Dynamic Multi-Sector CGE Modeling and the Specification of Capital

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Summary

Dynamic multi-sector CGE models often utilize investment aggregation according to fixed shares. The question is, do these models come to the same policy conclusions as CGE models that derive sectoral investment shares by optimal household decisions in a framework with *heterogeneous* capital ? Two models — a model with heterogeneous capital, and a second model with one capital aggregate and *fixed* investment shares — show that the impact of the same given tax shock is strikingly different in the two models. The paper also shows under which conditions the two models bring about "similar" policy conclusions.

1 Introduction

This paper is concerned with a sensitivity analysis of dynamic multi-sector computable general equilibrium (CGE) models with respect to the specification of capital and investment aggregation. The paper argues that simulation results of dynamic CGE models are quite sensitive to the respective specification of capital and investment aggregation. It identifies conditions under which the usual practice of investment aggregation according to fixed shares leads to substantial biases in the results of policy simulations.

Dynamic, computable general equilibrium (CGE) models, of both, the infinitely lived agent (ILA) as well as the overlapping generation (OLG) type, are among the most powerful tools in applied economics. Since the seminal book of Auerbach and Kotlikoff (1987) they have become a standard instrument for economic policy analyses. A decade ago, most dynamic CGE models represented a one sector version of either the Ramsey (1928) type of ILA model or the Diamond (1965) type of OLG model.

Over the last decade, a number of researchers have adopted a dynamic *multi-sector* framework for CGE models. The framework of dynamic *multi-sector* models either considers multiple capital goods (as there are multiple sectors) or it considers one capital aggregate. Some papers make use of the former approach. The other papers, however, follow the latter practice of aggregating sectoral investment by fixed shares to a composite capital aggregate. Examples of this approach include Ballard (1983), Ballard (1989), Farmer and Steininger (1999), Goulder and Summers (1989) or Wendner (2001), to cite a few. In these models, outputs of the industries combine, according to fixed coefficients, to produce a representative capital aggregate. However, the main question to be answered is: does the framework with one capital aggregate and fixed investment shares represent a "sufficiently close approximation" to an optimization model with multiple capital goods that determines investment shares by household optimization? If it does not, the use of a capital aggregate and fixed investment shares leads to biased policy simulations results.

This paper provides a first attempt to look deeper at the question of sensitivity of policy simulation results with regard to the specification of investment aggregation. It identifies the cases for which the fixed investment shares framework can be "safely" used as well as the conditions under which this framework is highly questionable.

We compare the performances of two models. First, we set up a model with one capital aggregate and with *f*ixed investment *s*hares (FS), which will be called *f*ixed shares *m*odel (FM). In this model, investment goods are combined, according to fixed shares, to produce one composite capital aggregate. This model is understood to represent the *typical* multi-sector CGE model with FS. It is often used for multi-sector CGE modeling in the literature.

Next, we set up a model with heterogeneous capital, i.e., with more than one capital good (and with optimal investment shares). This second model is considered to represent a natural *r*eference *m*odel (RM). In multi-sector analysis, if one is not employing a model with one capital aggregate, one has to employ a model with more than one capital good (heterogeneous capital). Thus, in the reference model, capital is heterogeneous and sectoral investment shares are determined by optimal household decisions. The model is otherwise identical to the FM. In the RM investment shares are not exogenously fixed but react to a change in relative prices as well as to a policy shock. Both models are subjected to the same sector-specific policy shocks, as is described below. In accordance with the main question, posed above, the conditions under which the results of the policy simulations differ are investigated.

Both models are OLG models. The reasons for considering OLG models rather than ILA models are twofold. First, the Auerbach and Kotlikoff (1987) OLG framework for tax policy analysis has gained great popularity in applied work. Examples include Ballard (1989), Broer (1997), Farmer and Steininger (1999), Keuschnigg and Kohler (1995), Rasmussen and Rutherford (2001), and Wendner (2001). Second, only the OLG approach can capture important intergenerational issues arising in the analysis of tax reform. Whenever a policy influences two generations in a different way (e.g., benefits the older generation alone) there is an impact on the aggregate savings rate, capital accumulation and economic growth. The ILA model cannot consider this type of effect. Nonetheless, all the results presented in this paper remain valid for an ILA framework as well.

So far, there has never been an attempt to analyze the sensitivity of the results of the same policy shock with respect to the specification of capital in a dynamic multi-sector CGE model. It will be demonstrated that simulation results are quite sensitive to the specification of capital (investment) aggregation. In spite of the substantial differences in the policy results between a model with a capital aggregate (and fixed shares) and a model with heterogeneous capital (and optimal shares), the choice of the one or the other specification is often arbitrarily. Surprisingly, the possible biases that are implied by the use of the fixed shares assumption are in no case pointed out to the reader.

Section 2 of the paper characterizes the typical model with fixed investment shares. Section 3 sets up a model with heterogeneous capital (and optimal investment shares). Section 4 compares the results of policy simulations for the two models. Section 5 identifies key algebraic relationships that explain the sensitivity of the policy outcome with respect to the specification of investment aggregation. Section 6 discusses the lessons to be learned from this analysis.

