Pension Systems and Population Growth: An Overlapping-Generations Model with Endogenous Birth Rates

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Abstract

We build an overlapping-generations model with endogenous fertility to analyse the effects of the pension-system design on population growth. Our model allows for any combination of pay-as-you-go and funded pension schemes and different pension levels. We show that there is a unique stable steady-state equilibrium by solving the relevant non-linear difference equation and derive the full transition dynamics. In the steady state, both population growth and welfare are positively affected by the degree of foundation of the pension and negatively affected by the pension level. But, there is no long-run increase in birth rate and welfare without a short-run decrease.

Keywords: Overlapping-generations models; Pension systems; Fertility

JEL classification: H55; J11; J13; J26

1 Introduction

In the last few years, pay-as-you-go (PAYGO) pension systems, where contributions are immediately paid out to pensioners, have been under review in many industrialised countries. A transition to fully or partially funded systems, where contributions are invested in the capital market and paid out to the respective contributors, is a hotly disputed issue, cf. e.g. Kotlikoff (1996a, 1996b), Feldstein and Samwick (1998) and Sinn (2000, 2001). Today, PAYGO systems seem less attractive to policy makers

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than decades ago because the birth rate of most industrialised countries is much lower today than it used to be. This has a direct impact on the rate of return of the pension system since - as we will show - the growth rate of population (in efficiency units) is the rate of return in a PAYGO system in the steady state, whereas the net marginal product of capital (the real interest rate) is the rate of return in a funded system.

Nevertheless, most economic models devoted to the study of pension systems model the birth rate, which, in fact, is a choice variable of modern families, as exogenously given. However, while we are not aware of any empirical work that assesses if PAYGO systems and funded systems have different impact on birth rates, empirical studies by Cigno and Rosati (1996), Ehrlich and Zhong (1998) and Cigno et al. (2000) suggest that fertility is affected by the level of pension benefits in countries with PAYGO systems.

In this paper, we build an overlapping-generations model where the rate of population growth is endogenously determined by the agent. The number of children affects the agents' economic situation in a variety of ways. On the one hand, we assume that children increase the felicity of their parents, i. e. the number of children has a positive effect on individual utility. On the other hand, raising children causes costs. There are two kinds of costs modelled in this paper: the direct effect of children's living expenses and an indirect effect of reduced working time. The latter is the opportunity cost of raising children: as time must be devoted to child care, labour supply decreases and income falls. In addition, there is an externality which affects the next generation: a low birth rate induces high social security contributions for the following generation in a PAYGO system, reducing consumption and savings accordingly.

A number of items distinguishes our model from the related work of Klanberg (1988), Werding (1998), Yoon and Talmain (2000) and von Auer and Büttner (2001), who have also modelled the effects of different pension systems on reproductive behaviour. First, the model introduces opportunity costs of child care in terms of reduced labour supply. This is a channel for the pension regime to affect the production side of the economy. Second, we allow for any combination of funded and unfunded pension schemes, which enables us to analyse e.g. pension reforms that establish a complementary funded scheme in addition to a prevalent PAYGO scheme, as it is about to be done in many industrialised countries. Third, we show that (under certain conditions) our model has a unique stable steady-state equilibrium. Moreover, we assess the transition towards a new steady state. Finally, we calibrate the model with German data in order to quantify (very roughly) the effects of the recent German pension reform, which adds a fully funded private pension scheme to the existing system.

We assume that our economy is a small open country with perfect capital mobility. Thus, the interest rate on capital is exogenously given and equals the world interest rate. We also treat productivity growth as exogenous in order to focus the analysis on the endogenous response of fertility to changes in the pension scheme. For a model of endogenous fertility and endogenous growth see Yoon and Talmain (2000).

The growth rate of effective population (workers times labour productivity) equals the growth rate of the sum of wages, which cannot exceed the interest rate in the long run, cf. Sinn (2001). Assuming that the interest rate is larger than the growth rate of effective population, our central results are as follows. In the steady state, population growth as well as welfare are higher the higher the weight of the funded pension system is and the lower the overall pension level is (unless the weight of the PAYGO pension is zero). This arises from the fact that any pension amount that is financed in a PAYGO system yields a return that is too low compared to the capital market and is therefore charged by an "implicit tax", which reduces the individuals' disposable income and makes it harder for them to renounce on consumption in favour of children. But, no Pareto-improvement can be achieved through a pension reform nor can a long-run increase of population growth without a short-run decline on the rate of population growth.

The rest of the paper is organised as follows: Section 2 presents the basic model. In section 3, we study transition dynamics by solving the central non-linear difference equation and proving uniqueness and stability of the steady state. In section 4, parameter values are calibrated. Section 5 studies the steady state and establishes the long-run dependencies of population growth on the degree of funding and other key features of the pension system. Section 6 examines some short-run properties of pension reform, and section 7 concludes.

2 The Model

Our model describes a *small open economy* where profit-maximising *firms* produce a homogenous good using a Cobb-Douglas technology. They employ capital and labour and face perfect competition in the factors and good markets. The *households* are represented by individuals that live three periods, "childhood", "youth" (working age) and "old age". In the model, children are assumed not to make any economic decisions. Young people allocate their time endowment to participate in the labour market in exchange for wages and devote some of their time to raise children. They decide on their consumption level and how many children to have, whereas old people (pensioners) just live off their savings and the pension they get. The individuals maximise their lifetime utility subject to an intertemporal budget constraint under perfect foresight. The *government*'s only function is to provide pensions. Hereby we distinguish between two parts of the pensions, one of which is funded and the other of which is not. The unfunded pension scheme[1], whose contributions are immediately paid out as benefits, is necessarily administrated by the government, whereas the funded scheme, whose contributions are invested in the capital market and paid out to

[1] We use the expression "unfunded system" and "PAYGO system" equivalently.

the respective contributors when they are pensioners, may also be privately managed. The pension benefits are financed by lump-sum contributions and - in the funded system - by interest revenues.

