\mathbf{x}_t$. Finally, the last term in (5.5) represents the stochastic (unpredictable) component of $\mathbf{x}_{T+n}$. The point forecasts of the endogenous variables conditional on the initial state of the system and the exogenous global variables are now given by

$$\begin{aligned} \mathbf{x}^*_{T+n} &= E(\mathbf{x}_{T+n} \mid \mathbf{x}_T, \cup_{\tau=1}^{n} \mathbf{d}_{T+\tau}) = \digamma^n \mathbf{x}_T + \sum_{\tau=0}^{n-1} \digamma^\tau \left[\mathbf{b}_0 + \mathbf{b}_1 (T+n-\tau) \right] + \\ & \sum_{\tau=0}^{n-1} \digamma^\tau \left[\boldsymbol{\Upsilon}_0 \mathbf{d}_{T+n-\tau} + \boldsymbol{\Upsilon}_1 \mathbf{d}_{T+n-\tau-1} \right]. \end{aligned} \tag{5.6}$$

The probability distribution function of $\mathbf{x}_{T+n}$, needed for the computation of the loss distribution of a given portfolio, can also be obtained under suitable assumptions concerning the probability distribution function of the shocks, $\boldsymbol{\varepsilon}_t$. Under the assumption that $\boldsymbol{\varepsilon}_t$ is normally distributed we have

$$\mathbf{x}_{T+n} \mid \mathbf{x}_T, \cup_{\tau=1}^{n} \mathbf{d}_{T+\tau} \backsim N(\mathbf{x}^*_{T+n}, \boldsymbol{\Omega}_n), \tag{5.7}$$

where $\mathbf{x}^*_{T+n}$ is given by (5.6), and

$$\boldsymbol{\Omega}_n = \sum_{\tau=0}^{n-1} \digamma^\tau \mathbf{G}^{-1} \boldsymbol{\Sigma} \mathbf{G}'^{-1} \digamma'^\tau, \tag{5.8}$$

where $\boldsymbol{\Sigma}$ is the $k \times k$ variance-covariance matrix of the shocks, $\boldsymbol{\varepsilon}_t$. Note that the (i,j) block of $\boldsymbol{\Sigma}$ is given by $\boldsymbol{\Sigma}_{ij}$ which is defined by (2.6). The estimation of $\boldsymbol{\Sigma}_{ij}$ and the other parameters will be addressed below.

[15] The analysis can be easily generalized to allow for the uncertainty of the exogenous global variables. This will be done in Section 10 when we discuss the effect of shocks on the loss distribution which is a non-linear function of the shocks. But for impulse response analysis, due to the linearity of the underlying relationships, the impulse response functions do not depend on the processes generating the global variables when they are strictly exogenous. The case where global variables are weakly exogenous is more complicated notationally although straightforward in principal.

The dynamic properties of the global model crucially depend on the eigenvalues of $\mathcal{F}$. In the trend-stationary case where all the roots of $\mathcal{F}$ lie inside the unit circle, $\mathbf{x}_{T+n}$ will have a stable distribution and will satisfy the following properties:

- The dependence of $\mathbf{x}_{T+n}$ on the initial values, $\mathbf{x}_T$, will disappear for sufficiently large values of n, the forecast horizon.
- The forecast covariance matrix, $\mathbf{\Omega}_n$, will converge to a finite value as $n \to \infty$.
- The point forecasts, $\mathbf{x}^*_{T+n}$, will exhibit the same linear trending property as the one specified in the underlying country-specific VAR models.

In contrast, when one or more roots of $\mathcal{F}$ fall on the unit circle none of the above properties hold.[16] The unit eigenvalues correspond to the unit roots and cointegrating properties of the various variables in the global VAR model.

- The multiplier matrix $\mathcal{F}^n$ converges to a non-zero matrix of fixed constants even if n is allowed to increase without bound, and the dependence of $\mathbf{x}^*_{T+n}$ on the initial values does not disappear as $n \to \infty$.
- The forecast covariance matrix, $\mathbf{\Omega}_n$, will rise linearly with n; indicating a steady deterioration in the precision with which values of $\mathbf{x}_{T+n}$ are forecast with the horizon, n.
- Finally, as noted in Section 4, the linear trend in the underlying VAR model when combined with a unit root in $\mathcal{F}$ generates a quadratic trend in the level of the variables.

Some of the above undesirable features can be avoided or by passed. For example, to avoid increasing forecast error variances one could focus on forecasting growth rates (using the GVAR in levels). As noted above quadratic trends can be eliminated by restricting the trend coefficients $\mathbf{b}_1$. Although, imposing these restrictions exactly would not be feasible when N is large relative to T, a partial solution can be achieved by imposing the restrictions, (4.5), on the trend coefficients of the country-specific models. This estimation problem is feasible and will be discussed in Section 7.

[16] The case where $\mathcal{F}$ has a root outside the unit circle leads to explosive forecasts and is of little interest and could indicate model mis-specification.

6 Impulse Response Analysis

One of the important tools in the analysis of dynamic systems is the impulse response function, which characterizes the possible response of the system at different future periods to the effect of shocking one of the variables in the model. For example, it may be of interest to work out the effect of a shock of a given size to the Yen/Dollar exchange rate on the evolution of real output in Germany. In carrying out such an analysis it is important that the correlation which exists across the different shocks, both within each country and across the different countries, are accounted for in an appropriate manner. In the traditional VAR literature this is accomplished by means of the orthogonalized impulse responses (OIR) à la Sims (1980), where impulse responses are computed with respect to a set of orthogonalized shocks, say $\boldsymbol{\xi}_t$, instead of the original shocks, $\boldsymbol{\varepsilon}_t$. The link between the two sets of shocks is given by $\boldsymbol{\xi}_t = \mathbf{P}^{-1}\boldsymbol{\varepsilon}_t$, where $\mathbf{P}$ is a $k \times k$ lower triangular Cholesky factor of the variance covariance matrix, $Cov(\boldsymbol{\varepsilon}_t) = \boldsymbol{\Sigma}$, namely

$$\mathbf{PP}' = \boldsymbol{\Sigma}. \tag{6.1}$$

Therefore, by construction $E(\boldsymbol{\xi}_t\boldsymbol{\xi}_t') = \mathbf{I}_k$. The $k \times 1$ vector of the orthogonalized impulse response function of a unit shock (equal to one standard error) to the j^{th} equation on $\mathbf{x}_{t+n}$ is given by

$$\boldsymbol{\psi}_j^o(n) = F^n\mathbf{G}^{-1}\mathbf{P}\mathfrak{s}_j, \; n = 0, 1, 2, ..., \tag{6.2}$$

where $\mathfrak{s}_j$ is a $k \times 1$ selection vector with unity as its j^{th} element (corresponding to a particular shock in a particular country) and zeros elsewhere. In the case of the global VAR model the orthogonalized impulse responses also depend on the order of factors in each region/country and the order in which the countries are stacked in $\mathbf{x}_t$. Mathematically, this non-invariance property of the orthogonalized impulse responses is simply due to the non-uniqueness of the Cholesky factor, $\mathbf{P}$.

The orthogonalized impulse response function is usually used for small systems that admit a natural causal ordering for the variables in the VAR. But in general such a natural ordering does not exist and the OIR functions are not unique and sometime depend *critically* on the order in which the variables are included in the VAR. The more recent literature emphasizes the use of "structural VAR" methodology to identify the shocks. This is achieved by imposing *a priori* restrictions on the covariance matrix of the shocks and/or on long-run impulse responses themselves. See, for example, Bernanke (1986), Blanchard and Watson (1986) and Sims (1986) who considered *a priori* restrictions on contemporaneous covariance matrix of shocks, and Blanchard and Quah (1989), and Clarida and Gali (1994) who consider restrictions on the long-run impact of shocks to identify the impulse responses. Although such a strategy may be operational when the VAR contains only a few variables, its application to the GVAR model does not seem to be feasible. In the GVAR model with $N+1$ countries and k_i endogenous variables per country, exact identification of the shocks will require $\sum_{i=0}^{N} k_i(k_i-1)$ restrictions. For example, in the case of the model to be considered empirically in Section 9 we would need to motivate 300 different theory-based restrictions. It is not clear to us how this could be achieved.

An alternative approach which is invariant to the ordering of the variables and the countries in the global VAR would be to use (5.5) directly, shock only one element, say the j^{th} shock in $\boldsymbol{\varepsilon}_t$, corresponding to the ℓ^{th} variable in the i^{th} country, and integrate out the effects of other shocks using an assumed or the historically observed distribution of the errors. This approach is advanced in Koop, Pesaran and Potter (1996), and Pesaran and Shin (1998) and yields the generalized impulse response (GIR) function.

$$\mathbf{GI}_{x:\varepsilon_{i\ell}}(n,\ \sqrt{\sigma_{ii,\ell\ell}},\ \mathcal{I}_{t-1}) = E\left(\mathbf{x}_{t+n}\right|\left.\varepsilon_{i\ell t}=\sqrt{\sigma_{ii,\ell\ell}},\ \mathcal{I}_{t-1}\right) - E\left(\mathbf{x}_{t+n}\right|\left.\mathcal{I}_{t-1}\right), \tag{6.3}$$

where $\mathcal{I}_t = (\mathbf{x}_t, \mathbf{x}_{t-1}, \dots)$ is the information set at time $t-1$, and $\mathbf{d}_t$ is assumed to be given exogenously. On the assumption that $\boldsymbol{\varepsilon}_t$ has a multivariate normal distribution and using (5.5) it is now easily seen

$$\boldsymbol{\psi}_j^g(n) = \frac{1}{\sqrt{\sigma_{ii,\ell\ell}}} F^n \mathbf{G}^{-1} \boldsymbol{\Sigma} \mathfrak{s}_j,\ n = 0, 1, 2, ..., \tag{6.4}$$

which measures the effect of one standard error shock to the j^{th} equation (corresponding to the ℓ^{th} variable in the i^{th} country) at time t on expected values of $\mathbf{x}$ at time $t+n$. $\boldsymbol{\psi}_j^g(n)$ will be identical to $\boldsymbol{\psi}_j^o(n)$ when $\boldsymbol{\Sigma}$ is diagonal or when the focus of the analysis is on the impulse response function of shocking the first element of $\boldsymbol{\varepsilon}_t$.

6.1 Impulse Response Analysis of Shocks to the Exogenous Variables

In this sub-section we derive generalized impulse response functions for a unit shock to the i^{th} exogenous variable, d_{it}. For this purpose we need to specify a dynamic process for the exogenous variables. Suppose $\mathbf{d}_t$ follows a first order autoregressive process:[17]

$$\mathbf{d}_t = \boldsymbol{\mu}_d + \boldsymbol{\Phi}_d \mathbf{d}_{t-1} + \boldsymbol{\varepsilon}_{dt}, \qquad \boldsymbol{\varepsilon}_{dt} \backsim i.i.d.\ (\mathbf{0}, \boldsymbol{\Sigma}_d), \tag{6.5}$$

[17] The analysis is easily extended to higher order processes.

where $\boldsymbol{\mu}_d$ is an $s \times 1$ vector of constants, $\mathbf{\Phi}_d$ is $s \times s$ matrix of lagged coefficients, $\boldsymbol{\varepsilon}_{dt}$ is an $s \times 1$ vector of shocks to the exogenous variables, and $\mathbf{\Sigma}_d$ is the covariance matrix of these shocks which we allow to be singular. This allows for the possibility that some of the elements of $\mathbf{d}_t$ could be perfectly predictable (such as linear trends, deterministic seasonal effects, etc.). As before the generalized impulse response function of the effect of a unit shock to the i^{th} exogenous variable on the vector of the endogenous variables n periods ahead is defined by:

$$\mathbf{GI}_{x:d_i}(n,\ \sigma_{d,ii},\ \mathcal{I}_{t-1}) = E\left(\mathbf{x}_{t+n} |\ d_{it} = \sqrt{\sigma_{d,ii}},\ \mathcal{I}_{t-1}\right) - E\left(\mathbf{x}_{t+n} |\ \mathcal{I}_{t-1}\right) \quad (6.6)$$

where $\sigma_{d,ii}$ is the $i-th$ diagonal element of $\mathbf{\Sigma}_d$. Using (5.3) it is now easily seen that

$$\begin{aligned} \mathbf{GI}_{x:d_i}(n,\ \sigma_{d,ii},\ \mathcal{I}_{t-1}) &= F\ \mathbf{GI}_{x:d_i}(n-1,\ \sigma_{d,ii},\ \mathcal{I}_{t-1}) + \mathbf{\Upsilon}_0 \mathbf{GI}_{d:d_i}(n,\ \sigma_{d,ii},\ \mathcal{I}_{t-1}) + \\ & \quad \mathbf{\Upsilon}_1 \mathbf{GI}_{d:d_i}(n-1,\ \sigma_{d,ii},\ \mathcal{I}_{t-1}), \end{aligned} \quad (6.7)$$

for $n = 0, 1, 2, ...$, where

$$\mathbf{GI}_{d:d_i}(n,\ \sigma_{d,ii},\ \mathcal{I}_{t-1}) = E\left(\mathbf{d}_{t+n} |\ d_{it} = \sqrt{\sigma_{d,ii}},\ \mathcal{I}_{t-1}\right) - E\left(\mathbf{d}_{t+n} |\ \mathcal{I}_{t-1}\right). \quad (6.8)$$

It is now easily seen that for $n < 1$, $\mathbf{GI}_{x:d_i}(n-1, \sigma_{d,ii}, \mathcal{I}_{t-1}) = \mathbf{GI}_{d:d_i}(n-1, \sigma_{d,ii}, \mathcal{I}_{t-1}) = 0$ and

$$\mathbf{GI}_{x:d_i}(0,\ \sigma_{d,ii},\ \mathcal{I}_{t-1}) = \mathbf{\Upsilon}_0 \mathbf{GI}_{d:d_i}(0,\ \sigma_{d,ii},\ \mathcal{I}_{t-1}).$$

Similarly,

$$\mathbf{GI}_{d:d_i}(0,\ \sigma_{d,ii},\ \mathcal{I}_{t-1}) = \frac{1}{\sqrt{\sigma_{d,ii}}} \mathbf{\Sigma}_d \mathfrak{e}_i,$$

where $\mathfrak{e}_i$ is a $s \times 1$ selection vector with its i^{th} element unity and other elements zero, and

$$\mathbf{GI}_{d:d_i}(n,\ \sigma_{d,ii},\ \mathcal{I}_{t-1}) = \mathbf{\Phi}_d \mathbf{GI}_{d:d_i}(n-1,\ \sigma_{d,ii},\ \mathcal{I}_{t-1}), \text{ for } n = 1, 2, ...$$

Hence

$$\mathbf{GI}_{d:d_i}(n,\ \sigma_{d,ii},\ \mathcal{I}_{t-1}) = \frac{1}{\sqrt{\sigma_{d,ii}}} \mathbf{\Phi}_d^n \mathbf{\Sigma}_d \mathfrak{e}_i, \text{ for } n = 0, 1, ...$$

Substituting this result in (6.7) we have

$$\mathbf{GI}_{x:d_i}(n, \sigma_{d,ii}, \mathcal{I}_{t-1}) = F\ \mathbf{GI}_{x:d_i}(n-1, \sigma_{d,ii}, \mathcal{I}_{t-1}) + \frac{1}{\sqrt{\sigma_{d,ii}}} (\mathbf{\Upsilon}_0 \mathbf{\Phi}_d + \mathbf{\Upsilon}_1) \mathbf{\Phi}_d^{n-1} \mathbf{\Sigma}_d \mathfrak{e}_i, \quad (6.9)$$

for $n = 1, 2, ...$, where

$$\mathbf{GI}_{x:d_i}(0,\ \sigma_{d,ii},\ \mathcal{I}_{t-1}) = \frac{1}{\sqrt{\sigma_{d,ii}}} \mathbf{\Upsilon}_0 \mathbf{\Sigma}_d \mathfrak{e}_i. \quad (6.10)$$

In the simple case where $\mathbf{d}$ is a scalar variable (such as the oil price) $\frac{1}{\sqrt{\sigma_{d,ii}}} \mathbf{\Sigma}_d \mathfrak{e}_i = \sqrt{\sigma_{d,ii}}$.