2 A Model with Fixed Investment Shares

In this section, we develop a model with one capital aggregate and fixed investment shares. The objective here is twofold. First, the model should represent a typical model in that it considers the basic characteristics of CGE models with FS. These include the following features. Consumption and aggregate savings are determined by household optimization; there are several production sectors that produce both consumption and investment goods; investment by sector of origin is determined by exogenously fixed investment shares; sectoral investments are aggregated to a composite capital aggregate; the policy under investigation is sector specific; revenues are recycled back into the economy. Second, in order to gain some insight leading beyond the results of numerical simulation, the model should be analytically tractable.

The simplest specification that is consistent with both aims is a two-sector model with one Cobb-Douglas (CD) and one Leontief (L) production sector. The tax program to be analyzed amounts to a simple type of differential consumption taxation. The tax rate is uniform across generations and does not change over time. Furthermore each generation receives transfers in proportion to its respective tax payment. The present value of tax payments of each generation equals the present value of its respective per capita transfers.[1]

There are two industries $i = x, y$ that produce both consumption and investment goods. The industries operate within a fully competitive environment and maximize profits. Production is specified according to a CD production function for the x-sector and according to L production function for the y-sector. X_t and Y_t denote the quantities produced of each good respectively. Production requires labor services, N_t^i as well as capital services, D_t^i.

Labor is perfectly mobile at the competitively determined nominal wage rate W_t. Initial level, L_0, and growth factor of the labor force, G^L, are exogenously fixed. Productivity increases uniformly across sectors by a factor of G^τ. For the sake of simplicity, we assume that $G^L = G^\tau = 1$. In this case we may set L_t equal to one. This simplification does not alter the results of the analysis.

Let the x-commodity represent the numeraire of the model. Then, the following relative prices can be defined: $p_t \equiv P_t^y/P_t^x$, $w_t \equiv W_t/P_t^x$, $q_t \equiv Q_t/P_t^x$. Relative prices are depicted as lowercase letters. Additionally, we shall express all quantities in per capita terms: $d_t^x \equiv D_t^x/N_t^x$, $d_t^y \equiv D_t^y/N_t^y$,

[1] A similar tax policy is investigated in Auerbach and Kotlikoff (1987, p. 60).

$x_t \equiv X_t/L_t$, $y_t \equiv Y_t/L_t$, $l_t^x \equiv N_t^x/L_t$, $l_t^y \equiv N_t^y/L_t$. The variable p_t represents the relative y-price, w_t denotes the wage rate in units of the x-commodity, and q_t is the capital rental rate in x-units. Variable l_t^x and l_t^y denote the percentages of labor employed in the x- and y-sector respectively.

With this notation at hand, firm behavior is characterized as follows.

$$x_t = l_t^x (d_t^x)^\alpha \quad 0 < \alpha < 1 \tag{1}$$

$$(1-\alpha)(d_t^x)^\alpha = w_t \tag{2}$$

$$\alpha (d_t^x)^{\alpha-1} = q_t \tag{3}$$

$$y_t = \min\{l_t^y/b_0, (d_t^y\, l_t^y)/b_1\} \quad 0 < b_0,\ 0 < b_1 < 1 \tag{4}$$

$$p_t = b_0\, w_t + b_1\, q_t \tag{5}$$

Commodity x is produced according to CD function (1). Coefficient α represents the production elasticity of capital services with respect to x-production. Since industries operate within a fully competitive environment, factors are paid their marginal product respectively ((2) and (3)). Commodity y is produced according to the L function (4). Coefficients b_0 and b_1 denote the labor and capital requirement per unit of y-output respectively. Equation (5) determines the relative y-price. From equation (4) it follows that

$$d_t^y = d^y = b_1/b_0\,. \tag{6}$$

In each period, there are two overlapping generations, the young generation and the old generation. Each generation lives for two periods. In the following, the young generation is indexed by subscript 1 (indicating the first period of life) and the old generation is indexed by subscript 2 (indicating the second period of life).

At the beginning of period t, the young generation offers labor services and receives a wage rate, w_t, in return. It allocates income to consumption of both commodities, $c^x_{t,1}$, $c^y_{t,1}$, and to savings. The old generation does not offer labor services. Consumption of both commodities, $c^x_{t,2}$, $c^y_{t,2}$, is financed by the old generation's rental income.

A simple tax program is considered. Consumption of the x-commodity is taxed at a rate τ. For $\tau > 0$, households receive transfers equal to their tax payments. The tax rate is uniform across generations. Once implemented, the tax program is permanent.[2]

Savings are realized by purchases of both investment goods, I^x_t and I^y_t. Variable I^i_t denotes investment (by sector of origin) of good i. Total nominal savings equals $P^x_t\, I^x_t + P^y_t\, I^y_t$. The value of the composite capital good is denoted as K_t. Then it holds that

$$K_{t+1} = P^x_t\, I^x_t + P^y_t\, I^y_t\,. \tag{7}$$

Thus, total savings in period t equals K_{t+1}. Let k_{t+1} denote the capital K_{t+1} in units of the x-commodity: $k_{t+1} \equiv K_{t+1}/P^x_t$. Then, the first period budget constraint becomes

$$c^x_{t,1}(1+\tau) + p_t\, c^y_{t,1} + k_{t+1} = w_t + t_{t,1}\,. \tag{8}$$

Variable $t_{t,1}$ denotes transfers received by the young generation in terms of the x-commodity. The second period budget constraint becomes

$$c^x_{t+1,2}(1+\tau) + p_{t+1}\, c^y_{t+1,2} = q_{t+1}\, k_{t+1} + t_{t+1,2}\,. \tag{9}$$