Firms: Aggregate output Y_τ is given by the production function $Y_\tau = K_\tau^\alpha (P_\tau L_\tau)^\beta$ ($\alpha + \beta = 1$), where K_τ is capital, L_τ employed labour and P_τ exogenous labour productivity in period τ. Let N_τ be the number of young people, i. e. the potential labour force, and π_τ the labour-force participation rate, then labour input equals $L_\tau = \pi_\tau N_\tau$. In order to solve the model, we transform variables in terms per efficiency unit of (potential) labour, $P_\tau N_\tau$. These quantities per effective worker are denoted by lower-case letters, $y_\tau \equiv \frac{Y_\tau}{P_\tau N_\tau}$ and $k_\tau \equiv \frac{K_\tau}{P_\tau N_\tau}$. Now, the production function becomes

$$y_\tau = k_\tau^\alpha \pi_\tau^\beta. \tag{1}$$

Profit maximisation gives the first-order conditions

$$w_\tau = \beta \frac{y_\tau}{\pi_\tau} \tag{2}$$

and

$$r_\tau + \zeta = \alpha \frac{y_\tau}{k_\tau}. \tag{3}$$

w_τ is the wage for one efficiency unit of *employed* labour, $P_\tau L_\tau$, r_τ and ζ are the rental and depreciation rates of capital, respectively, where, under the assumptions of a small open economy and no restrictions to capital mobility, the interest rate r_τ is exogenously given by the world capital market. The depreciation rate is also assumed constant.

Government: The government spends the amount G_τ as pension benefits per period and gets the contributions T_τ and the interest $r_\tau A_\tau$ on its assets A_τ as revenues on the other hand. It saves its budget surplus $B_\tau = T_\tau + rA_\tau - G_\tau$, which might also be negative. Let b_τ, t_τ and a_τ be the respective values of B_τ, T_τ and A_τ per effective *worker*, $b_\tau \equiv \frac{B_\tau}{P_\tau N_\tau}$, $t_\tau \equiv \frac{T_\tau}{P_\tau N_\tau}$ and $a_\tau \equiv \frac{A_\tau}{P_\tau N_\tau}$, g_τ, however, the pension payment per "effective *pensioner*" (that is an effective worker of the preceding generation), $g_\tau \equiv \frac{G_\tau}{P_{\tau-1} N_{\tau-1}}$. Then the budget constraint of the government in terms of effective workers of generation τ is given by

$$b_\tau = t_\tau + r_\tau a_\tau - \frac{g_\tau}{(1 + p_\tau)(1 + n_\tau)}, \tag{4}$$

where $p_\tau \equiv \frac{P_\tau}{P_{\tau-1}} - 1$ is the growth rate of labour productivity and $n_\tau \equiv \frac{N_\tau}{N_{\tau-1}} - 1$ is the growth rate of (the young) population.

Now, the pension system as a whole can be (at least theoretically) split up into a funded pension scheme (superscript K) and an unfunded scheme (superscript U).

Taking into consideration that assets are accumulated only in the funded system, the respective budget constraints become

$$b_\tau = t_\tau^K + r_\tau a_\tau - \frac{g_\tau^K}{(1+p_\tau)(1+n_\tau)} \tag{5}$$

and

$$t_\tau^U = \frac{g_\tau^U}{(1+p_\tau)(1+n_\tau)} \tag{6}$$

with $t_\tau^j = \frac{T_\tau^j}{P_\tau N_\tau}$ and $g_\tau^j = \frac{G_\tau^j}{P_{\tau-1}N_{\tau-1}}$, $j = K, U$; $t_\tau^K + t_\tau^U = t_\tau$ and $g_\tau^K + g_\tau^U = g_\tau$.

From equation (6), the return to contributions to the unfunded system can be derived by solving the equation for $g_{\tau+1}^U$ and dividing both sides by t_τ^U:

$$\frac{g_{\tau+1}^U}{t_\tau^U} = \frac{t_{\tau+1}^U}{t_\tau^U}(1+p_{\tau+1})(1+n_{\tau+1}). \tag{7}$$

Suppose that t_τ^U is constant, then for each unit of contribution, the contributor gets back $(1+p_{\tau+1})(1+n_{\tau+1})$ units of pension when he or she is old. That is, the rate of return to the unfunded pension scheme equals the growth rate of effective labour. For one unit of contribution to the funded system, in contrast, you get back

$$\frac{g_{\tau+1}^K}{t_\tau^K} = 1 + r_{\tau+1} \tag{8}$$

units of pension as it is invested in the capital market. Rate of return is the market interest rate.

Thus, the unfunded system is - absolutely as well as in comparison to the funded system - the more attractive the faster population grows. This fact may lead to the reverse conclusion that the incentive to get children is the stronger the higher the weight of the unfunded pension scheme is. Whether this is true or not will be investigated in the sequel.

Households: The representative individual's preferences are described by the lifetime-utility function

$$u_\tau = \log c_\tau^1 + \delta \log c_\tau^2 + \varepsilon \log m_\tau. \tag{9}$$

c_τ^1 and c_τ^2 is consumption in efficient units of an agent born in period $\tau - 1$, i. e. whose autonomous economic life begins in period τ when he or she reached working age, where superscript 1 denotes youth (i. e. in period τ) and superscript 2 denotes old age (in period $\tau + 1$).[2] $m_\tau = \frac{N_{\tau+1}}{N_\tau}$ is the number of children per *individual* of generation τ. $0 < \delta < 1$ is a discount factor and ϵ is assumed to be positive.

[2]Since m_τ is the number of children per *capita*, not per *effective* worker, we look at the optimisation problem of an *individual*, whose control variables are $P_\tau c_\tau^1$, $P_\tau c_\tau^2$ and m_τ. Maximising utility over c_τ^1 and c_τ^2 instead of $P_\tau c_\tau^1$ and $P_\tau c_\tau^2$ clearly yields the same results. Therefore, P_τ can be neglected in the utility function as can consumption during childhood.

(9) expresses the idea that children bring about happiness to their parents. On the other hand, they also cause costs. First, children must be provided with consumption goods. It shall be assumed that parents spend $P_\tau c^0$ units of income per child, where c^0 is constant. Second, parents have opportunity costs of foregone labour income as they are assumed to spend part of their potential working time with their children. The participation rate π_τ is the share of time they work. We assume that π_τ equals

$$\pi_\tau = 1 - \psi m_\tau \tag{10}$$

with $\psi > 0$ (as long as $\pi_\tau > 0$). Thus, if the representative individual has children, he or she earns only $\pi_\tau P_\tau w_\tau < P_\tau w_\tau$ units instead of $P_\tau w_\tau$.

In addition, he or she must pay the pension contributions $P_\tau \left(t_\tau^U + t_\tau^K\right)$ per individual, so that the young generation's budget constraint (in terms of effective workers) equals

$$s_\tau^1 = \pi_\tau w_\tau - t_\tau^U - t_\tau^K - c^0 m_\tau - c_\tau^1, \tag{11}$$

where s_τ^1 is the young generation's private saving.