7 Estimation of the GVAR Model as Individual Partial Systems

As was pointed out earlier a system estimation of the VAR model in (5.3) will not be feasible even for moderate values of N. The unconstrained estimation of (5.3) would involve estimating a large number of parameters often greater than the number of available observations! But the modeling approach set out above is feasible even for a relatively large number of countries/regions. This is due to the fact that the weights w_{ij}, $i, j = 0, 1, ..., N$ are not estimated simultaneously with the other country-specific parameters but are computed from cross-country data on trade and/or capital flow accounts. Also the estimation of the country-specific parameters is carried out on a country-by-country basis, rather than simultaneously. This is justified if N is sufficiently large and the following conditions hold:

1. (Stability) The global model, (5.3), formed from the country specific models is dynamically stable, namely the eigenvalues of matrix F defined by (5.4) are either on or inside the unit circle.

2. (Smallness) The weights used in the construction of foreign-specific variables, $w_{ij} \geq 0$, are small, being of order $1/N$ such that

$$\sum_{j=0}^{N} w_{ij}^2 \to 0, \text{ as } N \to \infty, \text{ for all } i,$$

3. (Weak dependence) The cross-dependence of the idiosyncratic shocks, if any, is sufficiently small so that

$$\frac{\sum_{j=0}^{N} \sigma_{ij,ls}}{N} \to 0, \text{ as } N \to \infty, \text{ for all } i, l, \text{ and } s,$$

where $\sigma_{ij,\ell s} = cov(\varepsilon_{i\ell t}, \varepsilon_{jst})$ is the covariance of the ℓ^{th} variable in country i with the s^{th} variable in country j.

These conditions are sufficient for $Cov(\mathbf{x}_{it}^{*}, \varepsilon_{it}) \to 0$ as $N \to \infty$.[18] They provide a formalization of the concept of "small-open-economy" from the perspective of econometric analysis. The need for conditions 1 and 2 is rather obvious. Clearly, condition 3 is satisfied when the country-specific shocks are purely idiosyncratic. But it is also satisfied for certain degree of dependence

[18] These conditions are derived for the relatively simple case where $k_i = 1$, and are available from the authors on request. We conjecture that the same type of results hold in the more general case.

across the idiosyncratic shocks. For example, the condition is met if there exists an ordering (j), seen from the viewpoint of country i, for which $\sigma_{i(j),ls}$ decay exponentially with $|i-(j)|$. It is not necessary for this ordering to be known and it need not be the same for other countries/regions. In this sense condition 3 allows for the idiosyncratic shocks to be "weakly correlated".

In practice it would not be possible to check the validity of these conditions directly, however. But, as shown below in sub-section 7.1, the implications of the weak exogeneity condition can be tested indirectly. Under weak exogeneity the parameters of the country-specific models can be estimated consistently by the ordinary least squares (OLS) or by the reduced rank procedure directly applied to (5.1). The OLS estimation is clearly much simpler but suffers from the shortcoming that it does not fully allow for the fact that one or more of the six factors used in the model may have unit roots; nor does it take into account the important possibility that the level of domestic and foreign variables may be tied together in the long-run (the phenomenon known as cointegration in the econometric literature). To deal with the unit root problem many researchers in the past have estimated the VAR model in first-differences (using rates of changes of the factors rather than their logarithms). But the first-differencing operation can be inefficient when there are in fact cointegrating relations amongst the factors and can be avoided by the reduced rank regression approach.

The reduced rank estimation procedure in the case where all the variables in the model are treated as endogenous I(1) has been developed by Johansen (1988, 1995).[19] But in the context of the GVAR model (2.1) for estimation purposes the foreign variables, $\mathbf{x}_{it}^{*}$, are treated as exogenous, and Johansen's approach needs to be modified to take this into account. Appropriate methods for estimating reduced rank regressions containing weakly exogenous regressors have been developed by Harbo, Johansen, Nielsen and Rahbek (1998), and Pesaran, Shin and Smith (2000). Here we provide some basic background to motivate the identification of the error correction terms and the weak exogeneity test which is discussed below.

To estimate the country-specific models subject to reduced rank restrictions first the error-correction equation (5.1) is re-written as

$$\Delta\mathbf{x}_{it} = \mathbf{a}_{i0} + \mathbf{a}_{i1}t - \mathbf{\Pi}_i\mathbf{v}_{i,t-1} + \mathbf{\Lambda}_{i0}\Delta\mathbf{x}_{it}^{*} + \mathbf{\Psi}_{i0}\Delta\mathbf{d}_t + \boldsymbol{\varepsilon}_{it}, \tag{7.1}$$

where

$$\mathbf{\Pi}_i = \left(\mathbf{A}_i - \mathbf{B}_i, -\mathbf{\Psi}_{i0} - \mathbf{\Psi}_{i1}\right), \tag{7.2}$$

[19] A variable is said to be $I(1)$, integrated of order 1, if it *must* be differenced *exactly* once before it becomes stationary, or $I(0)$.

and

$$\mathbf{v}_{i,t-1} = \begin{pmatrix} \mathbf{z}_{i,t-1} \\ \mathbf{d}_{t-1} \end{pmatrix}. \tag{7.3}$$

To avoid the problem of introducing quadratic trends in the level of the variables when $\mathbf{\Pi}_i$ is rank deficient as before we impose the restrictions, $\mathbf{a}_{i1} = \mathbf{\Pi}_i \boldsymbol{\kappa}_i$,which reduce to (4.5) when there are no global exogenous variables in the model.[20] Under these restrictions (7.1) becomes

$$\Delta \mathbf{x}_{it} = \mathbf{c}_{i0} - \mathbf{\Pi}_i \left[\mathbf{v}_{i,t-1} - \boldsymbol{\kappa}_i (t-1) \right] + \mathbf{\Lambda}_{i0} \Delta \mathbf{x}_{it}^* + \mathbf{\Psi}_{i0} \Delta \mathbf{d}_t + \boldsymbol{\varepsilon}_{it}, \tag{7.4}$$

where

$$\mathbf{c}_{i0} = \mathbf{a}_{i0} + \mathbf{\Pi}_i \boldsymbol{\kappa}_i, \tag{7.5}$$

$\mathbf{\Pi}_i$ is a $k_i \times (k_i + k^* + s)$ matrix and provides information on the long-run level relationships that may exist amongst the variables of the model. In the case where *all* the variables, $\mathbf{z}_{it}$ and $\mathbf{d}_t$, are $I(1)$ and are not cointegrated, then $\mathbf{\Pi}_i$ will be equal to zero and (7.4) reduces to the first differenced model

$$\Delta \mathbf{x}_{it} = \mathbf{a}_{i0} + \mathbf{\Lambda}_{i0} \Delta \mathbf{x}_{it}^* + \mathbf{\Psi}_{i0} \Delta \mathbf{d}_t + \boldsymbol{\varepsilon}_{it}. \tag{7.6}$$

It is interesting to note that this specification leads to random walk models (augmented by oil price changes) for the global variables, $\mathbf{z}_t$. Using the solution technique of Section 3, we have

$$\mathbf{G} \mathbf{\Delta} \mathbf{z}_t = \mathbf{a}_0 + \mathbf{\Psi}_0 \mathbf{\Delta} \mathbf{d}_t + \boldsymbol{\varepsilon}_t,$$

or

$$\mathbf{\Delta} \mathbf{z}_t = \mathbf{G}^{-1} \mathbf{a}_0 + \mathbf{G}^{-\mathbf{1}} \mathbf{\Psi}_0 \mathbf{\Delta} \mathbf{d}_t + \mathbf{G}^{-\mathbf{1}} \boldsymbol{\varepsilon}_t,$$

where $\mathbf{G}$ and $\mathbf{\Psi}_0$ are defined by (3.7) and (5.2), respectively. Therefore, as anticipated by the analysis of Section 4, there will be no long-run relationship in the global model if there are no long-run relations in the underlying regional models.

But in general, due to long-term inter-linkages that exist between domestic and foreign variables as well as between the domestic variables themselves, one would expect $\mathbf{\Pi}_i$ to be non-zero but rank deficient. The rank of $\mathbf{\Pi}_i$ identifies the number of long-run or cointegrating relationships. Rank deficiency arises when $Rank(\mathbf{\Pi}_i) = r_i$ and $r_i < k_i$. In the more general case where $\mathbf{\Pi}_i$ is non-zero but could (possibly) be rank deficient, the error-correction form of the country-specific model (7.4) needs to be estimated subject to the reduced rank restriction:

$$H_{r_i}: \qquad Rank(\mathbf{\Pi}_i) = r_i < k_i. \tag{7.7}$$

[20] The dimension of $\boldsymbol{\kappa}_i$ is now $(k_i + k_i^* + s) \times 1$.

Under the assumption that $Rank(\mathbf{\Pi}_i) = r_i$ one can write

$$\mathbf{\Pi}_i = \boldsymbol{\alpha}_i \boldsymbol{\beta}_i', \tag{7.8}$$

where $\boldsymbol{\alpha}_i$ is a $k_i \times r_i$ matrix of rank r_i and $\boldsymbol{\beta}_i$ is a $(k_i + k^* + s) \times r_i$ matrix of rank r_i.

For a given choice of $\boldsymbol{\beta}_i$, using (7.8) in (7.4) we have

$$\Delta \mathbf{x}_{it} = \mathbf{c}_{i0} - \boldsymbol{\alpha}_i \boldsymbol{\eta}_{it-1} + \mathbf{\Lambda}_{i0} \Delta \mathbf{x}_{it}^* + \mathbf{\Psi}_{i0} \Delta \mathbf{d}_t + \boldsymbol{\varepsilon}_{it}, \tag{7.9}$$

where

$$\boldsymbol{\eta}_{it} = \boldsymbol{\beta}_i' \mathbf{v}_{it} - \left(\boldsymbol{\beta}_i' \boldsymbol{\kappa}_i\right) t = \boldsymbol{\beta}_i' \mathbf{v}_{it} - \boldsymbol{\delta}_i t, \tag{7.10}$$

is an $r_i \times 1$ vector of long-run or (de-trended) cointegrating relations, also known as error-correction terms.

Identification and estimation of $\boldsymbol{\beta}_i$, and hence other parameters, is carried out in two steps: first the rank of $\mathbf{\Pi}_i$ is determined, for example, using the maximum eigenvalue or the trace statistics. Second $\boldsymbol{\beta}_i$ is estimated by imposition of suitable exact or possibly over-identifying restrictions on the elements of $\boldsymbol{\beta}_i$. Johansen's eigenvalue routine identifies $\boldsymbol{\beta}_i$ up to an $r_i \times r_i$ non-singular matrix. To investigate the identification conditions in the present application partition $\boldsymbol{\beta}_i$ as

$$\boldsymbol{\beta}_i = (\boldsymbol{\beta}_{ix}',\ \boldsymbol{\beta}_{ix^*}',\ \boldsymbol{\beta}_{id}')',$$

conformable to $\mathbf{v}_{it} = (\mathbf{x}_{it}', \mathbf{x}_{it}^{*\prime}, \mathbf{d}_t')'$. Then

$$\boldsymbol{\beta}_i' \mathbf{v}_{it} = \boldsymbol{\beta}_{ix}' \mathbf{x}_{it} + \boldsymbol{\beta}_{ix^*}' \mathbf{x}_{it}^* + \boldsymbol{\beta}_{id}' \mathbf{d}_t.$$

To identify $\boldsymbol{\beta}_i$ it is sufficient that $\boldsymbol{\beta}_{ix}$ (an $k_i \times r_i$ matrix), namely the part of $\boldsymbol{\beta}_i$ which corresponds to the endogenous variables, $\mathbf{x}_{it}$, is identified.[21] For this purpose we need a total of r_i^2 restrictions: r_i restrictions on each of the r_i columns of $\boldsymbol{\beta}_{ix}$. Notice that in the stationary case where $r_i = k_i$ the identification of the long-run relations can be achieved by setting $\boldsymbol{\beta}_{ix}' = \mathbf{I}_{k_i}$. In cases where $r_i < k_i$, $\boldsymbol{\beta}_i$ can be exactly identified by setting $\boldsymbol{\beta}_{ix}' = (\mathbf{I}_{r_i} \vdots \mathbf{Q}_i)$, where $\mathbf{Q}_i$ is an $r_i \times (k_i - r_i)$ matrix of fixed coefficients to be estimated freely. Other types of identifying restrictions based on *a priori* economic theory can also be entertained. But all exactly identifying restrictions yield the same estimate of $\mathbf{\Pi}_i$, and hence for forecasting and impulse response analysis the results will be invariant to the choice of exact identifying restrictions. In what follows we suggest using the exact identifying restrictions $\boldsymbol{\beta}_{ix}' = (\mathbf{I}_{r_i} \vdots \mathbf{Q}_i)$, which are relatively simple to implement.

[21] In general, it is also possible to identify $\boldsymbol{\beta}_i$ by placing restrictions on the other coefficients.

For simulation of portfolio loss distributions (and for impulse response analysis) we also need to estimate the covariance matrix of $\boldsymbol{\varepsilon}_t$. Denote the reduced rank regression estimates of $\boldsymbol{\varepsilon}_{it}$ by $\hat{\boldsymbol{\varepsilon}}_{it}$, then we have

$$\widehat{Cov}(\boldsymbol{\varepsilon}_{it}, \boldsymbol{\varepsilon}_{jt}) = T^{-1} \sum_{t=1}^{T} \hat{\boldsymbol{\varepsilon}}_{it} \hat{\boldsymbol{\varepsilon}}'_{jt}, \tag{7.11}$$

$$\widehat{Cov}(\boldsymbol{\varepsilon}_t) = \begin{pmatrix} \widehat{Cov}(\boldsymbol{\varepsilon}_{0t}, \boldsymbol{\varepsilon}_{0t}) & \widehat{Cov}(\boldsymbol{\varepsilon}_{0t}, \boldsymbol{\varepsilon}_{1t}) & \cdots & \widehat{Cov}(\boldsymbol{\varepsilon}_{0t}, \boldsymbol{\varepsilon}_{Nt}) \\ \widehat{Cov}(\boldsymbol{\varepsilon}_{1t}, \boldsymbol{\varepsilon}_{0t}) & \widehat{Cov}(\boldsymbol{\varepsilon}_{1t}, \boldsymbol{\varepsilon}_{1t}) & \cdots & \widehat{Cov}(\boldsymbol{\varepsilon}_{1t}, \boldsymbol{\varepsilon}_{Nt}) \\ \vdots & \vdots & & \vdots \\ \widehat{Cov}(\boldsymbol{\varepsilon}_{Nt}, \boldsymbol{\varepsilon}_{0t}) & \widehat{Cov}(\boldsymbol{\varepsilon}_{Nt}, \boldsymbol{\varepsilon}_{1t}) & \cdots & \widehat{Cov}(\boldsymbol{\varepsilon}_{Nt}, \boldsymbol{\varepsilon}_{Nt}) \end{pmatrix}, \tag{7.12}$$

$$\begin{aligned} \hat{\boldsymbol{\varepsilon}}_{it} &= \mathbf{x}_{it} - \hat{\mathbf{a}}_{i0} - \hat{\mathbf{a}}_{i1} t - \hat{\boldsymbol{\Phi}}_i \mathbf{x}_{i,t-1} - \\ & \quad \hat{\boldsymbol{\Lambda}}_{i0} \mathbf{x}^*_{it} - \hat{\boldsymbol{\Lambda}}_{i1} \mathbf{x}^*_{i,t-1} - \hat{\boldsymbol{\Psi}}_{i0} \mathbf{d}_t - \hat{\boldsymbol{\Psi}}_{i1} \mathbf{d}_{t-1}. \end{aligned} \tag{7.13}$$

where $\hat{\mathbf{a}}_{i0}$, $\hat{\mathbf{a}}_{i1}$, $\hat{\boldsymbol{\Phi}}_i$, $\hat{\boldsymbol{\Lambda}}_{i0}$, $\hat{\boldsymbol{\Lambda}}_{i1}$, $\hat{\boldsymbol{\Psi}}_{i0}$, and $\hat{\boldsymbol{\Psi}}_{i1}$ are the country-specific reduced-rank estimates.