Households are identical within as well as across generations. Preferences of a generation, entering the economy at time t, are characterized by the following intertemporal utility function:

$$U_{t,1} = \gamma\, \ln c^x_{t,1} + (1-\gamma) \ln c^y_{t,1} + \beta\, [\gamma\, \ln c^x_{t+1,2} + (1-\gamma) \ln c^y_{t+1,2}]\,. \tag{10}$$

[2]It should be emphasized again that the tax program is designed to be simple enough to allow for analytical insight of its impact on the economy later on. However, it is also designed to be typical (regarding CGE analyses) in that a sector specific tax shock is considered and tax revenues flow back to the households.

Parameter β denotes the time preference factor of the young generation. Parameter γ stands for the elasticity of utility with respect to x-consumption. Households maximize (10) subject to budget constraints (8) and (9) and — implicitly — subject to non-negativity constraints with respect to all decision variables. Optimal consumption and savings are given by (11) to (14).

$$c^x_{t,1} = \frac{\gamma}{(1+\beta)[1+\tau(1-\gamma)]}\, w_t\,, \quad c^y_{t,1} = \frac{(1-\gamma)(1+\tau)}{(1+\beta)[1+\tau(1-\gamma)]}\,\frac{w_t}{p_t} \tag{11}$$

$$c^x_{t,2} = \frac{\beta\,\gamma}{(1+\beta)[1+\tau(1-\gamma)]}\, q_t\, w_{t-1} = \frac{\gamma}{1+\tau(1-\gamma)} q_t\, k_t \tag{12}$$

$$c^y_{t,2} = \frac{\beta\,(1-\gamma)(1+\tau)}{(1+\beta)[1+\tau(1-\gamma)]}\,\frac{q_t}{p_t}\, w_{t-1} = \frac{(1-\gamma)(1+\tau)}{1+\tau(1-\gamma)}\,\frac{q_t}{p_t}\, k_t \tag{13}$$

$$k_{t+1} = \frac{\beta}{1+\beta}\, w_t \tag{14}$$

Investment by sector of origin is determined by exogenously fixed (nominal) shares of total savings, η.

$$I^x_t = \eta\, k_{t+1} \tag{15}$$

In (15) we take into account that total savings in units of commodity x equals k_{t+1}. From (7) and (15) it follows that $p_t\, I^y_t = (1-\eta)k_{t+1}$.[3] Figure 1 depicts the process of investment aggregation.

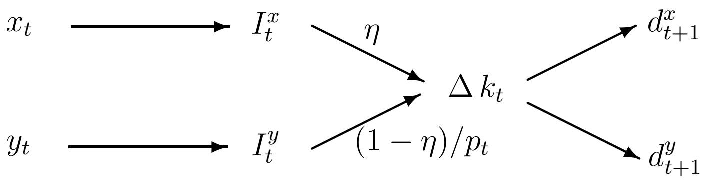

Figure 1. Investment Aggregation in the FM

Because of the competitive nature of the economy, the markets for labor, (16), for capital services, (17), and for both goods, (18), and (19), clear in

[3]Notice that $k_{t+1} = I^x_t + p_t\, I^y_t$. The transformation process follows a Cobb-Douglas technology $k_{t+1} = (I^x)^\eta\,(I^y)^{1-\eta}$ which gives rise to a fixed expenditure share η. To obtain a given increase in the capital stock one can derive cost minimizing investment demand by sector of origin. Equation (15) follows.

each period. On account of Walras' Law, one market clearing equation is redundant.

$$1 = l_t^x + l_t^y \tag{16}$$

$$k_t = d_t^x\, l_t^x + d^y\, l_t^y \tag{17}$$

$$x_t = c_{t,1}^x + c_{t,2}^x + I_t^x \tag{18}$$

$$y_t = c_{t,1}^y + c_{t,2}^y + I_t^y \tag{19}$$

Equations (16) to (19) complete the description of the model.