When people are old, they consume their savings (including interest) and their pension:

$$c_\tau^2 = \left(1 + r_{\tau+1}\right) s_\tau^1 + g_{\tau+1}. \tag{12}$$

Inserting equations (10) and (11) for s_τ^1 and π_τ, respectively, into equation (12) yields generation τ's lifetime-budget constraint, according to which the present value of total expenditure equals the present value of total revenues:

$$c^0 m_\tau + c_\tau^1 + \frac{c_\tau^2}{1 + r_{\tau+1}} + t_\tau^U + t_\tau^K = \left(1 - \psi m_\tau\right) w_\tau + \frac{g_{\tau+1}}{1 + r_{\tau+1}}. \tag{13}$$

The individual's problem is to maximise the utility function (9) subject to the budget equation (13). Combining the Lagrangean derivatives with respect to c_τ^1 and c_τ^2 gives the standard Euler equation

$$c_\tau^2 = \delta \left(1 + r_{\tau+1}\right) c_\tau^1, \tag{14}$$

which holds if saving an additional unit of income for the retirement period yields the same utility as consuming it in the working period. Inserting equation (14) into the budget (13) leads to the working period's consumption function (where c_τ^1 still depends on m_τ):

$$c_\tau^1 = \frac{1}{1 + \delta} \left(\pi_\tau w_\tau + \frac{g_{\tau+1}}{1 + r_{\tau+1}} - t_\tau^K - t_\tau^U - c^0 m_\tau\right). \tag{15}$$

In addition to these familiar equations, we get another condition when we combine the Lagrangean derivatives with respect to c_τ^1 and m_τ:

$$m_\tau = \frac{\varepsilon}{\psi w_\tau + c^0} c_\tau^1. \tag{16}$$

It is fulfilled if raising an additional child at the costs of $\psi w_\tau + c^0$ units of income brings about the same utility as consuming this amount.[3]

Accumulation and market clearing: The model is completed by the accumulation equations for the four stock variables v_τ (private wealth), a_τ (social security wealth), k_τ (domestic capital) and f_τ (net foreign assets),

$$(1 + p_{\tau+1})(1 + n_{\tau+1}) v_{\tau+1} = v_\tau + s_\tau, \tag{17}$$

$$(1 + p_{\tau+1})(1 + n_{\tau+1}) a_{\tau+1} = a_\tau + b_\tau, \tag{18}$$

$$(1 + p_{\tau+1})(1 + n_{\tau+1}) k_{\tau+1} = (1 - \zeta) k_\tau + i_\tau, \tag{19}$$

$$(1 + p_{\tau+1})(1 + n_{\tau+1}) f_{\tau+1} = (1 + r_\tau) f_\tau + x_\tau, \tag{20}$$

and the market clearing conditions for the good market,

$$y_\tau = c_\tau^1 + \frac{c_{\tau-1}^2}{(1 + p_\tau)(1 + n_\tau)} + c^0 m_\tau + i_\tau + x_\tau, \tag{21}$$

and capital market,

$$v_\tau + a_\tau = k_\tau + f_\tau, \tag{22}$$

where s_τ is private saving, i_τ is investment in domestic capital and x_τ is net exports. The labour market is implicitly assumed to be equilibrated as well (all supplied labour is employed). $s_\tau + b_\tau + \zeta k_\tau - r_\tau f_\tau$ is total domestic saving in period τ per effective worker. These savings are invested at home and abroad through i_τ and x_τ, respectively, so that the economy's total wealth $(v_\tau + a_\tau)$ equals the sum of domestic and net foreign assets $(k_\tau + f_\tau)$. As labour supply is not constant, neither is k_τ in spite of its constant rental rate.

Finally, note that population growth is determined by the number of children per capita,

$$n_{\tau+1} = m_\tau - 1, \tag{23}$$

and the pension benefits of the funded and the unfunded systems, respectively, are given by

$$g_\tau^K = \kappa_\tau g_\tau \tag{24}$$

and

$$g_\tau^U = (1 - \kappa_\tau) g_\tau, \tag{25}$$

which defines κ_τ (with $0 \leq \kappa_\tau \leq 1$) as the pension's degree of funding. κ_τ and g_τ are viewed as exogenous and as being the government's control variables, which characterise the pension regime (so that contributions result endogenously).

[3] Using equations (1), (2) and (3) , it can be shown that the general wage rate $w_\tau = \beta \left(\frac{\alpha}{r_\tau + \zeta} \right)^{\frac{\alpha}{\beta}}$ is independent of π and m because the positive effects of higher labour input on the optimal capital stock and on output for a given capital stock just balance out with the negative effect on marginal labour productivity.

3 Equilibrium

The **sequential market equilibrium** is given by a sequence of quantities of the endogenous variables m_τ, $n_{\tau+1}$, π_τ, k_τ, y_τ, w_τ, g_τ^U, g_τ^K, t_τ^U, t_τ^K, t_τ, b_τ, $a_{\tau+1}$, c_τ^1, c_τ^2, s_τ^1, s_τ, $v_{\tau+1}$, $i_{\tau-1}$, x_τ and $f_{\tau+1}$, which fulfil all the preceding equations. These quantities can be computed by inserting equations (15), (10), (8), (6), (24), (25) and (23) for c_τ^1 into equation (16). This yields a non-homogenous, non-linear first-order difference equation in m_τ:

$$m_\tau = \frac{\varepsilon}{w_\tau \psi + c^0} \frac{1}{1+\delta} \left[w_\tau (1 - \psi m_\tau) + \frac{(1-\kappa_{\tau+1}) g_{\tau+1}}{1 + r_{\tau+1}} - \frac{(1-\kappa_\tau) g_\tau}{(1+p_\tau) m_{\tau-1}} - c^0 m_\tau \right] \tag{26}$$

Note that this equation depends solely on m_τ, $m_{\tau-1}$ and on exogenous variables and parameters as w_τ is a function of r_τ and parameters only.

$m_{\tau-1}$ shows up because c_τ^1 (the denominator on the right-hand side) is affected by the contribution to the unfunded pension scheme which again depends on the ratio of old and young people, thus on the number of children per adult in period $\tau - 1$. This difference equation can be written as

$$m_\tau = \frac{\gamma_{3\tau}}{m_{\tau-1}} - \gamma_{2\tau}, \tag{27}$$

where $\gamma_{2\tau}$ and $\gamma_{3\tau}$ are functions of the model's parameters and exogenous variables (see appendix). Equation (27) determines the dynamics of the economy.[4] Given initial values m_0, v_1 and a_1, the sequence of m_τ and all the other endogenous variables of the sequential market equilibrium can be computed for all $\tau = 1, 2, ..., \infty$.

The **steady-state equilibrium** is given by quantities of these normalised variables that fulfil the model's equations and are constant over time for some initial m_0. They will be written without time indices.