7.1 Testing Weak Exogeneity of $\mathbf{x}^*_{it}$

Given the partial nature of the above analysis, it is important that the weak exogeneity of the foreign-specific variables are put to test. Following Johansen (1992) and Boswijk (1992) the weak exogeneity can be checked by testing the joint significance of the estimated error correction terms, namely $\hat{\boldsymbol{\eta}}_{i,t-1} = \hat{\boldsymbol{\beta}}'_i \mathbf{v}_{i,t-1} - \hat{\boldsymbol{\delta}}_i (t-1)$ defined by (7.10), in the marginal models for the foreign-specific variables. For example, to test the weak exogeneity of the ℓ^{th} element of $\mathbf{x}^*_{it}$ the relevant marginal model is

$$\Delta x^*_{it,\ell} = c_{i\ell} + \boldsymbol{\alpha}^*_{i\ell} \hat{\boldsymbol{\eta}}_{i,t-1} + \sum_{j=1}^{p_{i\ell}} \boldsymbol{\delta}'_{i\ell} \Delta \mathbf{z}_{i,t-j} + \zeta_{it,\ell}, \tag{7.14}$$

The lag order, $p_{i\ell}$, is set in the light of the empirical evidence and the available sample size. The weak exogeneity of $\Delta x^*_{it,\ell}$ can now be statistically evaluated by testing $\boldsymbol{\alpha}^*_{i\ell} = \mathbf{0}$, using standard F tests.[22]

Finally, it is worth noting that even if the weak exogeneity assumption is rejected, one could still obtain consistent estimates of the parameters of the GVAR model in two steps. First the country-specific models can be estimated treating all the domestic and foreign-specific variables (as well as the common global variables if deemed necessary) as endogenous. These parameter estimates

[22] A similar procedure is also advocated in Harbo, Johansen, Nielsen and Rahbek (1998, p. 395).

can then be used to obtain the parameters of the conditional models, $\mathbf{x}_i|\mathbf{x}_i^*$ separately, for $i = 0, 1, ..., N$, which can then be used to estimate the parameters of the full GVAR model.[23] This approach is, however, more data-intensive and will not be efficient if the weak exogeneity assumption is met in the case of one or more of the variables. A mixed estimation strategy is also clearly feasible, namely by treating some but not all of the foreign-specific variables as weakly exogenous.

8 Cross-Country Aggregation in Global VAR Modeling

One of the strengths of the global vector autoregressive modeling approach lies in its flexibility in taking account of the various inter-linkages in the global economy in the context of a truly multi-country setting. But it can be demanding in terms of data management, computations and data analysis when a large number of countries (say 100 or more) are included in the model. One possible way of making the analysis more manageable would be to apply the approach to a few key countries (say G7) individually, and then aggregate the remaining countries into 5-10 blocks or regions. This section considers how regional models can be constructed from the underlying country-specific models.[24]

Consider a given region i (South East Asia, North Africa, or the Middle East, for example) composed of N_i countries. Denote the vector of country-specific variables in region i by $\mathbf{x}_{i\ell t}$, and the associated foreign variable vector by $\mathbf{x}_{i\ell t}^*$, where $i = 0, 1, 2, ..., R$ and $\ell = 1, 2, ..., N_i$. We shall continue to assume that the reference country (or region) is denoted by 0.[25] The country-specific model for country ℓ in region i is given by

$$\mathbf{x}_{i\ell t} = \mathbf{a}_{i\ell 0} + \mathbf{a}_{i\ell 1}t + \mathbf{\Phi}_{i\ell}\mathbf{x}_{i\ell,t-1} + \mathbf{\Lambda}_{i\ell 0}\mathbf{x}_{i\ell t}^* + \mathbf{\Lambda}_{i\ell 1}\mathbf{x}_{i\ell,t-1}^* + \mathbf{\Psi}_{i\ell 0}\mathbf{d}_t + \mathbf{\Psi}_{i\ell 1}\mathbf{d}_{t-1} + \boldsymbol{\varepsilon}_{i\ell t}, \tag{8.1}$$

which is an adaptation of (2.1). The problem of aggregating the N_i countries within region i centers on the heterogeneity of the coefficient matrices $\mathbf{\Phi}_{i\ell}$, $\mathbf{\Lambda}_{i\ell 0}$, and $\mathbf{\Lambda}_{i\ell 1}$ associated with the country-specific variables. The cross-country heterogeneity of the remaining parameters does not pose any special problem. There will always be an aggregation problem so long as $\mathbf{\Phi}_{i\ell}$, $\mathbf{\Lambda}_{i\ell 0}$, and $\mathbf{\Lambda}_{i\ell 1}$ differ across the countries in the region. But in practice it is possible to reduce

[23] We are grateful to Soren Johansen for useful discussion regarding this approach.

[24] Note that this regional aggregation is a matter of convenience. It is not a logical requirement of the model.

[25] A region can be a reference country if it has a unified currency. Typically $N_0 = 1$.

the size of the aggregation error by using a weighted average of the variables $\mathbf{x}_{i\ell t}$ (and hence of $\mathbf{x}^*_{i\ell t}$); with the weights reflecting the relative importance of the countries in the region. Let $w^0_{i\ell}$ be the weight of country ℓ in the region i. Clearly, $\sum_{\ell=1}^{N_i} w^0_{i\ell} = 1$. Then aggregating the countries in the region using these weights we have

$$\begin{aligned} \mathbf{x}_{it} &= \mathbf{a}_{i0} + \mathbf{a}_{i1}t + \sum_{\ell=1}^{N_i} w^0_{i\ell}\mathbf{\Phi}_{i\ell}\mathbf{x}_{i\ell,t-1} + \sum_{\ell=1}^{N_i} w^0_{i\ell}\mathbf{\Lambda}_{i\ell 0}\mathbf{x}^*_{i\ell t} + \\ & \sum_{\ell=1}^{N_i} w^0_{i\ell}\mathbf{\Lambda}_{i\ell 1}\mathbf{x}^*_{i\ell,t-1} + \mathbf{\Psi}_{i0}\mathbf{d}_t + \mathbf{\Psi}_{i1}\mathbf{d}_{t-1} + \boldsymbol{\varepsilon}_{it}, \end{aligned} \tag{8.2}$$

where

$$\mathbf{x}_{it} = \sum_{\ell=1}^{N_i} w^0_{i\ell}\mathbf{x}_{i\ell t},\ \mathbf{a}_{i0} = \sum_{\ell=1}^{N_i} w^0_{i\ell}\mathbf{a}_{i\ell 0},\ \mathbf{a}_{i1} = \sum_{\ell=1}^{N_i} w^0_{i\ell}\mathbf{a}_{i\ell 1}, \tag{8.3}$$

$$\mathbf{\Psi}_{i0} = \sum_{\ell=1}^{N_i} w^0_{i\ell}\mathbf{\Psi}_{i\ell 0},\ \mathbf{\Psi}_{i1} = \sum_{\ell=1}^{N_i} w^0_{i\ell}\mathbf{\Psi}_{i\ell 1},\ \boldsymbol{\varepsilon}_{it} = \sum_{\ell=1}^{N_i} w^0_{i\ell}\boldsymbol{\varepsilon}_{i\ell t}. \tag{8.4}$$

Using (8.2) a regional model as specified in (2.1) can be obtained. In terms of the above notations we have:

$$\begin{aligned} \mathbf{x}_{it} &= \mathbf{a}_{i0} + \mathbf{a}_{i1}t + \mathbf{\Phi}_i\mathbf{x}_{i,t-1} + \mathbf{\Lambda}_{i0}\mathbf{x}^*_{it} + \mathbf{\Lambda}_{i1}\mathbf{x}^*_{i,t-1} + \\ & \mathbf{\Psi}_{i0}\mathbf{d}_t + \mathbf{\Psi}_{i1}\mathbf{d}_{t-1} + \mathbf{\Psi}_{i0}\mathbf{d}_t + \mathbf{\Psi}_{i1}\mathbf{d}_{t-1} + \boldsymbol{\xi}_{it}, \end{aligned} \tag{8.5}$$

where $\boldsymbol{\xi}_{it} = \boldsymbol{\varepsilon}_{it} + v_{it}$ is now composed of the equation errors, $\boldsymbol{\varepsilon}_{it}$, and the aggregation error is defined by

$$v_{it} = \sum_{\ell=1}^{N_i} w^0_{i\ell}(\mathbf{\Phi}_{i\ell} - \mathbf{\Phi}_i)\mathbf{x}_{i\ell,t-1} + \sum_{\ell=1}^{N_i} w^0_{i\ell}(\mathbf{\Lambda}_{i\ell 0} - \mathbf{\Lambda}_{i0})\mathbf{x}^*_{i\ell t} + \sum_{\ell=1}^{N_i} w^0_{i\ell}(\mathbf{\Lambda}_{i\ell 1} - \mathbf{\Lambda}_{i1})\mathbf{x}^*_{i\ell,t-1}. \tag{8.6}$$

The region-specific foreign variables, $\mathbf{x}^*_{it}$, can be constructed either using regional trade weights or country-specific trade weights as in (2.4). In the case of the latter y^*_{it} , for example, is defined as

$$y^*_{it} = \sum_{\ell=1}^{N_i} w^0_{i\ell} y^*_{i\ell t},\ i = 0, 1, 2, ..., R \tag{8.7}$$

where

$$y^*_{i\ell t} = \sum_{j=0}^{R}\sum_{k=1}^{N_j} w^y_{i\ell,jk} y_{jkt},\ \ell = 1, 2, ..., N_i,\ i = 0, 1, 2, ..., R, \tag{8.8}$$

$w^y_{i\ell,jk}$ is the share of country k in region j in the total trade of country ℓ in region i.

$$N = \sum_{i=0}^{R} N_i. \tag{8.9}$$

The importance of the aggregation error depends on the extent and nature of the differences in the coefficient matrices $\mathbf{\Phi}_{i\ell}$, $\mathbf{\Lambda}_{i\ell 0}$ and $\mathbf{\Lambda}_{i\ell 1}$ across the different countries in the region. The aggregation error can be minimized by choosing regions with similar economies (as far as possible) and by a sensible choice of the weights, $w_{i\ell}^0$. Importance of countries in a region is best measured by their output levels and for comparability it is important that they are measured in purchasing power parity (PPP) dollars. The weights $w_{i\ell}^0$ can be computed using PPP-adjusted GDP series for a given year or based on averages computed over several years. It may also be desirable to update the weights on a rolling basis; say by using five-yearly lagged moving-averages.

In view of the above analysis the regional variables can be constructed from country-specific variables using the following (logarithmic) weighted averages[26]

$$y_{it} = \sum_{\ell=1}^{N_i} w_{i\ell}^0 y_{i\ell t}, \;\; p_{it} = \sum_{\ell=1}^{N_i} w_{i\ell}^0 p_{i\ell t}, \;\; q_{it} = \sum_{\ell=1}^{N_i} w_{i\ell}^0 q_{i\ell t}, \qquad (8.10)$$

$$e_{it} = \sum_{\ell=1}^{N_i} w_{i\ell}^0 e_{i\ell t}, \;\; \rho_{it} = \sum_{\ell=1}^{N_i} w_{i\ell}^0 \rho_{i\ell t}, \;\; m_{it} = \sum_{\ell=1}^{N_i} w_{i\ell}^0 m_{i\ell t}. \qquad (8.11)$$

Notice that in constructing the regional variables $y_{it}, p_{it}, e_{it}, ...$ from the country-specific variables $y_{i\ell t}$, $p_{i\ell t}$, $e_{i\ell t}, ...$ one simply needs to use country-specific variables measured in their domestic currencies, bearing in mind that $e_{i\ell t}$ stands for the exchange rate of country ℓ in region i, measured in U.S. dollar.

9 An Empirical Application

9.1 Countries and Regions

In this section we illustrate our approach by estimating a global quarterly model over the period 1979Q1-1999Q1 comprising of USA, Germany, France, Italy, U.K., Japan, China and 20 other countries aggregated into 4 regions: Western Europe, South East Asia, Middle East, and Latin America. The details of these

[26] The weights $w_{i\ell}^0$ could be changed at fixed time intervals, say every five years, in order to capture secular changes in the composition of the regional output. However, changing these weights too frequently could mask the cyclical movements of the regional output being measured.

11 country/region classifications are given in Table 1.

Table 1

Countries/Regions in the GVAR Model

U.S.A.	**Germany**	**Japan**
Western Europe	**South East Asia**	**Latin America**
·Spain	·Korea	·Argentina
·Belgium	·Thailand	·Brazil
·Netherlands	·Indonesia	·Chile
·Switzerland	·Malaysia	·Peru
·Austria	·Philippines	·Mexico
	·Singapore	
Middle East	**China**	**France**
·Kuwait	**U.K.**	**Italy**
·Saudi Arabia		
·Turkey		

The output from these countries comprises around 70% of world GDP. They were chosen largely because the major banks in G-7 countries have most of their exposure in this set of countries. Noticeably absent are Scandinavian countries, Africa and Australia-New Zealand. Future extensions of the model will look to incorporate countries from these regions. Time series data on regions such as Latin America or South East Asia (SEA) were constructed from each country in the region weighted by the GDP share. For this we used purchasing power parity (PPP)-weighted GDP figures, which is thought to be more reliable than using weights based on U.S. dollar GDPs.[27] For modeling purposes we distinguish between the regions with developed capital markets namely U.S., Germany, Japan, Western European countries, South East Asia and Latin America, and the rest namely China and Middle East which over our sample period did not have fully functioning capital markets. Finally, as noted earlier, the U.S. dollar will be used as the numeraire exchange rate and its value in terms of the other currencies will be determined outside the U.S. model.

9.2 The Trade Weights

The first step in the global VAR modeling exercise is to construct the foreign country/region specific ("starred") variables from the domestic variables using

[27] Information on data sources and the construction of regional data series are provided in the Appendix. Also see Section 8 for details of regional aggregation.

the relations (2.4).[28] For the weights we decided to rely exclusively on trade weights based on the UN Direction of Trade Statistics. Information on capital flows were not of sufficiently high quality and tended to be rather volatile. The 11×11 matrix of the trade weights computed as shares of exports and imports over the 1996-98 period is presented in Table 2.