The model with a capital aggregate (and fixed investment shares) can be characterized by a single equation of motion in d^x.[4] Derivation and boundary conditions are shown in Appendix A.

$$\begin{aligned} &\frac{\alpha\,\gamma\,\sigma\,d^y}{\varphi}(d_{t+1}^x)^{-1}(d_t^x)^{\alpha} - \sigma\left(\frac{\gamma}{\beta\,\varphi}+\eta\right)d_{t+1}^x \\ &= \sigma\left(\frac{\alpha\,\gamma}{\varphi}-1\right)(d_t^x)^{\alpha} + \left[1-\sigma\left(\frac{\gamma}{\beta\,\varphi}+\eta\right)\right]d^y \\ &\text{with } \varphi \equiv 1+\tau(1-\gamma)\,, \quad \sigma \equiv (1-\alpha)\,\beta/(1+\beta) \end{aligned} \tag{20}$$

The steady state of the model with FS, where $k_t = k \Leftrightarrow d_t^x = d^x$, is implicitly defined by

$$(\varphi-\alpha\,\gamma)d^x + d^y[\gamma/\beta+\eta\,\varphi-\varphi/\sigma](d^x)^{1-\alpha} - (\gamma/\beta+\varphi\,\eta)(d^x)^{2-\alpha} + \alpha\,\gamma\,d^y = 0\,. \tag{21}$$

Since $d\,d_{t+1}^x/d\,d_t^x > 0$, the transition path follows a monotonic pattern (see Appendix A). Stability analysis shows that the dynamics of the model is asymptotically stable in the neighborhood of the steady state.

[4] Equivalently, the equation of motion can be expressed as an equation of k. For reasons of simplicity as well as comparability with the reference model, we prefer to express the equation of motion in terms of d^x. Observe that equation (20) can be easily transformed via the relation $k_{t+1} = (1-\alpha)\beta/(1+\beta)(d_t^x)^{\alpha}$.

3 The Reference Model

The FM with one capital aggregate represents a type of model that can often be found in dynamic, multi-sector CGE analyses. However, since the FS model uses one capital aggregate, the question arises as to whether policy results obtained by this type of model are robust with respect to the investment aggregation employed, whereby exogenously fixed investment expenditure shares are used. Does the framework with FS represent a sufficiently close approximation to the "true" optimization model with many capital goods, where investment *shares* are determined by household optimization? In order to be able to address this question, a reference model (RM) is developed against which the model with FS can be compared.

The natural and consistent framework for the appropriate RM is a growth model with *heterogeneous* capital.[5] As pointed out in the introduction, a multi-sector framework may take capital either as one capital aggregate or in form of multiple capital goods (capital aggregates) into account. While the FM considers the case of one capital aggregate, the reference model considers the case with two capital goods. All decisions are derived from optimization calculus and investment allocation by sector of origin follows from no-arbitrage and factor market clearing conditions. Except for the treatment of capital and investment aggregation, the RM parallels the model with FS. Therefore, only the differences of the RM with respect to the FM are presented below.

In the RM, capital goods *by sector of origin* represent true objects of choice. Households allocate savings to two assets, K_t^x and K_t^y. They differentiate the two assets according to their respective rate of return. Only if

[5] Capital is said to be *heterogeneous* if there are two capital goods whose services are factors of production in the economy under consideration.

a no-arbitrage condition — equalizing the rates of return of both assets — holds, will the households demand both assets: $Q^x_{t+1}/P^x_t = Q^y_{t+1}/P^y_t$. The variables Q^i, $i = x, y$ denote the nominal capital rental. Since we consider two capital goods in the RM, the capital rental is sector specific. The no-arbitrage condition (22) ensures an *interior* solution; i.e., households demand both assets, K^x_t as well as K^y_t. In units of the x-commodity, the no-arbitrage condition becomes:

$$q^x_{t+1} = q^y_{t+1}/p_t \quad \text{with } q^i_{t+1} \equiv Q^i_{t+1}/P^x_{t+1}\,. \tag{22}$$

The consideration of two capital goods rather than one composite capital good also modifies the household budget constraints. They become:

$$c^x_{t,1}(1+\tau) + p_t\, c^y_{t,1} + I^x_t + p_t\, I^y_t = w_t + t_{t,1}, \tag{23}$$

$$c^x_{t+1,2}(1+\tau) + p_{t+1}\, c^y_{t+1,2} = q^x_{t+1}\, K^x_{t+1} + q^y_{t+1}\, K^y_{t+1} + t_{t+1,2}\,. \tag{24}$$

Since $L_t = L = 1$, $k^i_{t+1} \equiv K^i_{t+1}/L_{t+1} = K^i_{t+1}$, $i = x, y$. By considering the no-arbitrage condition, (22), the same optimal consumption quantities as in the FM, (11) to (13), follow. Optimal savings become:

$$k^x_{t+1} + p_t\, k^y_{t+1} = \frac{\beta}{1+\beta}\, w_t\,. \tag{14'}$$

The sectoral capital stocks, k^i_t, develop according to

$$k^i_{t+1} = I^i_t \quad i = x, y\,. \tag{25}$$

The two rates of depreciation are set equal to one. The results of the paper do not change when considering other feasible depreciation rates.[6]

The *sector-of-destination* specific capital services are each composed of both capital goods. In order to keep our example as simple as possible, we

[6]Notice that each generation lives for two periods in this OLG economy. The length of a period, thus, is about thirty years. For a period length of thirty years, a rate of depreciation of one might not be considered too unrealistic.

adopt the assumption that capital goods are combined in fixed proportions to form the sector-of-destination specific capital services.

$$d_t^x = \min\left\{\frac{d_t^{xx}}{b_{xx}}, \frac{d_t^{yx}}{b_{yx}}\right\}, \quad d_t^y = \min\left\{\frac{d_t^{xy}}{b_{xy}}, \frac{d_t^{yy}}{b_{yy}}\right\} \tag{26}$$

Variable d^i denotes the quantity of the input of the capital service in the production of good i. Variables d^{ii} and d^{ji} represent the inputs of capital goods i and j for the generation of capital service i. As before, $d_t^{ij} \equiv D_t^{ij}/N_t^j$, where per capita variables are denoted in lowercase letters. The first index denotes the sector of origin, while the second index stands for the sector of destination. The coefficients b_{ij}, $i = x, y$, $j = x, y$ indicate the quantity of capital good i that is required to generate one unit of capital service j (sector j capital input). Figure 2 depicts the capital specification.