Removing time indices from equation (27), it becomes a quadratic equation on m, implying two steady-state solutions (see appendix). But:

Theorem 1 *For ε sufficiently large, a sequential market equilibrium converges to a unique steady-state equilibrium.*

Proof: See appendix. This theorem can be proved by transforming the difference equation (27) in a second-order linear difference equation and deriving its general solution.

[4] No additional dynamics that is not determined by the dynamics of equation (27) is induced by the accumulation equations (**??**) to (20). This is because of the following reasons: both types of wealth, v_τ and a_τ, are completely dissaved by the respective savers when they are old. Thus, they just equal the respective contributions of the preceding period, $(1 + p_{\tau+1})(1 + n_{\tau+1}) v_{\tau+1} = s_\tau^1$ and $(1 + p_{\tau+1})(1 + n_{\tau+1}) a_{\tau+1} = t_\tau^K$, which are driven by the dynamics of m_τ. k_τ is determined by the firm's optimality condition (3) - also dependent on m_τ - and f_τ can be computed as the residual of the capital market clearing condition (22).

4 Calibration

To be able to illustrate our results, we calibrate the model with German data as follows:

α	β	δ	ε	ζ	ψ	c^0	P^β	p	r
0.27	0.73	0.31	0.5	0.46	0.23	250000	113375	0.64	2.34

For $\beta = 1 - \alpha$ we take the average labour income share (Lohnquote) for 1991 to 2000. δ is chosen in such a way that a steady-state consumption ratio c^2_τ / c^1_τ of slightly more than 1 results, which seems plausible; $\delta = 0.31$ corresponds to a yearly rate of time preference of 4%. $c^0 = 250000$ DM in prices of 1995 is a rough estimate gained from the comparison of consumption data of differently sized households (cf. Statistisches Bundesamt, 547). ψ is estimated with OLS using German times series data from 1964 to 1998, where the estimation equation is (standard errors in brackets):

$$\pi_\tau = 1 - (\underset{(0.005)}{0.256} - \underset{(0.005)}{0.024}\, D90 - \underset{(0.006)}{0.045}\, D9198)\ m_\tau + \eta_\tau \tag{28}$$

with

$$\eta_\tau = \underset{(0.083)}{0.791}\ \eta_{\tau-1} + \xi_\tau. \tag{29}$$

m_τ is calculated as the ratio of people below 30 and between 30 and 60 (yearly average). π_τ is the ratio of labour-force participants between 30 and 60 and total population between 30 and 60, respectively. $D90$ (equals 1 in 1990) and $D9198$ (equals 1 from 1991 on) are dummy variables, which are justified by the German reunion at the end of 1989 and the switch from Western German to German data after 1990. For the error term η, it was allowed for first-order autocorrelation. ξ is white noise. Consequently, for ψ $0.256 - 0.045 \approx 0.21$ results.

The (normalised) labour productivity P_τ is computed as total factor productivity $P^\beta_\tau = \frac{Y_\tau}{K^\alpha_\tau (\pi_\tau N_\tau)^\beta}$ where N_τ is the population between 30 and 60 in 1998 (yearly average), Y_τ and K_τ are also values from 1998 - in prices of 1995. p_τ is growth rate of labour productivity (GDP per worker in prices of 1995) from 1992 to 1999[5] and r_τ is the 1992 to 1999 average real rate of return (4.1% per year) of long-term government bonds, both related to a 30-year interval. The depreciation rate $\zeta = 0.46$ resulted as the residual of equation (3). The present German pension system is completely unfunded, so that $\kappa_0 = 0$. The pension benefits were calculated as $g_0 = 600000$ by adding the respective values of the three public pension schemes (Rentenversicherung der Arbeiter, Rentenversicherung der Angestellten, knappschaftliche Rentenversicherung) and dividing by the number of people over 60; this yielded an amount of roughly of 20000 DM (of 1995) per year from 1995 to 1998. ε, finally, is chosen in such a way that - given the other parameters and exogenous variables in the initial state - the number of children per person is approximately $m_0 = 0.775$ as it is calculated as above for 1998.

[5] $p = 0.64$ corresponds to a yearly rate of 1.7%; cf. Statistisches Bundesamt, p. 638.

5 Steady-State Analysis of Pension Regimes

In this section, it shall be analysed how the pension regime affects population growth (and the other endogenous variables) in the steady state. Inserting the consumption function (15) for c_τ^1 into (16), replacing the pension contribution variables by equations (8), (6), (24) and (25), rearranging terms and removing time indices yields

$$\frac{\varepsilon}{m} = \frac{1}{\frac{1}{1+\delta}\left(w\pi - t^i - c^0 m\right)}\left(w\psi + c^0\right) \tag{30}$$

with

$$t^i \equiv t^U + t^K - \frac{g}{1+r} = \frac{(1-\kappa)g}{(1+p)m} - \frac{(1-\kappa)\,g}{1+r} \tag{31}$$

Equation (30) is the steady-state version of the difference equation (26) and is used to determine m. The left-hand side of equation (30) is the individual's marginal utility of children, which should equal, on the right-hand side, the marginal costs of children, $w\psi + c^0$, subjectively evaluated by the individual's marginal utility of income in terms of consumption, $\frac{1}{c^1}$. Equation (31) shows how these subjective marginal costs of children are influenced by the government's control variables κ and g. t^i is the present value difference between pension contributions and pension benefits. If t^i is positive, it is an implicit tax, imposed to the individual by the pension system. As can be easily seen, this is the case if the interest rate is larger than the growth rate of effective labour, $r > (1+p)(1+n) - 1 = (1+p)m - 1$.[6] In the following, we only consider this case for simplicity without loss of generality. If the growth rate of PN exceeded the interest rate, $(1+p)(1+n) - 1 > r$, t^i is negative and the effects of κ and g go just in the opposite direction.

Now, let us first consider the effect of the *degree of funding.*

Theorem 2 *If the interest rate is higher than the growth rate of effective labour and $g > 0$, then m increases with κ in the steady state.*

Proof. As is easily seen from equation (31), κ is multiplied by a negative factor if $(1+p)m < 1+r$. Thus, if κ increases for a given m, t^i decreases and the subjective marginal costs of m on the right-hand side of equation (30) decrease, too. Therefore, m must rise. It must also be noted, however, that t^i itself depends on m through t^U, which causes a secondary effect. If m increases, the term $\frac{g}{(1+p)m}$ decreases; this has two effects on t^i: on the one hand, it decreases t^i again and, thereby, reinforces the original effect of the increase of κ; on the other hand, it reduces the original effect as it diminishes the factor by which κ is multiplied. But, this latter effect can never overcompensate the original effect, since it is just caused by the fact that κ increases. Thus, there is a positive overall effect of κ on m. ■

[6] According to our calculations of section 4, this is clearly the case in Germany, where $r = 2.34$ (4.1% per year) and $(1+p)(1+n_0) - 1 = (1+p)\,m_0 - 1 = 0.27$ (0.8% per year)!