[Insert Table 2 about here]

The trade shares of each country/region is displayed by columns. This matrix plays a key role in linking up the models of the different regions together and shows the degree to which one country/region depends on the remaining countries. For example, not surprisingly the trade weights show that Latin America is much more integrated with the U.S. economy than the rest of the regions, while the Middle East is more integrated with the economies of Western Europe and Germany, and the bulk of China's trade is with the U.S., Germany, Japan and South East Asia.

9.3 Integration Properties of the Series

The second stage in the modeling process is to select appropriate transformations of the domestic and foreign variables for inclusion in the country/region specific cointegrating VAR models. The reduced rank regression techniques reviewed in Section 7 are based on the assumption that the underlying endogenous and exogenous variables to be included in the country/region specific models are approximately integrated of order unity. To ascertain the order of integration of the variables in the country/region specific models in Tables 3a and 3b we present augmented Dickey-Fuller (ADF) statistics for the levels, first differences and the second differences of the domestic and country/region specific foreign variables.

[Insert Tables 3a and 3b about here]

To ensure comparability all these statistics are computed over the same period, 1980Q2 to 1999Q1, starting with an underlying univariate autoregressive process of order 5, with a linear trend in the case of the levels (except for the interest rates) and an intercept term only in the case of the first and second-differences. The orders of the ADF test statistics reported in Tables 3a and 3b are selected according to the Akaike Information Criterion (AIC).

Generally speaking, the results of these unit root tests are in line with what is known in the literature. Interest rates (domestic and foreign) and real

[28] The details and the sources of the primary macro variables are provided in Appendix (A).

equity prices (domestic and foreign) are unambiguously $I(1)$ across all countries/regions. The same also applies to exchange rates with the notable exception of Latin America.[29] In the case of Latin America the hypothesis that exchange rate is $I(2)$ can not be rejected. Mainly as a consequence, it is also not possible to reject the hypothesis that the U.S.-specific foreign exchange rate variable defined by

$$e_{US}^{*}=\sum_{j=1}^{8} w_{US,j} e_j, \tag{9.1}$$

is an $I(2)$ variable. (See the last column of Table 3b). There are two possible ways of dealing with this problem. We could decide to model Δe instead of e, but this will most likely involve over-differencing and efficiency loss in the case of the seven remaining regional models. Another, arguably more attractive, alternative would be to include the real exchange rate $(e-p)$ in the regional models. The hypothesis that $e-p$ is $I(1)$ now prevails across all countries, and the hypothesis that $e^{*}-p^{*}$ is $I(1)$ is not supported in the case of U.K. and Latin America.

As far as the order of integration of the remaining three variables are concerned, the evidence is less clear cut, which is partly due to uneven data quality across the countries and the relatively short sample period under consideration. Using the 95% significance level, a unit root in real output is not rejected in any of the 11 regions. However, in the case of Japan and China the ADF statistics seem to suggest that real output could be $I(2)$! This is clearly implausible and again could be due to poor data quality in the case of China. The result for Japan is, however, difficult to explain, although Japan's national income statistics are not regarded as particularly reliable. A similar argument also applies to foreign output variables, y^{*} and real money balances, m and m^{*}. Overall, however, it seems appropriate for our purposes to treat all these variables approximately as $I(1)$. Finally, for the price variables the test results suggest that the general price level, p, is $I(1)$ in six regions and $I(2)$ in the remaining five regions. A similar outcome prevails with respect to p^{*}, which is $I(2)$ for six countries and $I(1)$ for the remaining four. (Recall that p^{*} is not included in the

[29] A number of modifications of the ADF test have also been proposed in the literature, for example, by Pantula, Gonzalez-Farias and Fuller (1994), Leybourne (1995), and Elliot, Rothenberg and Stock (1996), which have been shown to have better small sample power characteristics. To check the robustness of our conclusions to the choice of the test statistics we also computed Elliott et al.'s ADF-GLS statistics for all the series reported in Tables 3a and 3b. Overall, the test results support our general decisions regarding the unit properties of the various series. The test results based on ADF and ADF-GLS statistics differ only in a few cases and there seems to be no obvious patterns to these differences. The ADF-GLS test results are available from the authors on request.

U.S. model). Since over-differencing is likely to be less serious for the empirical analysis than wrongly including an $I(2)$ instead of an $I(1)$ variable, we shall be using inflation rates, Δp and Δp^*, that are at most $I(1)$, instead of the price levels.

9.4 Country/Region Specific Models

In view of the above results, the endogenous variables of the U.S. model were selected to be real output (y_{US}), the rate of inflation (Δp_{US}), the level of interest rate (r_{US}), the real money balances (m_{US}), and the real equity prices (q_{US}), all measured in logarithms as defined in (2.3). Within the GVAR framework the value of the U.S. dollar is determined outside the U.S. model, and the U.S.-specific real exchange rate variable, $e^*_{US} - p^*_{US}$, is then included as an I(1) weakly exogenous variable in the U.S. model.[30] Given the size of the U.S. economy and its importance for global economic interactions, no other foreign-specific exogenous variable was considered for inclusion in the U.S. model. But to control for important global political events, the logarithm of oil prices (p^o) were included as an exogenous I(1) variable in *all* the country/region specific models.[31]

In the case of U.K., Germany, France, Italy, rest of Western Europe, Japan, South East Asia and Latin America with advanced capital markets we chose $(y_j, \Delta p_j, r_j, e_j - p_j, m_j, q_j)$ and $\left(y^*_j, \Delta p^*_j, r^*_j, m^*_j, q^*_j, p^o\right)$ as their endogenous and exogenous variables, respectively. Notice that e^*_j is excluded from the set of exogenous variables on the grounds of its close relationship to e_j.[32] For the remaining regions (Middle East and China) the set of included endogenous and exogenous variables were $(y_j, \Delta p_j, r_j, e_j - p_j, m_j)$ and $\left(y^*_j, \Delta p^*_j, r^*_j, m^*_j, q^*_j, p^o\right)$, respectively.

The next step in the analysis is to estimate region-specific cointegrating VAR models and identify the rank of their cointegrating space. The order of the underlying VAR models was taken to be 1. This choice was dictated to us by the small number of time series observations that were available to us relative to the number of unknown parameters in each of the regional models. The "trace" and "maximum eigenvalue" test statistics for each of the 11 regions together with the associated 90% and 95% critical values are summarized in

[30] The weights $w_{US,j}$, $j = 1, 2, ..., 10$ are given in the first column of Table 2.

[31] The ADF statistics computed over the period 1980Q2-1999Q1 for the level and first-differences of oil prices were -2.27 and -4.74, respectively; thus providing empirical support for treating oil prices as an $I(1)$ variable.

[32] See Section 2 for a more detailed discusion.

Tables 4a-4c.[33]

[Insert Tables 4a-4c about here]

It is known that both of these statistics tend to over-reject in small samples, with the extent of over-rejection being much more serious for the maximum eigenvalue as compared to the trace test. Using Monte Carlo experiments it has also been shown that the maximum eigenvalue test is generally less robust to departures from normal errors than the trace test.[34] The latter point is particularly relevant to our applications since they contain equity prices, exchange rates and interest rates, all of which exhibit significant degrees of departures from normality. We shall therefore base our inference on the trace statistics. Accordingly, we found 5 cointegrating relations for the U.K., 4 for Germany and Japan and 3 for Italy, Western Europe, South East Asia, Latin America, Middle East and China, and 2 for France and the U.S. The result for the U.K. is in line with the full system estimates reported in Garratt et al. (2001) for the U.K. over the period, 1965q1-1995q4. For France the trace test when applied at 90% is very marginal and in view of the 3 or more cointegrating relations found for other Western European countries could be an underestimate. So in what follows we also assume that there are 3 cointegrating relations in the model for France. For the U.S. the test results seem quite conclusive and given the particular nature of the U.S. model we did not see any ground for doubting the 2 cointegrating relations that are suggested by the tests.

The cointegrating relations can be interpreted as long-run relations, either amongst the domestic variables and/or between the domestic and foreign variables. Long-run money demand equation (that relates m_{it} to ρ_{it} and y_{it}) is an example of the former, while the uncovered interest parity (that relates ρ_{it} to ρ_{it}^{*}) provides an example of the latter. These theoretical long-run relations suggest further (over-identifying) restrictions on the cointegrating relations which can be imposed and tested as in Garratt et al. (2001), for example. However, this will require detailed long-run structural analysis of the individual regions and will be beyond the scope of the present application.

[33] These statistics are computed using VAR(1) specifications with restricted trend coefficients. This is model IV in Pesaran, Shin and Smith (2000). Also see the discussion in Section 7. Computations are carried out using Microfit 4.1. See Pesaran and Pesaran (1997).

[34] See, for example, Cheung and Lai (1993).

9.5 Testing Weak Exogeneity of the Country-Specific Foreign Variables

One of the key assumptions underlying our estimation approach is the weak exogeneity of the country-specific foreign variables. But as noted earlier this assumption can be tested by running first-difference regressions of the foreign variables and testing the significance of the country-specific error correction terms in these regressions. See (7.14). For example, to the test the weak exogeneity of, for instance, foreign output in the U.K. model, y^*_{uk}, we need to test the joint hypothesis that

$$\delta_{uk,j} = 0, \ j = 1, 2, .., 5$$

in the regression

$$\Delta y^*_{uk,t} = a_{uk} + \sum_{j=1}^{5} \delta_{uk,j} ECM^{(j)}_{uk,t-1} + \boldsymbol{\phi}'_{uk} \Delta \mathbf{z}_{uk,t-1} + \phi_{uk,o} \Delta p^o_{t-1} + \zeta_{uk,t},$$

where $ECM^{(j)}_{uk,t-1}$, $j = 1, 2, ..., 5$ are the estimated error correction terms associated with the five cointegrating relations found in the U.K. model, $\Delta \mathbf{z}_{uk,t-1} = (\Delta \mathbf{x}'_{uk,t-1}, \Delta \mathbf{x}'^*_{uk,t-1}, \Delta \left(e^*_{uk,t-1} - p^*_{uk,t-1} \right), \Delta p^o_{t-1})'$. The F statistics for testing the weak exogeneity of all the country-specific foreign variables and the oil price variable are summarized in Table 5.

[Insert Table 5 about here]

Out of the 62 weak exogeneity tests carried out only 3 are statistically significant at the 5% level and none at 3% or less. The 3 rejections of weak exogeneity assumption relate to foreign output in France, real equity prices in Latin America and oil prices in the U.S. model. Arguably, the most convincing and plausible of these rejections is the weak exogeneity of oil prices in the U.S. model. So we re-estimated the U.S. model with oil prices as endogenous. This did not affect our main conclusion about the number of cointegrating relations in the U.S. model but confirmed the importance of possible feedback effects from the U.S. economy into oil prices. There seems little to choose between the two versions of the U.S. model, however. After careful considerations of the various issues involved we decided in favor of treating oil prices as exogenous throughout the GVAR model, considering the importance of geopolitical factors in determination of oil prices, and the desirability of retaining a flexible modeling approach suited to the analysis of special risks from international political events such as threat of wars and terrorism.

9.6 Other Features of the Country-Specific Models

Due to data limitations and the relatively large number of endogenous and exogenous variables involved, we have been forced to consider a VARX*(1,1) specifications for the country-specific models. It is therefore important to check the adequacy of the country-specific models in dealing with the complex dynamic inter-relationships that exist in the world economy. To this end in Table 6 we provide F statistics for tests of serial correlation of order 4 in the residuals of the error correction regressions for all the 63 endogenous variables in the GVAR model.

[Insert Table 6 about here]

Considering the relative simplicity of the underlying models it is comforting that 45 out of the 63 regressions pass the residual serial correlation test at the 95% level. Perhaps not surprisingly most of the statistically significant outcomes occur in the case of variables with known growth persistence characteristics, namely real money balances, interest rates and inflation. But even in these examples the degree of rejection in not uniform. For example, using the 1% significance level there are only 7 error correction equations that do not meet the requirement. Therefore, while there are cases of concern that need to be examined more carefully, overall the test results seem satisfactory. It is hoped that as more data become available, higher order VARX* models can be estimated and their results evaluated for residual serial correlation. This may require estimation of different order VARX* models for different countries. The GVAR methodology can accommodate both extensions, but these will not be pursued here.

These test results together with the weak exogeneity of the foreign variables also allow consistent estimation of the contemporaneous effects of foreign-specific variables on domestic variables (at least for the ones where the residual serial correlation test is not statistically significant). There are many estimates of interest that could be considered. Here we focus on the contemporaneous effects of the foreign variables on their domestic counterpart. For example, we could ask: what is the effect on German output if foreign output specific to Germany rises by 1%. Similarly, the effect of 1% increase in 'world' equity prices can be estimated on equity prices of the individual countries/regions. The estimates, best viewed as impact elasticities, from such an exercise are summarized in Table 7.[35] When statistically significant all the estimates have the expected

[35] This interpretation is due to the linear logarithmic nature of the country-specific models and the fact that foreign variables are formed as weighted averages with the weights adding

sign of being positive, except for the coefficients of Δm^* in South East Asia and Δr^* in Japan.

[Insert Table 7 about here]

The output elasticities are significant in the case of Germany, France, Italy, Western Europe, China and Japan. Equity price elasticities are statistically significant in the case of all countries/regions with a capital market. The patterns of statistical significance of inflation, interest rate and real money balances are more dispersed across countries. Perhaps not surprisingly equity markets show the closest degree of contemporaneous inter-dependence with the other channels playing a less prominent role by comparison.

9.7 Dynamic Properties of the Global Model

Due to the simultaneous nature of the country-specific models a more satisfactory approach to the analysis of dynamics and interdependencies (both on impact as well as over time) amongst the various factors would be via impulse response functions computed from the solution to the global VAR model. As discussed in Section 3, the global model can be obtained by combining the country-specific models. The total number of cointegrating relations in the global model can at most be equal to $r = \sum_{i=0}^{10} r_i = 36$.[36] The long-run and short-run dynamic properties of the global model are determined by the global cointegrating matrix, $\tilde{\boldsymbol{\beta}}$, given by (4.8), and the eigenvalues of $\mathcal{F} = \mathbf{G}^{-1}\mathbf{H}$, defined by (5.4). Since the global model contains 63 endogenous variables and the rank of $\tilde{\boldsymbol{\beta}}$ is at most 36, it then follows that $\mathcal{F}$ must have at least 27 $(= 63 - 36)$ eigenvalues that fall on the unit circle.[37] It is encouraging that our application does in fact satisfy this property. The matrix $\mathcal{F}$, estimated from the region-specific models has exactly 27 eigenvalues that fall on the unit circle with the remaining 36 eigenvalues having moduli all less than unity.[38] Amongst the latter set, the 3 largest eigenvalues (in moduli) are 0.9456, 0.8661, and 0.8575; thus ensuring a reasonably fast rate of convergence of the model to its steady state once shocked. These results also establish that the global model forms

up to unity.

[36]See Section 4 for further details.

[37]Notice that $rank(\tilde{\boldsymbol{\beta}}) = rank(\mathbf{G} - \mathbf{H})$, and for a non-singular matrix $\mathbf{G}$, then $rank(\mathbf{I} - \mathcal{F}) = rank(\tilde{\boldsymbol{\beta}})$.

[38]Out of these 36 eigenvalues, 28 (14 pairs) were complex, that produce the damped mildly cyclical character of the generalized impulse response functions discussed below. The eigenvalues with the three largest complex part are $0.3875 \pm 0.2495i$, $00.1023 \pm 0.1990i$, $0.7406 \pm 0.1624i$, , where $i = \sqrt{-1}$.

a cointegrating system with 36 long-run relations and a stable error-correcting representation. In particular, the effects of shocks on the long-run relations of the global economy will eventually disappear. The decay rate is bounded by 0.9456. However, due to the unit root properties of the global model (as characterized by the unit eigenvalues of F), global or regional shocks will have permanent effects on the levels of the variables such as real outputs, interest rates or real equity prices.