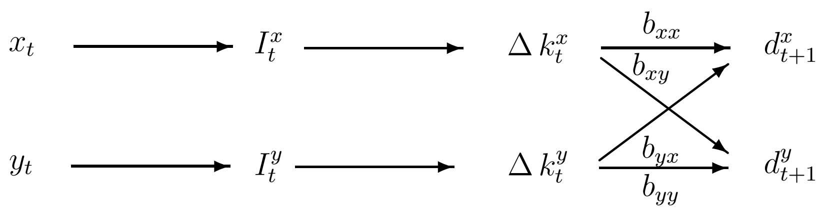

Figure 2. Heterogeneous Capital in the RM

Since capital services are each composed of both capital goods, the cost of a unit of x-capital is equal to $q_t^x\, b_{xx} + q_t^y\, b_{yx}$. Considering the no-arbitrage condition (22), the first order condition (3) becomes:

$$\alpha(d_t^x)^{\alpha-1} = q_t^x(b_{xx} + b_{yx}\, p_{t-1})\,. \tag{3'}$$

For the same reason, the cost of a unit of y-capital equals $q_t^x\, b_{xy} + q_t^y\, b_{yy}$. Therefore, the relative price becomes:

$$p_t = b_0\, w_t + b_1\, q_t^x(b_{xy} + b_{yy}\, p_{t-1})\,. \tag{5'}$$

In equation (5') the no-arbitrage condition (22) is considered. Observe that here, in contrast to the model with FS, the price equation is an intertemporal equation.

Due to the fact that two capital goods are being taken into account, there are two market clearing conditions for capital services in the RM.

$$k^x_t = d^x_t\, b_{xx}\, l^x_t + d^y\, b_{xy}\, l^y_t \tag{27}$$

$$k^y_t = d^x_t\, b_{yx}\, l^x_t + d^y\, b_{yy}\, l^y_t \tag{28}$$

Equations (27) and (28) complete the description of the reference model.

The dynamic system of the RM consists of three equations of motion, (29) to (31), in the variables k^x, d^x and p:[7]

$$d^x_{t+1} = \frac{d^y\, k^x_{t+1}(b_{xy} + p_t\, b_{yy}) - b_{xy}\,\sigma\, d^y (d^x_t)^\alpha}{k^x_{t+1}(b_{xx} + p_t\, b_{yx}) + p_t\,\Delta\, d^y - \sigma\, b_{xx}(d^x_t)^\alpha}\,, \quad \Delta \equiv b_{xx}\, b_{yy} - b_{xy}\, b_{yx}\,, \tag{29}$$

$$\begin{aligned} k^x_{t+1} =& \frac{(\varphi - \alpha\,\gamma)(d^x_t)^\alpha (k^x_t - b_{xy}\, d^y)}{(b_{xx}\, d^x_t - b_{xy}\, d^y)\,\varphi} - \frac{\gamma\,\sigma (d^x_t)^\alpha}{\beta\,\varphi} \\ &+ \frac{\gamma(k^x_t - b_{xx} d^x_t)[p_t - (1-\alpha)\, b_0\, (d^x_t)^\alpha]}{b_0\,(b_{xx}\, d^x_t - b_{xy}\, d^y)\varphi}\,, \end{aligned} \tag{30}$$

$$p_{t+1} = (1-\alpha)\, b_0\, (d^x_{t+1})^\alpha + \alpha\, b_1 (d^x_{t+1})^{\alpha-1} \frac{b_{xy} + b_{yy}\, p_t}{b_{xx} + b_{yx}\, p_t}\,. \tag{31}$$

The dynamic system (29) to (31) and the associated boundary conditions are derived in Appendix B. In (29), a value of $\Delta > 0$ means that both sectors primarily make use of the services of the capital stock produced in their own sector. If $\Delta < 0$, capital services of sector i will be used primarily in the other sector, j.

In contrast to the FM, which was described by a single equation of motion above, the RM has three dimensions. The reason is that the RM considers

[7] (k^x, k^y, p) space is the natural space to look at. However, there is an intraperiod relationship between d^x and k^y, which allows us to express the system equivalently in (k^x, k^y, p) as well as in (k^x, d^x, p) space. In order to be able to compare the results with those of the FM we prefer to look at (k^x, d^x, p) space here.

two capital stocks rather than one composite capital aggregate. Moreover, due to the no-arbitrage condition (22) — which governs the optimal allocation of a household's wealth among the heterogeneous capital goods — the price equation becomes an intertemporal equation.