The economic intuition behind this mechanism can be given as follows: From (6), (8), (25) and (24), contributions to the funded and unfunded systems are $t^K = \kappa\frac{g}{1+r}$ and $t^U = (1-\kappa)\frac{g}{(1+p)(1+n)}$, respectively. For one unit of the exogenously given pension benefit, the individual must contribute $\frac{1}{1+r}$ units to the funded system and $\frac{1}{(1+p)(1+n)} > \frac{1}{1+r}$ units to the unfunded system. But, κ and $1-\kappa$ are the weights of the two schemes. Thus, the larger κ is, the less the representative individual must contribute to the "more expensive" unfunded system, the lower contributions he or she pays in total, the lower t^i is, the higher his/her disposable lifetime income is accordingly. And higher income means higher consumption in both periods of life (compare equations (15) and (14)), given the number of children. But, the higher the level of consumption is, the lower the marginal utility, $\frac{1}{c^1}$, of consumption (or income) is, by which marginal costs of children are evaluated in equation (30), consequently the larger the number of children. In other words: the higher the weight κ of the "cheaper" funded pension scheme, the easier it is for the individual to renounce on labour income and consumption in favour of children. This finding is just opposite to the initially mentioned speculation that a *lower* degree of funding would provide a stronger incentive to raise children since according to (7), the rate of return of the pension system shrinks when $m = 1+n$ rises.

A higher number of children means a higher growth rate of population and the economy, too. On the other hand, more children lead (according to equation (10)) to a lower labour-force participation rate and in that way to lower marginal productivity of capital, a smaller capital stock, lower production and lower labour income, πw, per (potential) effective worker as well as per capita. This has a negative effect on consumption, c^1 and c^2, which is, however, secondary since the higher m is just due to higher consumption (so that the overall effect on consumption is still positive).

Theorem 3 *If the interest rate is higher than the growth rate of effective labour and $\kappa < 1$, then m decreases with increasing g in the steady state.*

Proof. This proof is largely the same as the proof of theorem 2. g is multiplied by a positive factor in equation (31) if $(1+p)m < 1+r$. Thus, if g increases for a given m, t^i and the subjective marginal costs of m also increase. Therefore, m must decline. As above, there is a secondary effect since t^U depends on m. If m decreases, the term $\frac{1-\kappa}{(1+p)m}$ becomes larger; this again has two effects on t^i: first, it increases t^i again and, thereby, reinforces the original effect of the increase of g; second, it *enlarges* the factor by which g multiplied. This latter effect is different from its counterpart in the preceding proof as it also reinforces the original effect. Thus, there is a strict negative effect of g on m. ■

The higher g, the larger the difference is between its present value and the efficiently - in the funded pension system - invested part of its financing, $\frac{(1-\kappa)g}{1+r}$. This gap must be financed by the following generation through the "inefficient" unfunded system by the amount $\frac{(1-\kappa)g}{(1+p)(1+n)}$ which is larger than amount $\frac{(1-\kappa)g}{1+r}$; and the difference

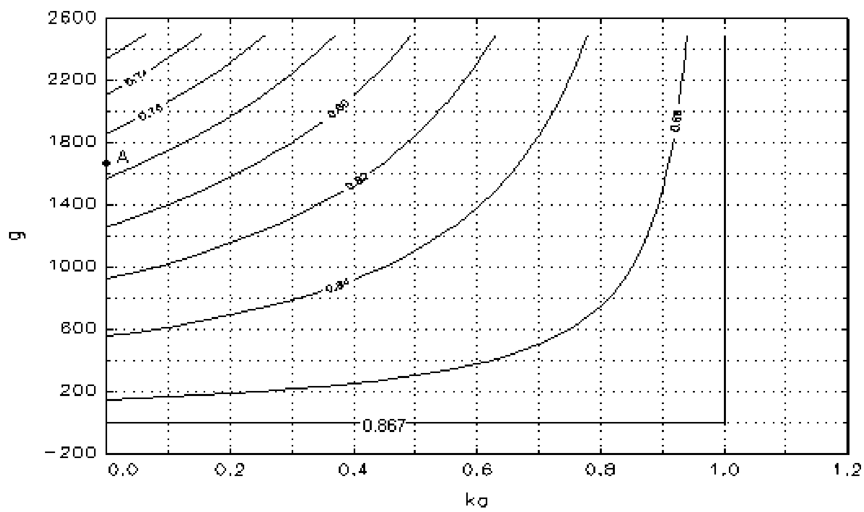

Figure 1: Isoquants of children per capita

between these two amounts increase in g, see equation (31). Thus, a higher g leads to a higher implicit tax, lower disposable income and the well-known consequences for the other variables.

An exception is given, however, if $\kappa = 1$, as can be seen from equation (31); in this case, the level of g does not matter because there is no unfunded system, which could cause any implicit tax. And trivially, κ does not matter if $g = 0$. But otherwise, a higher κ can be offset by a higher g as the set of m-isoquants of figure 1 illustrates.

This diagram is drawn using the parameter values of section 4. Point A denotes the status quo. For illustration, g is given as *monthly* DM-amounts, where $g = 600000$ per 30-year-period corresponds to $g = 1667$ per month. m-isoquants can also be interpreted as indifference curves since a higher m is always connected with higher consumption levels c^1 and c^2 and consequently with higher utility. For any g, growth as well as welfare is higher in a fully funded pension regime ($\kappa = 1$) than for any other combination of $\kappa < 1$ and $g > 0$ since there is no implicit tax! Note that even in this best case (on the right-angled isoquant), m is still far below 1 ($m = 0.867$), which would - neglecting immigration and the increase in life expectancy - be necessary to keep the population constant! This would mean that, given the calibrated preference and technology parameters, a constant population cannot be achieved through a change in the pension system alone (at least if a negative pension and a degree of funding of more than 100% are excluded).