The time-profiles of the effects of a variety of shocks of interest on the global economy can now be computed by means of generalized impulse response functions (GIRFs) discussed in Section 6 which identifies the shocks as intercept shifts in the various equations using historical variance-covariance matrix of the errors for estimation of impact effects. This approach is particularly suited for the analysis of dynamics of the transmission of shocks across regions, since GIRFs are invariant to the ordering of the countries/regions in the global VAR model. (See Section 6). Also, while it is true that it may not be possible to provide "structural" *economic* interpretation of these shocks as 'demand', 'supply' or 'policy' shocks,[39] the GIRFs provide a historically consistent account of the inter-dependencies of the idiosyncratic shocks particularly across different regions. Given that specific-country models condition on weakly exogenous foreign variables, it is reasonable to expect that there should remain only a modest degree of correlations across the shocks from different regions, and hence it is more reasonable to believe that the GVAR helps identify regional shocks as compared to shocks that can be given a satisfactory economic interpretations. For example, the GVAR approach could provide a plausible account of the transmission of shocks from the U.S. (modelled almost as a closed economy) to the rest of the world. Accordingly, we shall consider the following shock scenarios with emphasis on their regional transmissions:

- A one standard error negative shock (a negative "unit" shock) to U.S. equity prices.
- A one standard error positive shock to German output.
- A one standard error negative shock to equity markets in South East Asia.

We could examine the time profiles of the effects of these shocks either on the endogenous variables of a particular region, or on a given variable across all the regions.

[39] The number of possibilities is huge, 63! in total.

9.7.1 A Negative Shock to U.S. Equity Prices

Figure 1 displays the impacts of shocks to U.S. equity market on equity prices worldwide.

[Insert Figure 1 about here]

On impact, a fall in the U.S. equity prices causes prices in all equity markets to fall as well but by smaller amounts: 3.5% in the U.K., 4.5% in Germany, 2.4% in Japan, 2.6% in South East Asia, and 4.8% in Latin America, as compared to a fall of 6.4% in the U.S. (See Table 8).

[Insert Table 8 about here]

However, over time the fall in equity prices across the regions start to catch up with the U.S. and gets amplified in the case of Italy and Latin America. The U.K. presents an interesting exception to this pattern, although these point estimates should be viewed with caution. They are likely to be poorly estimated with large standard errors, particularly those that refer to long forecast horizons.[40] Nevertheless, the relative position and pattern of the impulse response functions could still be quite informative. For example, they confirm the pivotal role played by the U.S. stock market in the global economy, and suggest that in the longer run scope for geographic diversifications across equity market might be somewhat limited. See Figure 1.

The time profiles of the effects of the shock to the U.S. equity market on real output across the different regions are shown in Figure 2. The second panel of Table 8 provides the associated point estimates for a number of selected horizons.

[Insert Figure 2 about here]

The impact effects of the fall in the U.S. equity market on real output are negative for most regions, but rather small in magnitude. After one year real output shows a fall of around -0.31% in the U.S., -0.25% and -0.29% in Germany and U.K., respectively, -0.26% in Latin America, and -0.12% in South East Asia. Japanese output only begins to be negatively affected by the adverse U.S. stock market shock much later. The two regions without capital markets are either

[40] It is possible to compute standard errors for the generalized impulse responses using bootstrap techniques. See, for example, Garratt et al. (2001). But this would be a highly computer intensive exercise and it is not clear to us that it will add much to our overall conclusions.

not affected by the shock (Middle East) or even show a rise in output (in the case of China). Once again these point estimates should be treated with caution.[41]

9.7.2 A Positive Shock to German Output

The effects of a one standard error rise in output in Germany on equity prices and real output across the different regions are summarized in Table 9 and displayed in Figures 3 and 4.[42]

[Insert Table 9 and Figures 3&4 about here]

Table 9 also provides the point estimates of the effects of the shock on inflation, interest rates and real exchange rates for selected horizons. On impact the effect of the increase in Germany's output is to increase German equity prices by 2.50%, followed by 1.20% in South East Asia and 0.90% in France, with mixed outcomes for the remaining regions. The impact effects are in fact negative on U.S., U.K. and Japan equity prices, although these are rather small compared to standard errors of shocks to equity prices in these economies. Over time the effect of the positive output shock is to increase equity prices in France, Italy and the rest of Western Europe in line with the increase in Germany's equity prices, although the effects on U.K. and U.S. equity prices continue to remain negative but very small. This shows the high degree of integration of the European economies with Germany with the notable exception of the U.K. equity market which seems to follow the U.S. market instead.

A similar story also emerges if the effects of the shock to Germany's output on other countries' output are considered. (see the second panel of Table 9). After one year the effect of the shock on U.S. and U.K. output is almost zero while it is still sizeable on France, Italy and the rest of Western Europe. These differences become further pronounced at horizons beyond one year.

The effects of the shock on other variables are mixed. They are mostly small and transient in the case of inflation and interest rates, but quite sizeable as far as exchange rate and real money balances are concerned, at least in the case of some of the countries, notably Germany, Italy and the rest of Western Europe.

[41] Table 8 also provides point estimates of the time profiles of the effects of the adverse US stock market shock on inflation, interest rates and real exchange rates. Overall the pattern of the impulse responses across the regions seem plausible, although space does not permit a detailed discussion of these results here.

[42] A one standard error shock here converts to around 2.96% per annum increase in output.

9.7.3 A Negative Shock to Equity Markets in South East Asia

Given the interest in the effects of the 1997 South Asian Crisis and its possible contagion effects, we consider here the generalized impulse response functions for a one standard error negative shock to equity prices in South East Asia (SEA).[43] The one standard error shock is equivalent to 8.2% decline in SEA's equity prices and on impact has small positive effects on Japan's and U.S. equity prices (1.30% and 0.31%, respectively) and relatively small negative effects on equity prices in other countries. See the first panel of Table 10 and Figure 5.

[Insert Table 10 and Figure 5 about here]

But over time these effects accumulate and after two years all markets are adversely affected with the exception of the U.S. The U.S. equity market (and to a lesser extent the U.K. and Japanese markets) seems to have been reasonably robust to the South Asian Crisis. It is also interesting to note that in the longer run the Western European (except for the U.K.) equity markets seem to be more vulnerable to the South Asian Crisis than Japan.

As to be expected the output effects of the negative shock to the SEA's equity markets is much more muted when compared to its effects on equity prices. Even after one year adverse effects of the shock is only sizeable in the case of European economies (with the exception of U.K.) with the largest effect, perhaps not surprisingly, being on South East Asia itself. See the second panel of Table 8 and Figure 6.

[Insert Figure 6 about here]

Once again the impulse responses suggest that Japan, U.S. and U.K. are likely to be reasonably robust to adverse shocks from South East Asian equity markets. At first this result seems rather surprising considering the relatively strong trade links that exists between South East Asia, Japan and U.S. (see Table 2). However, this result largely reflects the apparently weak links that exists between the equity markets of these economies as can be seen from the first panel of Table 10 and discussed above. The impulse responses of the effects of the negative shock to SEA's equity markets on inflation, interest rates and exchange rates are summarized in the bottom three panels of Table 8. Other implications of the South Asian Crisis (such as an adverse shock to exchange rates) can also be investigated using the global VAR modeling tools developed in this paper.

[43] Our framework can also be used to investigate the contagion effects within the South East Asian region. However, this would have required a much higher degree of regional disaggregation and could be the subject of a separate study.

10 Conditional Loss Distributions

In this section we show how to use the GVAR to generate conditional loss distributions. We begin with a characterization of a firm's change in value as a function of systematic and idiosyncratic components. We adopt a somewhat simplified version of the more general model developed in Pesaran, Schuermann, Treutler and Weiner (2003), hereafter PSTW. Following an approach which is structurally similar to Arbitrage Pricing Theory (APT), a firm's change in value (or return), conditional on information available up to time t, $\mathcal{I}_t$, can be decomposed as

$$r_{ji,t+1} = \mu_{jit} + \xi_{ji,t+1}, \tag{10.1}$$

where μ_{jit} is the (forecastable) conditional mean, and $\xi_{ji,t+1}$ is the (non-forecastable) innovation component of the return process. Consistent with the distributional assumptions of the GVAR model, the innovation has a conditional Gaussian distribution[44]

$$\xi_{ji,t+1} \mid \mathcal{I}_t \sim N(0, \omega^2_{\xi,ji}). \tag{10.2}$$

Linking the firm return expression (10.1) into the GVAR model, we can specify the conditional mean process more precisely. Thus firm returns depend on changes in the underlying macroeconomic factors, say k_i region-specific macroeconomic variables, the exogenous global variables ($\mathbf{d}_t$, in our application oil prices), together comprising the systematic components and the firm-specific idiosyncratic shocks:

$$r_{ji,t+1} = \alpha_{ji} + \sum_{\ell=1}^{k_i} \beta_{ji,\ell} \Delta x_{i,t+1,\ell} + \sum_{\ell=1}^{s} \gamma_{ji,\ell} \Delta d_{t+1,\ell} + \eta_{ji,t+1}, \; t = 1, 2, ..., T, \tag{10.3}$$

where $r_{ji,t+1}$ is the equity return from t to $t+1$ for firm j ($j = 1, ..., nc_i$) in region i. α_{ji} is a regression constant for company j in region i, k_i is the number of macroeconomic factors (drivers) in region i, $\beta_{ji,\ell}$ is the factor loading corresponding to the change in the ℓ^{th} macroeconomic variable for company j in region i, $\Delta x_{i,t+1,\ell}$ is the log difference of the ℓ^{th} macroeconomic factor in region i, $d_{t+1,\ell}$ is the ℓ^{th} global factor, $\gamma_{ji,\ell}$ is its associated coefficient, and $\eta_{ji,t+1}$ is

[44] The normality assumption could be a good approximation for quarterly returns, but it is relatively easy to adapt the analysis to allow for fat-tailed distributions such as Standard t with low degrees of freedom; in fact we do so in PSTW. The assumption that the conditional variance of returns are time-invariant also seems reasonable for quarterly returns, although it would need to be relaxed for returns measured over shorter periods, such as weeks or days.

a firm-specific shock.[45] This can be written more compactly as

$$r_{ji,t+1} = \alpha_{ji} + \boldsymbol{\beta}'_{ji}\Delta\mathbf{x}_{i,t+1} + \boldsymbol{\gamma}'_{ji}\Delta\mathbf{d}_{t+1} + \eta_{ji,t+1}, \tag{10.4}$$

where $\mathbf{x}_{i,t+1}$ and $\mathbf{d}_{t+1}$ are the $k_i \times 1$ and $s \times 1$ vectors of macroeconomic and global factors, which are precisely the variables in the country-specific models defined by (2.1) or (5.1). The main advantage of using the GVAR as a driver for a credit portfolio model is that it provides the correlation structure among macroeconomic variables of the global economy. If the model captures all systematic risk, the idiosyncratic risk components of any two companies in the model should be uncorrelated.

Accordingly, we assume that the firm-specific shocks, $\eta_{ji,t+1}$, have mean zero, a constant (time-invariant) variance, $\omega^2_{\eta,ji}$, are serially uncorrelated and are distributed independently of the macroeconomic factors. Further for the simulation of the loss distribution, we shall assume that these shocks are also independently distributed across firms as normal variates, namely $\eta_{ji,t+1} \backsim IIN\left(0, \omega^2_{\eta,ji}\right)$.[46]

In any given time period, the probability of default for firm j in region i will be correlated, through the influence of common macro effects (or systematic risk factors) in region i, and globally, with the probability of default of other firms in the bank's portfolio. Most credit portfolio models share this linkage of systematic risk factors to default and loss; they differ in specifically how they are linked (see Figure 7).[47]

[Insert Figure 7 about here]

[45] Given the international nature of some of the firm's operations it would also be relevant to add country-specific foreign variables in the APT regressions. This will complicate the exposition but can be readily accommodate within our framework. For details of such an extension see PSTW.

[46] Relaxing the distributional assumption for $\eta_{ji,t+1}$ is no more difficult than it is for $\xi_{ji,t+1}$. Another alternative is to sample directly from the actual APT regression residuals $\hat{\eta}_{jit}$, assuming the available sample periods across the different companies is sufficiently large. In our application we have at most 80 data points per company and for some of the companies in the loan portfolio we have considerably less, and resampling does not promise to provide a more accurate picture of the true distribution of the residuals. See also the discussion in Section 10.3.

[47] For detailed comparisons, see Koyluoglu and Hickman (1998), Crouhy et al. (2000), Gordy (2000) and Saunders and Allen (2002).

10.1 The Merton Model, Default Thresholds and Credit Ratings

Before expected loss due to default can be computed we need a procedure for determining a default threshold, c_{ji}, with respect to which the default state can be defined. We follow a standard approach in the literature by making use of the Merton (1974) option-based model of firm default.[48] In that model, shareholders effectively hold a put option on the firm, while the debt-holders hold a call option.[49] If the value of the firm falls below a certain threshold, the shareholders will put the firm to the debt-holders. We follow a typical adaptation of the Merton model by using asset returns and their volatility instead of total value of assets and their volatility. But since asset returns and their volatility are difficult to observe directly, we use equity returns and their volatility as proxies.[50]

In the Merton model default occurs if the value of the firm j in region i at time t falls below a given fixed threshold value, c_{ji}. The separation between a default and a non-default state can now be characterized using the indicator variable $I\left(r_{ji,t+1} < c_{ji}\right)$ such that

$$\begin{aligned} I\left(r_{ji,t+1} < c_{ji}\right) &= 1 \text{ if } r_{ji,t+1} < c_{ji} \Longrightarrow \text{ Default}, \\ I\left(r_{ji,t+1} < c_{ji}\right) &= 0 \text{ if } r_{ji,t+1} \geq c_{ji} \Longrightarrow \text{ No Default}. \end{aligned} \tag{10.5}$$

In standard implementations of the Merton model the percentage changes in asset value are taken to be normally distributed.[51] Moreover, this class of models places a specific interpretation on credit ratings from rating agencies, namely as a distance to default metric. Assuming that changes in asset value are normally distributed, the default probability can be expressed as the probability of a standard normal variate falling below some critical value.

Conceptually it is useful to anchor the default process by fixing the default threshold, for instance at the end of the sample period, thereby allowing the

[48] For a discussion of the power of Merton default prediction models see Falkenstein and Boral (2001) who find that the Merton model generally does well in predicting default, but should be combined with other measures such as balance sheet ratios. Duffee (1999) points out that due to the continuous time diffusion processes underlying the Black Scholes formula, short-term default probabilities may be underestimated. Amongst others, Jarrow and Turnbull have developed intensity-based models, also called "reduced form" models, where model assumptions are imposed on the prices of the firm's liabilities only. See Duffee (1999).

[49] Through put-call parity, one could also conceptualize this as shareholders holding a call option on the firm's assets, while the debtholders' pay-off is isomorphic to writing a put option.

[50] Arguably equity returns are even preferred since they allow for non-constant liabilities within the Merton framework.