A fixed point of the dynamic system, where $k_t^x = k^x$, $d_t^x = d^x$, $p_t = p$, defines a steady state of the two-sector OLG model with heterogeneous capital and consumption taxation. Numerical analysis as well as algebraic analysis for special cases shows that if a steady state exists it is unique. At the steady state, local stability analysis indicates that from the three eigenvalues of the dynamic system two are lower and one is larger than unity in absolute value. Thus, the steady state is saddle-path stable.

By taking the total derivative of (31) and solving for $d\,p/d\,d^x$, we find that in a steady state the relative price decreases with rising d^x if $(b_{xx} + b_{yx}\,p)d^x - (b_{xy} + b_{yy}\,p)d^y < 0$ and otherwise increases:

$$\frac{d\,p}{d\,d^x} = \frac{\alpha(1-\alpha)(d^x)^{\alpha-1}\,p\,b_0\,B/d^x}{b_{yx}\,p^2 + (1-\alpha)b_0\,b_{xx}(d^x)^{\alpha} + \alpha\,b_1\,b_{xy}(d^x)^{\alpha-1}} \tag{32}$$
$$\text{with } B \equiv (b_{xx} + b_{yx}\,p)d^x - (b_{xy} + b_{yy}\,p)d^y\,.$$

The term B displays the *generalized capital intensity condition* of the two-sector model with heterogeneous capital and optimal sectoral investments.[8] If $B > 0$, sector x is capital intensive and sector y is labor intensive. The relative price of y-output increases with rising d^x if the x-sector is more capital intensive than the y-sector, i.e. if more composite capital services per unit of labor are required in the x- than in the y-sector. Intuitively, a rise in d^x increases the wage-rental ratio, w/q^x, and makes the more labor intensive good more expensive compared to the other good. Thus, if $B > 0$, y-production is more labor intensive than x-production. Consequently the relative y-price, p, rises.

[8]The capital intensity in the model with FS is simply given by the term $d^x - d^y$.

4 Comparative Policy Simulations

Now we are well-prepared to go after the main question, i.e. does the model with one capital aggregate represent a "sufficiently close" approximation to the reference model, in which investment shares are determined by household optimization? As we shall see, the answer depends on which policy results we are interested in. If we are interested in the long-term impact of a tax reform, the answer is probably yes. If we are interested in the short-term impact, the answer is no. If we look at small policy shocks or at the impact on aggregate variables, the answer is probably yes. If we look at large policy shocks or at sectoral variables, the answer is no.

In this section we employ a number of numerical simulations to gain an idea about the significance of the specification of investment aggregation with regard to the differences in RM and FM policy results. In the following section, we provide economic explanations of the (qualitative) policy impact on various economic variables. We identify key algebraic characteristics that explain the sensitivity of the policy outcome with respect to the specification of investment aggregation.

The two models (RM and FM) are calibrated for a steady state such that in a base case all variables and common parameters are identical. This implies that for given values of α, β, b_0, b_1, η, γ, and τ the model with FS determines d^x. Given d^x (along with α, β, b_0, b_1, η, γ, and τ) the additional parameters in the RM are calibrated such that $k^x/(k^x + p\,k^y) = \eta$. In particular, this procedure allows for exogenously fixing one of the parameters b_{ij}. The remaining three parameters are determined by the steady state relations of the dynamic system of the RM. This calibration procedure ensures that all variables and common parameters of the two models are identical in a base case, i.e. before the policy shock.

Four different parameter sets — PS A to PS D — are investigated. They give rise to four different base cases. PS A differs from the other parameter sets in that the capital intensity of x-production falls short of y-capital intensity. Thus, $(d^x - d^y) = B < 0$ in the PS A base case.[9] In PS B both shares, η and γ, are high ($\eta = \gamma = 0.8$). In PS C the consumption share, γ, is small ($\gamma = 0.4$). In PS D the investment share, η, is small ($\eta = 0.3$). The parameters β, b_0, b_1, and b_{xy} are identical across parameter sets. The parameters b_{xx}, b_{yx}, and b_{yy} are calibrated such that the respective base cases of the RM are identical to those of the FM. Table 4 in Appendix C shows all four parameter sets and associated base cases.[10]

Two policy shocks are simulated. First, the tax on x-consumption, τ, is increased from zero to 20%. The impact of this tax rise on aggregate and sectoral variables is compared for the RM and the FM for all four parameter sets. Second, τ is raised from zero to 10%, to 20%, to 30%, and to 40%. The impact of an increasing strength of the tax shock on aggregate and sectoral variables is compared for the RM and the FM for one parameter set.

In order to compare differences in the effects of the tax shocks between the RM and the FM a simple distance measure, d, is employed. The distance for a variable z_t in a period t is defined to be equal to the percentage change of z_t in the FM (due to the policy shock) minus the percentage change of z_t in the RM: $d_{z_t} \equiv \%z_t|_{FM} - \%z_t|_{RM}$. Moreover, we shall employ two further measures, a function measuring the sum of absolute values of the distances of individual variables, $d_{\sum}$, as well as a function measuring the

[9]Technically, the production elasticity, α, is higher in PS A ($\alpha = 0.4$) than in the other parameter sets ($\alpha = 0.2$).