Tables 1 and 2 show some simulated values of selected endogenous variables for different κ- and g-values, respectively.

kappa	m	pi	y	y+r*f	t/(pi*w)	c^1	GDP growth
0	0,774	0,823	8552	7803	0,211	3714	0,0080
0,2	0,796	0,818	8499	7967	0,182	3820	0,0089
0,4	0,816	0,813	8452	8122	0,154	3916	0,0097
0,6	0,834	0,809	8408	8271	0,129	4004	0,0105
0,8	0,851	0,805	8367	8413	0,105	4086	0,0112
1	0,867	0,801	8329	8551	0,082	4163	0,0118

Table 1

The values of table 1 are calculated for $g = 1667$ (monthly). y, $y + rf$ and c^1 are also expressed in DM (1995) per month. $y + rf$ is total income per effective worker, that is GNP, and "GDP growth" is the yearly growth rate of effective labour or of GDP, which equals $\sqrt[30]{(1+p)(1+n)} - 1$. Note that GNP increases with κ, whereas GDP (y) decreases. This implies that net foreign assets increase (or net foreign debt decreases), whereas domestic capital decreases (because of decreasing labour input) with κ. The total contribution rate $\frac{t^U + t^K}{\pi w}$ decreases drastically with κ, although π decreases as well (and w remains constant). The first line of table 1 represents the status quo and the last line the best state, which corresponds to point A and a point on the right-angled isoquant at the same level of g in figure 1, respectively.

g	m	pi	y	y+r*f	t/(pi*w)	c^1	GDP growth
1100	0,810	0,815	8466	8076	0,135	3888	0,0095
1300	0,798	0,817	8494	7983	0,161	3830	0,0090
1500	0,785	0,820	8525	7886	0,188	3768	0,0085
1700	0,771	0,823	8557	7786	0,216	3703	0,0079
1900	0,757	0,827	8592	7682	0,245	3633	0,0072
2100	0,741	0,830	8630	7572	0,275	3557	0,0065

Table 2

Table 2 shows the same variables as in table 1 for different g-values and for $\kappa = 0$. The effect of g is opposite to the effect of κ.

Theorems 1 and 2 as well as the isoquants of figure 1 suggest that it is always possible to compensate the negative effect of an increase in g by an adequate increase in κ.

Theorem 4 *A fully funded pension increase does not affect m.*

Proof. If some additional amount of pension is fully funded, κ and g both rise in such a way that the amount of unfunded pension, $(1 - \kappa) g$, which causes the implicit tax, remains unchanged. From equations (31) and (30), it can be easily seen that in this case, t^i is not affected and neither is m. ■

This is because the present value of the additional pension benefits and the corresponding contributions are the same. The only effect is that voluntary private saving is crowded out by forced saving in the pension system one by one. Of course, the same argument is valid for a fully funded decrease of pension.

A combined change of κ and g can always be divided into a fully funded pension increase or decrease (movement along a certain isoquant in figure 1) and an additional change of κ or g (movement along one of the axes).

6 Transition to a New Steady State

If only different steady states were considered, a pure funded system would clearly be preferred to a regime with any degree of funding smaller than one and a lower pension level would be better than a higher one. But of course, for the case of a change of pension level or degree of funding, transition dynamics must also be taken into account. We state:

Theorem 5 *If the interest rate is higher than the growth rate of effective labour, then there is no long-run increase in* m_τ *due to a change of* κ_τ *or* g_τ *without a short-run decrease.*

Proof: From theorem 4, we know that it is sufficient to show the validity of theorem 5 for any isolated change of κ and g, respectively (which occurs in addition to a fully funded change). Let us again consider a version of equation (26) with marginal utility of children on the left-hand side and subjective marginal costs on the right-hand side:

$$\frac{\varepsilon}{m_\tau} = \frac{w_\tau \psi + c^0}{\frac{1}{1+\delta}\left[w_\tau \pi_\tau - t^i_\tau - c^0 m_\tau\right]}, \tag{32}$$

where the implicit tax is given by

$$t^i_\tau = t^U_\tau + t^K_\tau - \frac{g_{\tau+1}}{1+r_{\tau+1}} = \frac{(1-\kappa_\tau)\, g_\tau}{(1+p_\tau)(1+n_\tau)} + \frac{\kappa_{\tau+1} g_{\tau+1}}{1+r_{\tau+1}} - \frac{g_{\tau+1}}{1+r_{\tau+1}}. \tag{33}$$

First, suppose that a long-run increase of m_τ shall be achieved by a higher κ_τ from period τ_0+1 onwards. Then $\Delta\kappa_{\tau_0+1} > 0$ and $\Delta t^i_{\tau_0} = \frac{g_{\tau_0+1}}{1+r_{\tau_0+1}}\Delta\kappa_{\tau_0+1} > 0$. In period τ_0, contributions to the funded system must rise whereas contributions to the PAYGO system cannot be reduced since they are needed to finance its benefits $(1-\kappa_{\tau_0})\, g_{\tau_0}$. If $\Delta g_{\tau_0+1} < 0$, then $\Delta t^i_{\tau_0} = -\frac{1-\kappa_{\tau_0+1}}{1+r_{\tau_0+1}}\Delta g_{\tau_0+1} > 0$. The present value surplus of pension benefits over contributions to the funded system decreases, but again, $t^U_{\tau_0}$ remains constant as does $(1-\kappa_{\tau_0})\, g_{\tau_0}$. And a higher implicit tax means an increase of the subjective marginal costs of children and thus a reduction of m. ■

How the transition dynamics looks under the sequential market equilibrium shall be described in the following for an increase of pension funding and a given pension level. The diagrams are drawn using the parameter, r and g values as calibrated above. This is to come close to the situation in Germany where the recent pension reform requests people to save 4 percent of their income (which corresponds to $\kappa = 0.5$ in our model) for their pensions in addition to the governmental pension scheme that is an unfunded PAYGO system. That is, they should start to pay contributions not only to the old unfunded system but also to a new funded system. Although these contributions are not compulsory we view them as if they were because they are heavily subsidised by the government[7].

[7] These subsidies are tax financed and thus do not affect disposable income.