[51] For example, credit portfolio management models such as CreditMetrics adjust the asset returns to be standard normally distributed.

loss distribution to shift in response to macroeconomic factors. Define $PD_{jit} = \Pr(r_{ji,t+1} < c_{ji} \mid \mathcal{I}_t)$ as shorthand notation for the probability of default of company j in region i at time t. Then using (10.1) and (10.5), this default probability can be written as

$$PD_{jit} = \Phi\left(\frac{c_{ji} - \mu_{jit}}{\omega_{\xi,ji}}\right), \tag{10.6}$$

where $\Phi(\cdot)$ is the standard normal distribution function. There are no direct observations on PD_{jit}. Instead what we do have is a credit rating $\mathcal{R}_{jit}$ for a set of large companies, namely those that were assigned a rating by one of the rating agencies such as Moody's or Standard & Poor.[52] Importantly we have the rating histories $\{\mathcal{R}_{jit}\}_{t=1}^{T}$ for all companies $j = 1, 2, ..., nc_i$, $i = 0, 1, ..., N$ in the credit portfolio that we shall be considering. We may use these histories, plus histories for all other companies with a rating at the beginning of period t, to estimate the default probability for each rating for each time period, $PD_{\mathcal{R}_t}$.[53] For example, the estimated probability of default for companies rated '$\mathcal{BBB}$' in period t may be 22 basis points ($PD_{\mathcal{BBB}_t} = 22$bp), while in period t' it may rise to 37bp ($PD_{\mathcal{BBB}_{t'}} = 37$bp). We are then able to assign that default probability in period t for rating $\mathcal{R}$ to all firms with that rating in that period.

Given sufficient data for a particular region or country i (the U.S. comes to mind), one could in principle have PDs varying over i. However, since a particular firm j's default is only observable once, multiple (serial) bankruptcies notwithstanding, it makes less sense to allow PD to vary across j. Empirically, then, we will abstract from possible variation in default rates across countries i, so that probabilities of default vary only across credit ratings and over time.

Thus for a particular credit rating $\mathcal{R}_{jit}$ for firm j in region i at time t, (say '$\mathcal{BBB}$'), we assign the corresponding default probability estimate $PD(\mathcal{R}_{jit})$ which varies over time and across rating types but not across firms individually. Therefore, two different firms with the same credit rating in period t will have the same default probability estimates. Specifically

$$\Pr(r_{ji,t+1} < c_{ji} \mid \mathcal{I}_t) = PD(\mathcal{R}_{jit})$$

and therefore

$$c_{ji} = \mu_{jit} + \omega_{\xi,ji} DT(\mathcal{R}_{jit}), \tag{10.7}$$

where $DT(\mathcal{R}_{jit}) = \Phi^{-1}(PD(\mathcal{R}_{jit}))$ is the 'default threshold' associated with

[52] $\mathcal{R}$ may take on values such as 'Aaa', 'Aa', 'Baa',..., 'Caa' in Moody's terminology, or 'AAA', 'AA', 'BBB',..., 'CCC' in S&P's terminology.

[53] See PSTW for details on how to obtain default frequency, or default probability, estimates from rating histories.

the estimated default probability $PD\left(\mathcal{R}_{jit}\right)$, and $\Phi^{-1}(\cdot)$ denotes the inverse cumulative standard normal distribution.

Suppose now that we have time series data over the sample period $t = 1, 2, ..., T$, and we wish to obtain an estimate of the default threshold at T to be used in computation of conditional loss distribution over the period T to $T+1$. Averaging the relations (10.7) over $t = 1$ to T we obtain

$$c_{ji} = \bar{\mu}_{ji} + \omega_{\xi,ji}\overline{DT}_{\mathcal{R}ji},$$

where

$$\bar{\mu}_{ji} = \frac{1}{T}\sum_{t=1}^{T}\mu_{jit}, \text{ and } \overline{DT}_{\mathcal{R}ji} = \frac{1}{T}\sum_{t=1}^{T}DT\left(\mathcal{R}_{jit}\right).$$

A model-free estimate of $\bar{\mu}_{ji}$ is given by $\bar{r}_{ji}$, the average return over the sample period. As noted above, estimates of $PD\left(\mathcal{R}_{jit}\right)$ (and hence $DT\left(\mathcal{R}_{jit}\right)$) can be obtained using time series observations of rating histories from credit rating agencies such as Moody's or Standard & Poor, and $\omega_{\xi,ji}$ can be estimated (as shown below) using the GVAR model and the parameters of the APT regressions. Alternatively, an unconditional (model-free) estimate of the return variance, say $\omega_{ji}^2 = Var\left(r_{ji,t+1}\right)$, could be used. The results are unlikely to be much affected by which of the two estimated error-variances is used. But the model-free estimate has the advantage of being simple and could fit better with the rating agencies' own approach of not putting too much weight on the business cycle factors in arriving at their credit ratings.[54] Adopting the model-free estimation approach, c_{ji} can be consistently estimated at time T by

$$\hat{c}_{ji} = \bar{r}_{ji} + \hat{\omega}_{ji}\overline{DT}_{\mathcal{R}ji}, \tag{10.8}$$

where

$$\hat{\omega}_{ji}^2 = \frac{\sum_{t=1}^{T}\left(r_{jit} - \bar{r}_{ji}\right)^2}{T-1}.$$

Clearly, it would be possible to update this estimate in a recursive fashion as more data becomes available, either by using an expanding or a rolling observation window.

We would say that *conditional* on information we have at time T, default occurs when $r_{ji,T+1} < \hat{c}_{ji}$, or equivalently if

$$r_{ji,T+1} < \bar{r}_{ji} + \hat{\omega}_{ji}\overline{DT}_{\mathcal{R}ji}. \tag{10.9}$$

[54] We use rating histories from Moody's to estimate $PD_{\mathcal{R}_t}$ and hence $\hat{c}_{ji}$, and "Moody's believes that giving only a modest weight to cyclical conditions best serves the interests of the bulk of investors." See Moody's (1999, pp. 6-7).

By treating the critical value as constant, we implicitly assume constant liability growth. Thus we continue to make assumptions about the capital structure of the firm, but ones that are less restrictive and more realistic.[55]

In the Merton default prediction model, accounting data (book value of callable liabilities), the market value of equity and the volatility in the market value of equity are used to derive $PD(\mathcal{R}_{jit})$.[56] We do the inverse: using an existing measure of expected default probability, we determine the critical value $\hat{c}_{ji}$.

There are several reasons to think this approach is less than ideal. Putting aside issues of the structural Merton model per se (e.g. the assumption that the value of liabilities remains unaltered even if the market value of assets may double), mappings from credit ratings to default probabilities are typically obtained using corporate bond rating histories over many years. The reason is simple: default events for investment grade firms are quite rare: less than 0.5% per year. However, there is substantial evidence that default rates are tied to the business cycle (Nickell, Perraudin and Varotto (2000), Bangia et al. (2002)). The difficult task of endogenizing the default threshold is a fruitful area for future research.

10.2 Expected Loss Due to Default

Given the value change process for firm j, defined by (10.4), and the default threshold, $\hat{c}_{ji}$, we now consider the conditions under which the firm goes bankrupt and is thus no longer able to repay its debt obligations. Specifically, we need to define the expected loss to firm j at time T given information available to the lender (e.g. a bank) at time T, which we denote by $\mathcal{I}_T$. Default occurs when the firm's value (return) falls below some threshold $\hat{c}_{ji}$ (e.g. when the value of a firm's assets falls below the value of its callable liabilities). Expected loss at time T, $E_T(L_{ji,T+1}) = E(L_{ji,T+1} \mid \mathcal{I}_T)$, is given by

$$\begin{aligned} E_T(L_{ji,T+1}) &= \Pr(r_{ji,T+1} < \hat{c}_{ji} \mid \mathcal{I}_T)\, E_T(\mathcal{X}_{ji,T+1}) E_T(\mathcal{S}_{ji,T+1}) \\ &\quad + [1 - \Pr(r_{ij,T+1} < \hat{c}_{ji} \mid \mathcal{I}_T)] \times \tilde{L} \end{aligned} \tag{10.10}$$

where $\hat{c}_{ji}$ is given by (10.8), $\mathcal{X}_{ji,T+1}$ is the maximum loss exposure assuming no recoveries for company j in region i (typically the face value of the loan)

[55] While the standard Merton model assumes liability growth to be zero, the adapted version can incorporate other growth rates. Still, assuming constant liability growth may be more realistic than allowing for no fluctuation of liability values at all.

[56] This approach is taken by KMV to generate what they call *EDF*s (expected default frequencies).

and is known at time T, $\mathcal{S}_{ji,T+1}$ is the percentage of exposure which cannot be recovered in the event of default, and $\tilde{L}$ is some future loss in the event of non-default at $T+1$ (which we set to zero for simplicity).[57] Typically $\mathcal{S}_{ji,T+1}$ is not known at time of default and will be treated as a random variable over the range $[0,1]$. In the empirical application we assume that $\mathcal{S}_{ji,T+1}$ are draws from a beta distribution with given mean and variance calibrated to (pooled) historical data on default severity. Substituting (10.4) into (10.10) and setting $\tilde{L}$ to zero we now obtain:

$$
\begin{aligned}
E_T\left(L_{ji,T+1}\right) &= \Pr\left(\alpha_{ji}+\boldsymbol{\beta}_{ji}'\Delta\mathbf{x}_{i,T+1}+\boldsymbol{\gamma}_{ji}'\Delta\mathbf{d}_{T+1}+\eta_{ji,T+1}<\hat{c}_{ji}\mid\mathcal{I}_T\right) \\
&\quad E_t(\mathcal{X}_{ji,T+1})E_t(\mathcal{S}_{ji,T+1}) \qquad (10.11)
\end{aligned}
$$

To compute the *conditional* default probability,

$$
\pi_{ji,T}=\Pr\left(\alpha_{ji}+\boldsymbol{\beta}_{ji}'\Delta\mathbf{x}_{i,T+1}+\boldsymbol{\gamma}_{ji}'\Delta\mathbf{d}_{T+1}+\eta_{ji,T+1}<\hat{c}_{ji}\mid\mathcal{I}_T\right), \qquad (10.12)
$$

we make use of the solution to the GVAR model given by (5.3) and (5.4), and note that

$$
\Delta\mathbf{x}_{i,T+1}=\mathbf{S}_i\left[\mathbf{b}_0+\mathbf{b}_1\left(T+1\right)-\left(\mathbf{I}_k-\digamma\right)\mathbf{x}_T+\boldsymbol{\Upsilon}_0\Delta\mathbf{d}_{T+1}+\left(\boldsymbol{\Upsilon}_0+\boldsymbol{\Upsilon}_1\right)\mathbf{d}_T+\mathbf{G}^{-\mathbf{1}}\boldsymbol{\varepsilon}_{T+1}\right],
$$

where $\mathbf{S}_i$ is a $k_i\times k$ selection matrix such that $\mathbf{x}_{it}=\mathbf{S}_i\mathbf{x}_T$. In the case where the macroeconomic variables are stacked by countries, as in $\mathbf{x}_T=(\mathbf{x}_{0T}',\mathbf{x}_{1T}',....,\mathbf{x}_{NT}')'$, then $\mathbf{S}_i=\left(\mathbf{0}_{k_0},..,\mathbf{0}_{k_{i-1}},\mathbf{I}_{k_i},\mathbf{0}_{k_{i+1}},..,\mathbf{0}_{k_N}\right)$. To take account of the uncertainty associated with the global exogenous variables, $\mathbf{d}_{T+1}$, we adopt the autoregressive specification defined by (6.5) and note that

$$
\Delta\mathbf{d}_{T+1}=\boldsymbol{\mu}_d-\left(\mathbf{I}_s-\boldsymbol{\Phi}_d\right)\mathbf{d}_T+\boldsymbol{\varepsilon}_{d,T+1}. \qquad (10.13)
$$

Hence

$$
\Delta\mathbf{x}_{i,T+1}=\mathbf{S}_i\left[\begin{array}{c}\mathbf{b}_0+\boldsymbol{\Upsilon}_0\boldsymbol{\mu}_d+\mathbf{b}_1\left(T+1\right)-\left(\mathbf{I}_s-\digamma\right)\mathbf{x}_T+\left(\boldsymbol{\Upsilon}_0\boldsymbol{\Phi}_d+\boldsymbol{\Upsilon}_1\right)\mathbf{d}_T\\ +\boldsymbol{\Upsilon}_0\boldsymbol{\varepsilon}_{d,T+1}+\mathbf{G}^{-\mathbf{1}}\boldsymbol{\varepsilon}_{T+1}\end{array}\right], \qquad (10.14)
$$

where $\boldsymbol{\varepsilon}_{d,T+1}\backsim i.i.d.\ (\mathbf{0},\boldsymbol{\Sigma}_d)$, and by assumption is distributed independently of the macroeconomic shocks, $\boldsymbol{\varepsilon}_{T+1}$, and the firm's idiosyncratic shock, $\eta_{ji,T+1}$. Using this result in (10.12) and after some simplifications we have

$$
\pi_{ji,T}=\Pr\left(\xi_{ji,T+1}<\hat{c}_{ji}-\mu_{ji,T}\mid\mathcal{I}_T\right), \qquad (10.15)
$$

[57] One would expect loss severity to be higher in recessions than expansions. Bankruptcies are pro-cyclical, flooding the market with distressed assets which drive down their price (or increasing severity). However, for simplicity we are assuming that exposure and severity are independently distributed.

where

$$\xi_{ji,T+1} = \eta_{ji,T+1} + \boldsymbol{\theta}'_{ji}\boldsymbol{\varepsilon}_{T+1} + \boldsymbol{\theta}'_{ji,d}\boldsymbol{\varepsilon}_{d,T+1}, \tag{10.16}$$

$$\boldsymbol{\theta}'_{ji} = \boldsymbol{\beta}'_{ji}\mathbf{S}_i\mathbf{G}^{-1},\ \boldsymbol{\theta}'_{ji,d} = \boldsymbol{\gamma}'_{ji} + \boldsymbol{\beta}'_{ji}\mathbf{S}_i\boldsymbol{\Upsilon}_0, \tag{10.17}$$

and

$$\begin{aligned}\mu_{ji,T} \quad = \quad & \alpha_{ji} + \boldsymbol{\gamma}'_{ji}\boldsymbol{\mu}_d + \boldsymbol{\beta}'_{ji}\mathbf{S}_i\left(\mathbf{b}_0 + \mathbf{b}_1 + \boldsymbol{\Upsilon}_0\boldsymbol{\mu}_d\right) \qquad (10.18) \\ & + \boldsymbol{\beta}'_{ji}\mathbf{S}_i\left[\mathbf{b}_1 T - \left(\mathbf{I}_k - \digamma\right)\mathbf{x}_T\right] + \left[\boldsymbol{\beta}'_{ji}\mathbf{S}_i\left(\boldsymbol{\Upsilon}_0\boldsymbol{\Phi}_d + \boldsymbol{\Upsilon}_1\right) - \boldsymbol{\gamma}'_{ji}\left(\mathbf{I}_s - \boldsymbol{\Phi}_d\right)\right]\mathbf{d}_T.\end{aligned}$$

Therefore, there are three types of shocks that affect firm's probability of default: its own shock, $\eta_{ji,T+1}$, macroeconomic shocks, $\boldsymbol{\varepsilon}_{T+1}$, and the global exogenous shock, $\boldsymbol{\varepsilon}_{d,T+1}$ (in our model the oil price shock). Note that although the firm in question operates in country/region i, its probability of default could be affected by macroeconomic shocks worldwide. Under the assumption that all these shocks are jointly normally distributed and the parameter values are given, we have the following expression for the probability of default over T to $T+1$ formed at T[58]

$$\pi_{ji,T} = \Phi\left[\frac{\hat{c}_{ji} - \mu_{ji,T}}{\sqrt{Var\left(\xi_{ji,T+1} \mid \mathcal{I}_T\right)}}\right], \tag{10.19}$$

where

$$Var\left(\xi_{ji,T+1} \mid \mathcal{I}_T\right) \equiv \omega^2_{\xi,ji} = \omega^2_{\eta,ji} + \boldsymbol{\theta}'_{ji}\boldsymbol{\Sigma}\boldsymbol{\theta}_{ji} + \boldsymbol{\theta}'_{ji,d}\boldsymbol{\Sigma}_d\boldsymbol{\theta}_{ji,d}. \tag{10.20}$$

Both of these restrictions (given parameter values and joint normality) can be relaxed. Parameter uncertainty can be taken into account by integrating out the unknown parameters using their posterior or predictive likelihoods, as in Garratt et al. (2002). In the presence of non-normal shocks one could also employ non-parametric stochastic simulation techniques by re-sampling from the residuals of the GVAR model to estimate $\pi_{ji,T}$. These and other related developments are beyond the scope of the present application, which is primarily intended as an illustration of the use of the GVAR modeling approach in credit risk analysis.