[10]Actually, the authors investigated forty widely different parameter sets of which the four parameter sets reported in this paper are representative ones. A working paper that discusses the whole set of parameter sets more comprehensively is available from the authors upon request.

average distance of a variable, $d_\varnothing$: $d_\sum \equiv \sum_{i=1}^{n} |d_{z^i}|$, $d_\varnothing \equiv d_\sum/n$. It is obvious that these measures are crude ones. However, they approximate information concerning the difference in the policy results between RM and FM, and they facilitate ranking according to (dis)similarity across parameter sets and policy shocks.

Table 1 shows steady state results of an increase of the x-consumption tax rate, τ, from zero to 20%. For each of the four parameter sets, the three columns show the percentage changes of the variables of the RM and the FM with respect to the base case values and the distance measure. The table differentiates between aggregate and sector specific variables. The last row shows the sum of absolute values of the distances of individual variables as well as the average distance of a variable for the four parameter sets. The table displays five key results.

Capital intensity. The four parameter sets differ in the variables α, γ, η, and — implicitly — in b_{ij}. The output elasticity of capital services, α, mainly determines x-capital intensity. For a given value of d^y, the lower α is, the higher the x-capital intensity, d^x. Parameter set A is associated with a relatively high value of α. Thus, the rate of interest is large and d^x is small compared to the other parameter sets. In particular, $d^x - d^y < 0$ in the FM and $B < 0$ in the RM.[11] Table 1 shows that the difference between RM and FM in the impact of the policy shock on all variables is substantially larger when the y-sector is more capital intensive than the x-sector. This is true regardless of the values of γ and η. While the distance sum, $d_\sum$, equals 28 for PS A, it amounts to between 10 and 4 for the other three parameter sets. The same is true for distances in individual variables. In no case is an individual variable's d for PS A smaller than for any of the other parameter

[11]Observe that $B \equiv (b_{xx} + b_{yx}\,p)d^x - (b_{xy} + b_{yy}\,p)d^y$ displays the generalized capital intensity condition in the RM.

Table 1. Steady State Results (percentage changes)

	A			B			C			D		
Agg. Variables	RM	FM	d	RM	FM	d	RM	FM	d	RM	FM	d
Real GDP	-1.193	-0.754	0	0.053	0.050	0	0.257	0.237	0	0.082	0.112	0
Consumption	-0.526	-0.269	0	-0.038	-0.033	0	-0.018	-0.015	0	-0.046	-0.030	0
Investment (k)	-3.373	-2.338	1	0.227	0.208	0	0.758	0.694	0	0.320	0.374	0
w	-3.373	-2.338	1	0.227	0.208	0	0.758	0.694	0	0.320	0.374	0
p	3.085	1.498	-2	0.059	0.052	0	0.301	0.260	0	0.099	0.120	0
Utility	-0.946	-0.607	0	-0.244	-0.233	0	-0.573	-0.535	0	-0.296	-0.301	0
Sectoral Variables												
x	-4.758	-3.691	1	-2.738	-2.470	0	-5.372	-4.749	1	-2.973	-3.160	0
y	9.684	9.355	0	11.154	10.077	-1	6.591	5.857	-1	5.065	5.434	0
d^x	-8.220	-5.744	2	1.141	1.045	0	3.846	3.519	0	1.608	1.883	0
I^x (k^x in RM)	-6.086	-2.338	4	-0.560	0.208	1	-0.493	0.694	1	1.569	0.374	-1
I^y (k^y in RM)	4.264	-3.780	-8	3.314	0.157	-3	5.443	0.433	-5	-0.314	0.253	1
l^x	-1.434	-1.385	0	-2.959	-2.673	0	-6.084	-5.406	1	-3.282	-3.521	0
l^y	9.684	9.355	0	11.154	10.077	-1	6.591	5.857	-1	5.065	5.434	0
q (q^x in RM)	5.025	3.613	-1	-0.912	-0.828	0	-2.986	-2.729	0	-1.345	-1.481	0
c_1^x	- 7.089	-6.095	1	-3.628	-3.646	0	-10.038	-10.095	0	-3.539	-3.487	0
c_1^y	8.157	11.023	3	15.578	15.565	0	7.631	7.607	0	15.639	15.677	0
c_2^x	-2.421	-2.702	0	-4.507	-4.444	0	-12.724	-12.548	0	-4.836	-4.916	0
c_2^y	13.591	15.034	1	14.524	14.608	0	4.417	4.671	0	14.084	13.963	0
η	-2.808	–		-0.785	–		-1.241	–		1.245	–	
$d_{\sum}$ ($d_{\varnothing}$)	28 (1.5)			7 (0.4)			10 (0.5)			4 (0.2)		

sets where $d^x - d^y > 0$ ($B > 0$). On average, the difference of a variable's change between the RM and the FM is 1.5 percentage points in PS A and between 0.2 and 0.4 percentage points in the parameter sets where $d^x - d^y > 0$ ($B > 0$). One direct explanation for this behavior is given by the magnitudes of d^x and d^y. A given change in τ requires a bigger percentage change in d^x — and implicitly in η — the smaller d^x (the bigger α) is. Thus, since η is fixed in the FM, the difference of the policy impacts between RM and FM increases the smaller d^x is. Thus, the difference in the policy result between RM and FM is larger for PS A than for the other three parameter sets. The smaller the capital intensity of the taxed sector the bigger are the differences in the policy results between RM and FM.