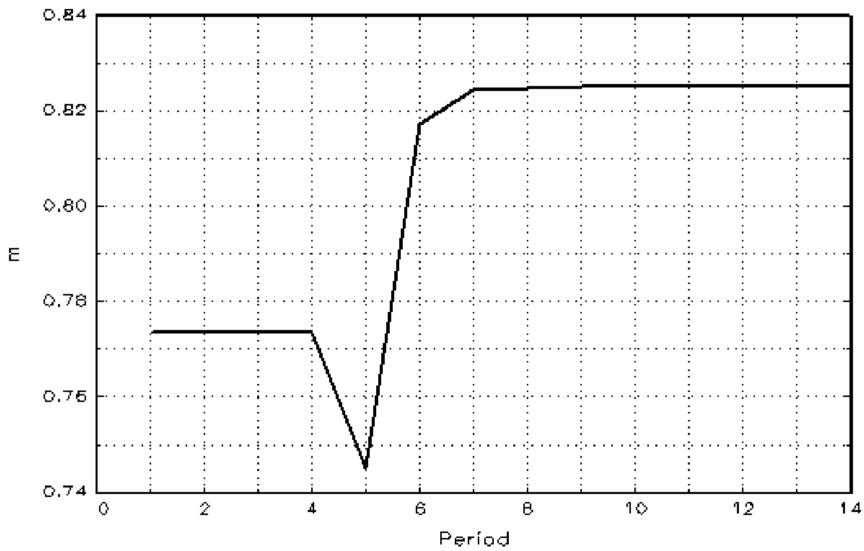

Figure 2: Children per capita

We assume that the degree of funding is raised from $\kappa_5 = 0$ to $\kappa_6 = 0.5$ which means that in period 6, 50 percent of the pension are financed by social security assets and interest revenues for the first time. These assets are accumulated by the contributions of generation 5 (the pensioners of period 6). Thus, in *period 5*, contributions to the funded system must already be paid, whereas the contributions to the unfunded scheme cannot yet be reduced as the pensioners of period 5 are still fully dependent on them. Therefore, generation 5 must bear a double burden: they finance completely the pension of their parents in addition to half of their own. They get the same pension as the generation before, but they must pay a higher contribution, so that their disposable lifetime income is lower. This means lower consumption during both youth and old age, a higher marginal utility of consumption or income and higher subjective marginal costs of children. The number of children is *lower* in period 5 than in the steady state before.

In *period 6*, the funded system provides benefits for the first time, whereas the benefits of the unfunded system get reduced by one half. These quantities remain constant thereafter. For generation 6, this means that the contributions to the funded system are the same as for generation 5, whereas the contributions to the unfunded system are lower. Because of their worse returns, they are even reduced by more than the contributions to the funded scheme amount at all. This implies a higher disposable income, higher consumption and a larger number of children than in the initial steady state.

In *period 7*, the contributions to the funded and unfunded systems would remain unchanged if the population growth rate were the same. But that is higher due to the increased number of children per capita of generation 6. This fact causes the contributions to the unfunded system to fall even further, so that also the total contributions and the implicit tax are even lower than in period 6. Again, higher disposable income means more consumption and more children. In all the following periods, the even higher growth rate makes this process repeat like in period 7, even if the effects become very small. The transition is actually finished in period 8.

The trajectories of the remaining variables can be computed for the given trajectory of m_τ; they are always connected essentially as in the steady state, compare section 5. To give some numerical examples: The total contribution t_τ is about 19% higher in period 5, but 34% lower in the long run compared to the initial state. This corresponds to a yearly rate of return to t_τ of 0.8% initially, 0.2% in period 5 and 2.2% in the long run and a contribution rate of 21.1%, 25.0% and 14.1%, respectively. m_τ, c_τ^1 and c_τ^2 are all 3.8% lower in the 5th period but 6.7% higher in the long run. π_τ, k_τ and y_τ all are 0.8% *higher* first but 1.4% *lower* in the new steady state[8]; this counteracting effect is obviously relatively small. Finally, GDP grows by 0.80% per year initially, 0.67% from period 5 to 6 and 1.01% in the long run.

As we have shown, a long-run increase in population growth cannot be achieved without a short-run growth reduction. This leads to the interesting question, after how many periods the population stock will catch up with the value it would have without the change of the pension regime. Unfortunately, this question cannot be answered generally. But in our case, population is already higher in period 7, with the degree of funding risen in period 6 from 0 to 0.5 (as in the simulations above) or to 1, respectively. That is, population is smaller just for one generation. Figure 3 shows the respective population stocks relative to keeping the pure PAYGO system.

7 Concluding Remarks

The goal of this paper was to analyse the impact of different pension systems on population growth. Given that the growth rate of population in efficiency units is smaller than the real interest rate - which in the long run must be the case, see Sinn (2001) - the contributions to a PAYGO system yield a lower return than they would yield if they were invested in the capital market, as contributions to a funded system are. Thus, an implicit tax is imposed on the PAYGO system, such that lower disposable income is attained than under funded systems, implying less consumption and hence a higher marginal utility of consumption. If children are a normal good, i. e. other things being equal, more children induce more felicity to their parents, then equating marginal utilities implies that less children are raised with lower levels of consumption than in a funded system. The same applies for higher pension levels financed at least partly in a PAYGO system since each additional amount of a PAYGO pension is taxed with additional implicit tax. Hence, a transition to a more highly

[8]These values are always the same because k_τ and y_τ are proportional to π_τ.

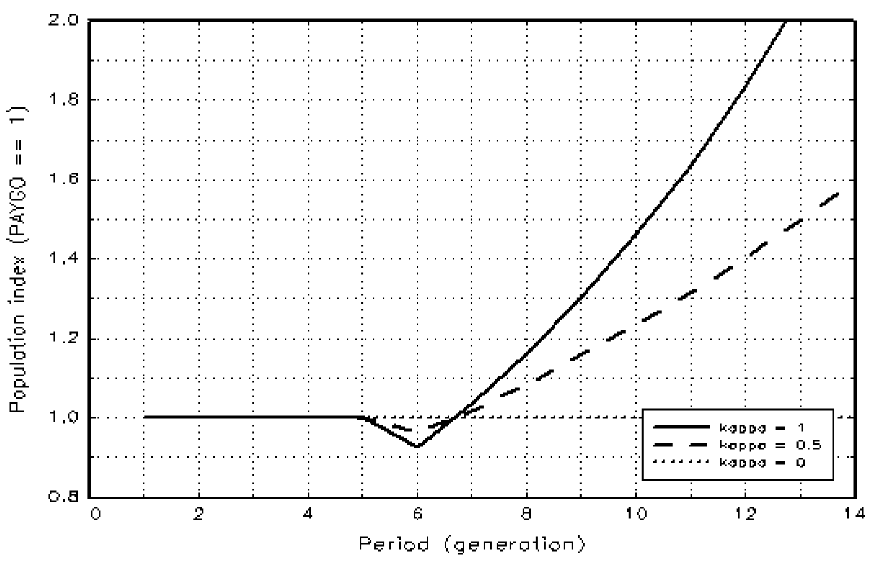

Figure 3: Population index

funded system or a decrease of the pension level in an at the most partly funded system should increase the birth rate in the long run. These results are fully consistent with the empirical findings of Cigno and Rosati (1996), Ehrlich and Zhong (1998) and Cigno et al. (2000), who found that in countries with PAYGO systems, the pension level had negative impact on fertility.

Along the transition path, however, the birth rate decreases temporarily in both cases due to income losses of at least one generation. If the degree of funding is increased, one generation has to finance the same part of their parents' pensions as before and a higher part of their own pensions additionally. If the pension level is decreased, then one generation has to suffer a reduction in their pensions without paying lower contributions. Some subsequent generations might also be worse off if the birth rate catches up slowly, inducing low rates of return of the PAYGO system.