The expected loss due to default of a loan (credit) portfolio can now be computed by aggregation of the expected losses across the different loans. Denoting the loss of a loan portfolio over the period T to $T+1$ by L_{T+1} we have

$$E_T\left(L_{T+1}\right) = \sum_{i=0}^{N}\sum_{j=1}^{nc_i}\pi_{ji,T}E_T(\mathcal{X}_{ji,T+1})E_T(\mathcal{S}_{ji,T+1}), \tag{10.21}$$

[58] Joint normality is sufficient but not necessary for $\xi_{ji,t+1}$ to be approximately normally distributed. This is due to the fact that $\xi_{ji,t+1}$ is a linear function of a large number of weakly correlated shocks (63 in our particular application).

where nc_i is the number of obligors (could be zero) in the bank's loan portfolio resident in country/region i.

10.3 Simulation of the Loss Distribution

The expected loss as well as the loss distribution can also be computed by stochastic simulation using draws from the joint distribution of the shocks, $\epsilon_{T+1} = (\boldsymbol{\eta}'_{T+1}, \boldsymbol{\varepsilon}'_{T+1}, \boldsymbol{\varepsilon}'_{d,T+1})'$, where $\boldsymbol{\eta}_{T+1}$ is the vector of firm-specific shocks. As noted earlier these draws could either be carried out parametrically from normal or t-distributed random variables, or if sufficient data points are available can be implemented non-parametrically using re-sampling techniques. Under the parametric specification the variance covariance matrix of ϵ_{T+1} is given by

$$Cov\left(\epsilon_{T+1}\right) = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Theta \end{pmatrix}, \tag{10.22}$$

where Θ is a diagonal matrix with elements ω_{ji}^2, $j = 1, 2, ..., nc_i$, $i = 0, 1, ..., N$.

Denote the r^{th} draw of this vector by $\epsilon_{T+1}^{(r)}$, and compute the firm-specific return, $r_{ij,T+1}^{(r)}$, noting that

$$r_{ij,T+1}^{(r)} = \mu_{ji,T} + \xi_{ji,T+1}^{(r)}, \tag{10.23}$$

where $\mu_{ji,T}$ is given by (10.18) and

$$\xi_{ji,T+1}^{(r)} = \eta_{ji,T+1}^{(r)} + \boldsymbol{\theta}'_{ji}\boldsymbol{\varepsilon}_{T+1}^{(r)} + \boldsymbol{\theta}'_{ji,d}\boldsymbol{\varepsilon}_{d,T+1}^{(r)}. \tag{10.24}$$

Then simulate the loss in period $T+1$ using (known) loan face values, say $FV_{ji,T}$, as exposures, and draws from a beta distribution for severities (as described above):

$$L_{T+1}^{(r)} = \sum_{i=0}^{N}\sum_{j=1}^{nc_i} I\left(r_{ij,T+1}^{(r)} < \hat{c}_{ji}\right) FV_{ji,T}\mathcal{S}_{ji,T+1}^{(r)}. \tag{10.25}$$

The simulated expected loss due to default is given by (using R replications)

$$\bar{L}_{R,T+1} = \frac{1}{R}\sum_{r=1}^{R} L_{T+1}^{(r)}. \tag{10.26}$$

When $\epsilon_{T+1}^{(r)}$ are drawn from a multivariate normal distribution with a covariance matrix given by (10.22), then

$$\bar{L}_{R,T+1} \xrightarrow{p} E_T\left(L_{T+1}\right), \text{ as } R \to \infty.$$

The simulated loss distribution is given by ordered values of $L_{T+1}^{(r)}$, for $r = 1, 2, ..., R$. For a desired percentile, for example the 99%, and a given number of replications, say $R = 10,000$, credit value at risk is given as the 100^{th} highest loss.

10.4 Expected Loss Given Shocks

In credit risk analysis we may also be interested in evaluating quantitatively the relative importance of changes in different macroeconomic factors on the loss distribution. To this end the loss distribution conditional on a given shock can be compared to a baseline distribution without such a shock. As with all counterfactual experiments it is important that the effects of the shock on other macroeconomic factors are clearly specified. One possibility would be to assume that the other factors are displaced according to their historical covariances with the variable being shocked. This is in line with the GIRF analysis discussed above. In this set-up if factor ℓ in country i is shocked by one standard error (i.e. $\sqrt{\sigma_{ii,\ell\ell}}$) in the period from T to $T+1$, the vector of the macroeconomic factors would be displaced by

$$\frac{1}{\sqrt{\sigma_{ii,\ell\ell}}}\mathbf{G}^{-1}\boldsymbol{\Sigma}\mathfrak{s}_\ell,$$

where as before $\mathfrak{s}_\ell$ is a $k\times 1$ selection vector with its element corresponding to the ℓ^{th} variable in country i being unity and zeros elsewhere. Such a shock has no effect on the global exogenous variables and the firm-specific shocks. In the absence of any macroeconomic shocks, namely when $\boldsymbol{\varepsilon}_{T+1}=\mathbf{0}$, firm-specific returns are given by

$$r^0_{ij,T+1}=\mu_{ji,T}+\eta_{ji,T+1}+\boldsymbol{\theta}'_{ji,d}\varepsilon_{d,T+1},$$

and with a one standard error shock to $x_{i,T+1,\ell}$ we have (see (10.22) and (10.23))

$$r^\ell_{ij,T+1}=\mu_{ji,T}+\eta_{ji,T+1}+\frac{1}{\sqrt{\sigma_{ii,\ell\ell}}}\boldsymbol{\theta}'_{ji}\boldsymbol{\Sigma}\mathfrak{s}_\ell+\boldsymbol{\theta}'_{ji,d}\varepsilon_{d,T+1}.$$

The loss distributions associated with these two scenarios can now be simulated using these returns in (10.25).

The above counterfactual, while of some interest, will underestimate the expected loss under both shock scenarios since it abstracts from volatility of the macroeconomic factors and assumes that the return variance stays constant under the shock. To allow for the volatility of macroeconomic factors in the analysis consider the case where $\boldsymbol{\theta}'_{ji}\boldsymbol{\varepsilon}_{T+1}$ and $\varepsilon_{i,T+1,\ell}=\mathfrak{s}'_\ell\boldsymbol{\varepsilon}_{T+1}$ are jointly normally distributed. It is then easily seen that

$$\boldsymbol{\theta}'_{ji}\boldsymbol{\varepsilon}_{T+1}\left|\varepsilon_{i,T+1,\ell}=\sqrt{\sigma_{ii,\ell\ell}}\right.\sim IIN\left(\frac{1}{\sqrt{\sigma_{ii,\ell\ell}}}\boldsymbol{\theta}'_{ji}\boldsymbol{\Sigma}\mathfrak{s}_\ell,\ \omega^2_{ji,\ell}\right),$$

where[59]

$$\omega^2_{ji,\ell}=\boldsymbol{\theta}'_{ji}\boldsymbol{\Sigma}\boldsymbol{\theta}_{ji}-\boldsymbol{\theta}'_{ji}\boldsymbol{\Sigma}\mathfrak{s}_\ell\left(\mathfrak{s}'_\ell\boldsymbol{\Sigma}\mathfrak{s}_\ell\right)^{-1}\mathfrak{s}'_\ell\boldsymbol{\Sigma}\boldsymbol{\theta}_{ji}. \tag{10.27}$$

[59] Note that $\mathfrak{s}'_\ell\boldsymbol{\Sigma}\mathfrak{s}_\ell=\sigma_{ii,\ell\ell}$.

Also, recalling that $\boldsymbol{\varepsilon}_{T+1}$, $\boldsymbol{\varepsilon}_{d,T+1}$, and $\boldsymbol{\eta}_{T+1}$ are independently distributed it is then easily seen that

$$\xi_{ij,T+1} \left| \varepsilon_{i,T+1,\ell} = \sqrt{\sigma_{ii,\ell\ell}} \sim IIN \left(\frac{1}{\sqrt{\sigma_{ii,\ell\ell}}} \boldsymbol{\theta}'_{ji} \boldsymbol{\Sigma} \mathfrak{s}_\ell, \omega^2_{\xi,ji,\ell} \right) \right. \tag{10.28}$$

where

$$\omega^2_{\xi,ji,\ell} = \omega^2_{\eta,ji} + \omega^2_{ji,\ell} + \boldsymbol{\theta}'_{ji,d} \boldsymbol{\Sigma}_d \boldsymbol{\theta}_{ji,d}, \tag{10.29}$$

and $\omega^2_{ji,\ell}$ is already defined by (10.27).

Therefore, to allow for volatility of the shocks (macroeconomic as well as idiosyncratic shocks), the simulation of the loss distribution needs to be carried out using the draws

$$r^{l,(r)}_{ij,T+1} = \mu_{ji,T} + \frac{1}{\sqrt{\sigma_{ii,\ell\ell}}} \boldsymbol{\theta}'_{ji} \boldsymbol{\Sigma} \mathfrak{s}_\ell + \omega_{\xi,ji,\ell} \ \mathcal{Z}^{(r)} \tag{10.30}$$

where $\mathcal{Z}^{(r)} \sim IIN\,(0,1)$. The baseline loss distribution in this case can also be simulated directly using the draws

$$r^{(r)}_{ij,T+1} = \mu_{ji,T} + \omega_{\xi,ji} \mathcal{Z}^{(r)}. \tag{10.31}$$

Default occurs if the r^{th} simulated return, baseline ($r^{(r)}_{ij,T+1}$) or shock-conditional ($r^{l,(r)}_{ij,T+1}$), falls below the threshold $\hat{c}_{ji}$:

$$\begin{aligned} \text{Baseline:} \qquad & r^{(r)}_{ij,T+1} < \hat{c}_{ji} \Longrightarrow \text{ Default}, \\ \text{Shock-Conditional:} \qquad & r^{l,(r)}_{ij,T+1} < \hat{c}_{ji} \Longrightarrow \text{ Default}. \end{aligned} \tag{10.32}$$

Using these results in (10.25) the loss distribution can be simulated for any desired level of accuracy by selecting R, the number of replications, to be sufficiently large.

Finally, it might also be of interest to compare the base line default probability, $\pi_{ji,T}$, given by (10.19) with the default probability that results under the shock to $x_{i,T+1,\ell}$, which we denote by $\pi_{ji,\ell,T}$. We have

$$\pi_{ji,T} = \Phi \left(\frac{\hat{c}_{ji} - \mu_{ji,T}}{\omega_{\xi,ji}} \right),$$

and

$$\pi_{ji,\ell,T} = \Phi \left(\frac{\hat{c}_{ji} - \mu_{ji,T} - \frac{1}{\sqrt{\sigma_{ii,\ell\ell}}} \boldsymbol{\theta}'_{ji} \boldsymbol{\Sigma} \mathfrak{s}_\ell}{\omega_{\xi,ji,\ell}} \right). \tag{10.33}$$

where $\omega_{\xi,ji}$ and $\omega_{\xi,ji,\ell}$ are defined by (10.20) and (10.29), respectively.

10.5 Results

10.5.1 The Sample Portfolio

We analyze the effects of economic shocks on a hypothetical sample of large-corporate loan portfolio comprised of 119 companies, dispersed over 10 regions, with a current face value of \$1bn.[60] Table 11 provides the individual company details, with a summary by regions given in Table 12. The column to the right indicates the inception of the equity series available for APT-type regression analysis. We wanted to mimic (broadly) the portfolio of a large, internationally active bank. Arbitrarily picking Germany as the bank's domicile country, the portfolio is relatively more exposed to German firms than would be the case if exposure were allocated purely on a GDP share (in our "world" of 26 countries). For the remaining regions, exposure was more in line with GDP share. Within a region, loan exposure is randomly assigned. The expected severity for loans to U.S. companies is the lowest at 20%, based upon studies by Citibank, Fitch Investor Service and Moody's Investor Service.[61] All other severities are based on assumptions, reflecting the idea that severities are higher in less developed countries. Table 11 gives the portfolio composition, regional weights, individual exposures and expected (μ_β) and unexpected (σ_β) severities.[62]

[60] We restricted ourselves to major, publicly traded firms which had a credit rating from either Moody's or S&P. Thus, for example, Chinese companies are not included for lack of a credit rating. Further details are provided in PSTW.

[61] As cited in Saunders and Allen (2002).

[62] Mean severity is assumed to be slightly lower in Germany (as compared to France or U.K., for example), since Germany is taken to be the bank's domicile country and hence the bank may have some local advantages in the recovery of distressed assets. Unexpected severity refers to standard deviation of severity distribution assumed here to be Beta distributed.

Table 12
The Composition of the Sample Portfolio for Regions

Region	# Obligors	Equity Series[1] Quarterly	Credit Rating[2] Range	Portfolio Per cent	Severity[3] Mean	S.D.
					(μ_β)	(σ_β)
U.S.	14	79Q1 - 99Q1	AAA to BBB-	20	20%	10%
U.K.	9	79Q1 - 99Q1	AA to BBB+	6	35%	15%
Germany	18	79Q1 - 99Q1	AAA to BBB-	21	30%	15%
France	8	79Q1 - 99Q1	AA to BBB	8	35%	15%
Italy	6	79Q1 - 99Q1	A to BBB-	8	35%	15%
W. Europe	12	79Q1 - 99Q1	AAA to BBB+	8	35%	15%
Middle East	4	90Q3 - 99Q1	B-	2	60%	20%
S.E. Asia	23	89Q3 - 99Q1	A to B	10	50%	20%
Japan	13	79Q1 - 99Q1	AAA to B+	10	35%	15%
L. America	12	89Q3 - 99Q1	A to B-	5	65%	20%
Total	119	-	-	100	-	-

1. Equity prices of companies in emerging markets are not available over the full sample period used for the estimation horizon of the GVAR.
2. The sample contains a mix of Moody's and S&P ratings, although S&P rating nomenclature is used for convenience.
3. Severity is drawn from a beta distribution with mean μ_β and standard deviation σ_β.

10.6 Conditional Loss Distributions

The systematic risk in our model is captured empirically through the APT regressions where firm returns are regressed on changes in all domestic variables and oil prices. Around 80% of those regressions were significant (using the F-test) at the 5% level, with real equity prices being the most important (statistically) regressor, followed by the oil price and the real exchange rate variables.[63]

We then generated loss distributions for two different horizons: one-quarter and four-quarters ahead. A one year horizon is typical for credit risk management and thus of particular interest. For each horizon we examined the impact of several shock scenarios including those presented in Section 9. They are[64]

[63] More formal model selection criteria are explored in PSTW, where we include foreign (starred) variables.
[64] 2.33 σ corresponds, in the Gaussian case, to the 99% Value-at-Risk, a typical range in

- a –2.33 σ shock to U.S. equity, corresponding to a quarterly drop of 14.28%
- a +2.33 σ shock to real German output corresponding to a quarterly rise of 2.17%
- a –2.33 σ shock to SEA equity corresponding to a quarterly drop of 24.77%

In addition we present a symmetric positive shock to SEA equity prices – but the impact on losses will not be symmetric.