As a higher number of children of a generation is always connected with a higher consumption level of this generation during youth and old age, a higher rate of population growth also means higher utility. Thus, there is no Pareto-improving transition from one steady state to another. Even worse, it can be shown that the present value of disposable income of all generations - starting with the one that is negatively concerned - can generally not be increased by any reform.[9]

A solution might be - according to Sinn (2000) and von Auer and Büttner (2001) -

[9]Disposable income is gross labour income minus the implicit tax. Whereas the discounted sum of implicit taxes cannot be changed at all, cf. Sinn (2000), the discounted sum of gross labour income is affected by the rate of population growth over the labour force participation rate and the discount factor (which considers population growth). The sign of the total effect is not clear; in the example of section 6, it is negative.

to internalise the intergenerational externality that an increasing birth rate decreases the social security contributions to be paid by the following generation. This can be done by charging each individual by contributions that are reversely proportional to that individual's number of children, in fact to such a degree that the individual would have to pay the same amount as before the reform with the same number of children as before. Since increasing the number of children reduces social security contributions, the individual marginal costs of children are lower, implying a higher birth rate, which compensates the governmental bugdet for the lower contributions per person. These items are to be treated in detail in a subsequent paper.

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1 Convergence to a Steady-State Equilibrium

In this appendix, we are going to prove theorem 1. For this purpose, we solve difference equation (27), which can be written as

$$m_\tau m_{\tau-1} + \gamma_{2\tau} m_{\tau-1} = \gamma_{3\tau}, \tag{34}$$

where

$$\gamma_{2\tau} = -q_\tau \frac{(1-\kappa_{\tau+1})\, g_{\tau+1}}{1+r_{\tau+1}} - q_\tau w_\tau \tag{35}$$

and

$$\gamma_{3\tau} = -q_\tau \frac{(1-\kappa_\tau)\, g_\tau}{1+p_\tau} \tag{36}$$

with

$$q_\tau = \frac{\varepsilon}{(1+\delta+\varepsilon)\,(\psi w_\tau + c^0)} \tag{37}$$

and

$$w_\tau = \beta \left(\frac{\alpha}{r_\tau + \zeta} \right)^{\frac{\alpha}{\beta}}. \tag{38}$$

Removing time-indices from equation (34) gives a quadratic equation, which has the two (steady-state) solutions

$$m_{1/2} = \frac{-\gamma_2 \pm \sqrt{\gamma_2^2 + 4\gamma_3}}{2} \tag{39}$$

for m. We are going to show, however, that only one steady-state equilibrium can be stable.

Equation (34) can be transformed to the homogenous second-order linear difference equation

$$\gamma_{3\tau} z_{\tau+1} + \gamma_{2,\tau-1} z_\tau - z_{\tau-1} = 0, \tag{40}$$

where m_τ is replaced by

$$m_\tau = \frac{z_\tau}{z_{\tau+1}} - \gamma_{2\tau}. \tag{41}$$

If $\gamma_{2\tau}$ and $\gamma_{3\tau}$ are constants (which is the case in the long run), (40) has the general solution

$$z_\tau = D_1 \lambda_1^\tau + D_2 \lambda_2^\tau, \tag{42}$$

where the eigenvalues λ_1 and λ_2 are given by

$$\lambda_{1/2} = \frac{-\gamma_2 \pm \sqrt{\gamma_2^2 + 4\gamma_3}}{2\gamma_3} \tag{43}$$

and D_1 and D_2 are some arbitrary constants. Inserting (42) in (41) gives

$$m_\tau = \frac{1 - \gamma_2 \lambda_1 + (1 - \gamma_2 \lambda_2) D \Lambda^\tau}{\lambda_1 (1 + D \Lambda^{\tau+1})} \tag{44}$$

with $\Lambda = \frac{\lambda_2}{\lambda_1}$ and $D = \frac{D_2}{D_1}$.

In order to assess m_τ's asymptotic properties, we distinguish four cases:

1) $\lambda_1, \lambda_2 \in R$ and $\Lambda > 1$; then, using L'Hôpital's rule

$$\lim_{\tau \to \infty} m_\tau = \lim_{\tau \to \infty} \frac{(1 - \gamma_2 \lambda_2) D \Lambda^\tau \log \Lambda}{\lambda_1 D \Lambda^{\tau+1} \log \Lambda} = \frac{1 - \gamma_2 \lambda_2}{\lambda_1 \Lambda} = \frac{1}{\lambda_2} - \gamma_2; \tag{45}$$

2) $\lambda_1, \lambda_2 \in R$ and $\Lambda < 1$; then

$$\lim_{\tau \to \infty} m_\tau = \frac{1 - \gamma_2 \lambda_1}{\lambda_1} = \frac{1}{\lambda_1} - \gamma_2; \tag{46}$$

3) $\lambda_1 = \lambda_2 =: \lambda \in R$; then $\Lambda = 1$ and

$$m_\tau = \frac{(1 - \gamma_2 \lambda)(1 + D)}{\lambda (1 + D)} = \frac{1}{\lambda} - \gamma_2; \tag{47}$$

4) $\lambda_1, \lambda_2 \in C$; then $\lambda_1 = \overline{\lambda_2}$ and

$$\Lambda = \frac{re^{i\varphi}}{re^{-i\varphi}} = e^{2i\varphi} = \cos 2\varphi - i \sin 2\varphi \tag{48}$$

$$\Rightarrow |\Lambda| = (\cos 2\varphi)^2 + (\sin 2\varphi)^2 = 1. \tag{49}$$

In the complex case (4), there would be permanent fluctuations and no steady state. But in each of the real cases (1) - (3), there is a unique stable steady-state equilibrium.[10] According to equation (43), case (4) is avoided if $\gamma_2^2 + 4\gamma_3 \geq 0$, or, using (35), (36) and (37) without time indices,

$$q \geq \frac{4 \frac{(1-\kappa) g}{1+p}}{\left[\frac{(1-\kappa) g}{1+r} + w\right]^2} =: \overline{q} \tag{50}$$

[10] By inserting expression (43), it can be easily shown that the solutions (46) and (45) equal those given in (39) (as long as $\gamma_2^2 + 4\gamma_3 > 0$). But only one of the two cases can be valid.

or

$$\varepsilon \geq \frac{\overline{q}\,(1+\delta)\,(\psi w + c^0)}{1-\overline{q}\,(\psi w + c^0)} := \overline{\varepsilon}, \tag{51}$$

where $\overline{q}$ is given by equation (50) and w by (38). Thus, if the preference for children is sufficiently high ($\varepsilon \geq \overline{\varepsilon}$), the model converges to a unique stable steady state, q. e. d.