We generated 50,000 simulations for each case.[65] For the one-quarter ahead forecast and shock scenarios, we computed expected loss results, both theoretical (using (10.21)) and simulated (10.26). The two sets of estimates turn out to be very close indeed so we only report the simulated ones. The simulated expected loss results together with the unexpected counterparts (*S.D.*) are summarized in Table 13.

Table 13
Simulated Mean and Standard Deviation of Losses for One-Quarter and Four-Quarters Ahead (in Basis Points Exposure)

Shock Scenarios	One-Quarter Ahead		Four-Quarter Ahead	
	Mean	S.D.	Mean	S.D.
-2.33σ U.S. Equity	4.37	12.57	8.71	17.33
-2.33σ SEA Equity	3.24	11.15	7.53	16.47
Baseline	2.45	9.52	6.43	15.05
+2.33σ German Output	2.38	9.37	6.40	14.98
+2.33σ SEA Equity	2.14	8.77	5.87	14.22

The U.S. equity price shock seems rather severe at first: expected loss is nearly double than what is expected under the baseline (no shock) scenario while unexpected loss (i.e. the loss standard deviation) is about one-third higher. By the time one moves to the tail (99% and beyond) of the loss distribution (Figure 8), the absolute differences are less pronounced.

[Insert Figure 8 about here]

For the baseline, there is a 1% chance of losing about 49.7bp of the face value of the portfolio after one quarter, while conditional on the -2.33σ U.S. real

risk management.

[65] To ensure convergence, we also performed simulations up to 200,000 runs; the results were indistinguishable.

equity price shock the loss is closer to 53.8bp. The two loss scenarios diverge further out in the tail such that at the 99.7% level, with a loss of about 55.7bp for the baseline and 68.0bp for the U.S. equity price shock scenario, only to converge again past the 99.8% level (and re-diverge past about 99.9%).[66] This nonlinearity is a direct consequence of the nonlinearity of the credit risk model which can only be uncovered in the loss distribution through simulation. The positive German output shock has little bearing on the loss distribution, either in terms of expected and unexpected loss, or even the shape of the loss distribution itself. In fact, the positive shock to S.E. Asian real equity prices is more beneficial. Thus from the perspective of a German risk manager, the perspective we are trying to mimic, given this portfolio, positive shocks to German output are less cause for excitement than positive shocks to S.E. Asian equity prices.

Symmetric shocks do not translate to symmetric loss outcomes. The loss curve in Figure 8 for the negative S.E. Asian equity shock lies further above the baseline than the positive equity shock curve lies below it. This also is a result of the nonlinear credit loss model.

The four quarter loss distribution was generated one quarter at a time sequentially with defaulted firms in each replication being eliminated from the portfolio for the second quarter simulations, and so on. Therefore, the default process and the resulting loss distribution are path-dependent. Mean and standard deviation of the annual simulated loss distributions are presented in the second panel of Table 13. The loss distributions for the baseline and the four shock scenarios are displayed in Figure 9.

[Insert Figure 9 about here]

The expected loss for the U.S. equity shock scenario is now about one-third higher than the baseline at the four quarter horizon, and the pattern of the loss curves are broadly in line with the curves for the one-quarter losses, except that the loss distributions for the favorable shocks are now relatively closer to the baseline distribution. The four-quarter is also somewhat smoother then the one-quarter loss distribution, lacking the "elbow" in the 99.7 to 99.8% range.

What might be the impact on losses of a severe shock, say to U.S. equity prices? From their peak in 2000 to a recent low (in early October, 2002), the S&P500 has dropped about 49%. That also corresponds to the largest quarterly drop in the index since 1928 (which occurred during February to April of 1932). Such a large drop corresponds to 8.02σ, and the impact on the loss distribution

[66] As simulations for far tail events are increasingly less reliable the further out into the tail one goes, the numbers should be interpreted with some care.

of our portfolio can be seen in Figure 10, where we present the one-quarter ahead loss distribution of this stress scenario and the baseline; we also include the previous, less severe, U.S. equity shock plus an intermediate shock of -5σ for comparison.

[Insert Figure 10 about here]

Indeed, such a shock would result in rather large losses. We would expect to lose 151.4bp (or 1.5%) of total loan exposure, and there is a 1% chance that 3.55% of the portfolio would be wiped out. Note that total U.S. exposure in the loan book is 20%. The non-linear impact of shocks on losses is quite pronounced: the -8.02σ shock is only 60% higher than the -5σ, in units of σ, of course, but the unexpected loss after one quarter is more than double (73.0bp vs. 23.6bp) and the 99% loss $3\frac{1}{2}$ time as much (3.55% vs. 1.02%).

11 Concluding Remarks

In this paper, we develop an operational framework for global macroeconomic modeling. Our approach aggregates regional cointegrated systems into a unified global system. We demonstrate the feasibility of this approach by linking up eleven separate vector error-correcting regional models estimated using quarterly observations over the period 1979Q1-1999Q1. Each of the regional models contain foreign variables that are weighted averages of the domestic variables for other regions, constructed to match the international trade pattern of the country under consideration. The individual country models are then combined in a consistent and cohesive manner to generate forecasts for *all* the variables in the world economy simultaneously.

This resultant model is shown also to be error-correcting with dampened cyclical properties. We outline conditions of weak exogeneity of the foreign variables, a key assumption of the model. We then test these conditions, where we include the global variable (price of oil) in the exogeneity regressions as well. Of the 63 exogeneity tests carried out, only 3 are statistically significant at the 5% level and none at the 3% level. Finally, using generalized impulse response analysis, we examine the propagation of shocks across factors and regions.

The focus of the model is very much on constructing a compact and coherent representation of factor and regional interdependencies, while tackling the problem of limited data in large-scale models such as these. Our model allows for interaction amongst the different economies through three separate but interrelated channels:

1. Direct dependence of the relevant macro-factors on their region-specific foreign counterparts and their lagged values;

2. Dependence of the region-specific variables on common global weakly exogenous variables such as oil prices and possibly other variables controlling for major global political events;

3. Certain degree of dependence of idiosyncratic shocks across regions as captured via the cross-region covariances.

Thus, for instance, we are able to account for inter-linkages between equity market movements in South East Asia and output in Germany. The use of our regional weighting scheme allows for efficient use of all available data.

The original motivation for developing this model was the need for a macro-based risk management tool for commercial, and perhaps even central, banks. By engaging in commercial lending to companies whose fortunes fluctuate with aggregate demand, a bank is ultimately exposed to macroeconomic fluctuations. This can be mitigated through international diversification. However, precisely because economic fluctuations are correlated across factors and countries, it fosters the need for a compact global macroeconometric model which explicitly allows for such interdependencies. To demonstrate the value in constructing such a model as a basis for portfolio risk management, we use a simplified version of a Merton-style credit risk model, developed fully in Pesaran, Schuermann, Treutler and Weiner (2003), which has explicit links to the macroeconomic factors in the GVAR model, thereby allowing us to generate scenario-based loss distributions for a credit portfolio. Using a portfolio of loans to 119 firms in ten of the eleven regions (China was left out due to poorly developed equity markets), we generated loss distributions for one and four quarters ahead, both under a baseline forecast as well as under a set of shock scenarios. The simulated losses are shown to converge quickly to their analytical counterparts. We show that symmetric shocks do not result in symmetric loss outcomes due to the nonlinearity of the credit model. Our results may be thought of as demonstrating the value of hedging credit risk with market risk, an idea that is quickly gaining traction amongst practitioners today.

Because of the focus on modeling interlinkages, the model can readily be used to shed light on the analysis of a variety of transmission mechanisms, contagion effects, and testing of long-run theories (for instance, purchasing power parity) in a global as well as other settings. Several other applications of our methodology come to mind:

- "New economic geography": a literature which sets the stage for explicitly incorporating geography into the models of economic activity through either domestic or international trade (see Krugman, 1993, for an introduction to the topic, and Fujita, Krugman and Venables, 1999, for a more formal treatment)

- Regional and urban economics: models of inter-regional linkages, either through city-suburb economic ties (Voith, 1998) or linkages between cities as in the "systems of cities" literature (Henderson, 1988)

- Labor mobility: consider a longer horizon, lower frequency issue of labor mobility responding to regional economic shocks; for instance, auto workers migrating from Michigan to Texas in response to oil-price shocks in the early 1980s (Blanchard and Katz, 1992).

This list is by no means exhaustive and is designed to stimulate interest, and research, of applying the GVAR framework to problems of modeling economic inter-linkages.

A Data Appendix

A.1 Variables and Data Sources

The primary variables (disaggregated by country/region when applicable) used in this study are:

Y :	Gross Domestic Product (GDP)
P :	General Price Index
Q :	Equity Price Index
E :	Exchange Rate
R :	Interest Rate
M :	Money Supply
PO :	Oil Price

A.2 Output (GDP)

The source for all 27 countries is the International Monetary Fund's International Financial Statistics (IFS) GDP (1990) series. France, Germany, Italy, Japan, Mexico, the Netherlands, Spain, Switzerland, UK and USA are all from series BR, and the remaining countries are from series BP.

Where quarterly data were not available (ie, for Brazil, China, Indonesia, Kuwait, Malaysia, Saudi Arabia, Singapore, Thailand and Turkey), quarterly series were interpolated linearly from the annual series. For Singapore, Malaysia and Thailand, interpolated series were used only during the periods 1979-1992, 1979-1996 and 1979-1995, respectively. Quarterly output series were available for the subsequent periods.

For the period before German reunification, in 1990Q4, West German growth rates were used. The growth rate from 1988Q3 to 1990Q3 was used to compute a 'unified' output series for 1990Q4.

The data for Kuwait and Peru were rebased to 1990 using CPI for those countries.

The data for Argentina and Singapore were seasonally adjusted.

A.3 General Price Indices

The data source for all countries except China was the IFS Consumer Price Index Series '64'. A full sample was available for all countries except Brazil, where 1979 data was unavailable, and a backcast using the average growth rate of prices for 1980 was employed.

A.4 Equity Price Indices

There were no data for China or Saudi Arabia. For Austria we used Morgan Stanley Capital International (MSCI) series.

For Belgium, Indonesia, Italy, Malaysia, Singapore, Spain, Switzerland, Thailand, and Turkey we used Datastream, using quarterly averages from daily observations. However, we used quarterly average of weekly datapoints, as opposed to daily observations, for Argentina. The data for Malaysia was market cap weighted.

We used IFS data for Brazil, Chile, France, Germany, Japan, Korea, Mexico, The Netherlands, Peru, Philippines, UK and USA. Indices for share prices (IFS code "62") generally related to common shares of companies traded on national or foreign stock exchanges. Monthly indices were obtained as simple arithmetic averages of the daily or weekly observations ("ZF").

These nominal equity price indices were deflated by the non-seasonally adjusted general price indices. The resultant real series were then adjusted for(possible) seasonal variations.

A.5 Exchange Rates

IFS series 'rf' was used for all countries.

A.6 Interest Rates

Interest Rate data was taken from IFS Series '60B', the money market rate, with the following exceptions: for Argentina, Chile, China, Saudi Arabia and Turkey we used the IFS deposit rate; for Peru we used the IFS discount rate; for the Philippines we used the IFS Treasury rate.

A.7 Money Supply

The Money Supply data source for all countries was the sum of IFS series 34 (money) and series 35 (quasi-money). All series were seasonally adjusted. The data for Argentina, Brazil, Peru and Turkey required a decimal place adjustment to make the Money:GDP ratio reasonable.

For Belgium, we used quarterly data for all quasi-money; for money we used annual data converted to quarterly through interpolation up to 1990, and quarterly data from 1990Q4 to 1999Q1.

We used annual data converted to quarterly through interpolation for the Philippines; for the Philippines this was necessary for the period 1984-1986 only since quarterly data were available thereafter. There were no quarterly data for Saudi Arabia for 1983, and therefore annual data were used for that year.

A.8 Oil Price Index

For oil prices we used monthly averages of Brent Crude series from Datastream.

B Construction of Regional Data Series: Domestic and Foreign

Time series observations at the regional level were constructed as weighted averages of corresponding country-specific series as set out in equations (A.1) and (A.2) below. Specifically, the regional variables are constructed from country-specific variables using the following (logarithmic) weighted averages[67]

$$y_{it} = \sum_{\ell=1}^{N_i} w_{i\ell}^0 y_{i\ell t}, \; p_{it} = \sum_{\ell=1}^{N_i} w_{i\ell}^0 p_{i\ell t}, \; q_{it} = \sum_{\ell=1}^{N_i} w_{i\ell}^0 q_{i\ell t}, \tag{A.1}$$

$$e_{it} = \sum_{\ell=1}^{N_i} w_{i\ell}^0 e_{i\ell t}, \; \rho_{it} = \sum_{\ell=1}^{N_i} w_{i\ell}^0 \rho_{i\ell t}, \; m_{it} = \sum_{\ell=1}^{N_i} w_{i\ell}^0 m_{i\ell t}. \tag{A.2}$$

Notice that in constructing the regional variables $y_{it}, p_{it}, e_{it}, \ldots$ from the country-specific variables $y_{i\ell t}, p_{i\ell t}, e_{i\ell t}, \ldots$ one simply needs to use country-specific variables measured in their domestic currencies. Notice that $e_{i\ell t}$ stands for the exchange rate of country ℓ in region i, in terms of US dollars.

For weights we used the GDP shares of each country in the region, computed as that country's PPP-adjusted GDP divided by the total PPP/USD GDP of the region. In order to avoid the use of time-varying weights, we choose a relatively recent time period for which PPP data is available, namely 1996.

Not all time series were available for all countries over the entire sample period. As a result the composition of the regional series is allowed to change as data on specific countries become available. For example, if data is not available for a given country over the first few periods in the sample, a zero weight is attached to this country with the weights of the remaining countries in the region adjusted to ensure that the sum of the weights add up to unity. Once data becomes available for the country in question, the weights are redistributed and the new information is 'folded into' the dataset.

[67] The weights $w_{i\ell}^0$ could be changed at fixed time intervals, say every 5 years, in order to capture secular changes in the composition of the regional output. However, changing these weights too frequently could mask the cyclical movements of the regional output being measured.

Foreign variables are constructed uniquely for each region. For example, foreign money supply m^* for Western Europe is different from m^* for Latin America. We use the trade shares to appropriately weight the influence of foreign regions on a specified region's economy. Using an inter-regional trade matrix, we first compute the trade shares for each region with a given country (eg. the percent of Argentina's trade originating from the Western Europe), and then aggregate across countries based on the trade weights of the countries within the region.

The weights used to aggregate, across countries, the foreign variables need to be constructed with care. Since each *starred* variable is a weighted average of regional *starred* variables, if a given region's x variable is not available, then the weighted average must be adjusted to reflect the fact that the foreign variable is not comprised of all the x variables. This can easily be accomplished. For example, suppose that we are computing the German $q^*$and that x% of Germany's trade is with Turkey. However, Turkey's equity index is not available. When we take a weighted average of Germany's trading partners' equity indices, we will be effectively only weighting (1-x)%, since the Turkish index is unavailable. We can then divide our result by (1-x)% to yield the appropriate q^* for Germany. Finally, for regions with more than one member country, there exists 'intra-regional' trade (ie. trade between countries in the same region) that will not appear in the 'foreign' (*starred*) variables. As such, the weights may sum to less than one.

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