The x-consumption share. Second, parameter sets B and C allow us to assess the impact of γ on the differences in policy results for RM and FM. The x-consumption share is high in parameter set B ($\gamma = 0.8$) while it is low in parameter set C ($\gamma = 0.4$). In both parameter sets $\eta = 0.8$.[12] The lower the consumption share, the bigger is the difference between RM and FM. The distance measure (sum) equals 10 for PS C while it equals 7 for PS B. Economically, the tax shock enters the economy via $\varphi \equiv 1 + \tau(1-\gamma)$, which affects optimal consumption. Variable φ is the more strongly affected the lower the γ is. If x-consumption is assigned a low weight in the utility function, a given tax shock will imply a strong response in c^x; otherwise, with γ being big, the same tax shock will imply a small response in consumption. Therefore, the policy results diverge more strongly between RM and FM for PS C than for PS B. Generally, in the case of a consumption tax, the policy results differ more between the RM and the FM the lower the weight, γ, is for the taxed consumption good in the utility function.

[12] Calibration requires a rise in γ to be associated with a decline in b_{xx}.

The investment share. Third, parameter sets B and D allow us to assess the impact of η on the differences of the tax shock on the variables between RM and FM. Observe that in PS B η, b_{xx}, and k^x are large and b_{yx}, and k^y are small. The opposite is true for PS D. The smaller the η, the less the policy results for RM and FM differ.

Here, economic intuition is not as straightforward as it is concerning the consumption share. The technological parameter Δ determines the (intraperiod) relationship between d^x, k^x, and k^y according to: $d^x = [(b_{xy}\, k^y - b_{yy}\, k^x) d^y]/[b_{xx}\, k^y - b_{yx}\, k^x - \Delta\, d^y]$.[13] A given increase in d^x is usually associated with a decline in k^x and a rise in k^y if $\Delta \equiv (b_{xx}\, b_{yy} - b_{xy}\, b_{yx}) > 0$, and it is associated with a rise in k^x and a decline in k^y if $\Delta < 0$. The policy shock brings about an increase in d^x.

In PS B, k^x declines because $\Delta > 0$. Since k^x is very large, $l^x = (k^x - b_{xy}\, d^y)/(b_{xx}\, d^x - b_{xy}\, d^y)$ also falls and l^y (starting from a low value) strongly increases. This brings about a strong increase in $y = l^y/b_0$. In spite of a rise in y-consumption, y-investment has to increase strongly to clear markets. This response is very different from that implied by the FM because due to (15) both sectoral investments must always move in the same direction as d^x.

In PS D, k^x is *small* (because of a small η) and rises (because $\Delta < 0$) by a small amount. Consequently, l^y as well as y-production change by less compared to PS B. Therefore, a part of the increase in y-consumption demand is compensated by a decline in y-investment and the other part is compensated by an increase in y-production. The decline in y-investment is necessarily smaller than the change in y-investment in PS B.

Since the reference model's responses of I^x and I^y are unequal in sign the

[13]This relationship follows directly from the two capital services market clearing conditions (27) and (28).

more moderate responses implied by PS D resemble more closely the responses shown in the FM than the policy responses implied by PS B. The more general result is that the larger the fixed investment share of the taxed sector, the more different are the policy responses between RM and FM. Moreover, as is shown below, the more different the parameters b_{xx}, b_{xy}, and η are, the more different are the policy results for RM and FM.

Fourth, except for PS A, the policy responses between RM and FM are very similar for all *aggregate* variables. Even for PS A, the differences shown for aggregate variables are never more than two percentage points. Most of the differences occur in sectoral variables. The distances are largest for sectoral investment expenditures as well as sectoral production. This result calls into question the relevance of FS models since multi-sector models are often specifically employed in order to investigate sector-specific effects of a policy shock.

Fifth, a glimpse at the distance sums shows that steady state results are generally "similar". Overall, the average difference of the policy responses between individual variables of the RM and the FM amounts to between 1.5 and 0.2 percentage points. However, policy responses of sectoral investment expenditures implied by the FM and by the RM regularly differ in sign.

Table 2 shows the short run results of the same policy shock. The results in the table are average percentage changes with respect to the base case value over the first three periods (starting with the period of policy implementation). Alternatively, one could also compare the policy responses in the period of policy implementation. This comparison, however, would essentially reveal large differences between RM and FM as the dynamic systems of the two models differ greatly. Consequently, the initial conditions require different variables to be fixed in the period of policy implementation.

Consideration of the average percentage changes over the first three periods is intentional since it provides a comparison which favors greater similarity of the short run policy responses. Table 5 in Appendix C shows *first period policy effects* for both models.

The table shows that the differences in short run policy responses between RM and FM are far bigger than those reported for steady state policy effects. The difference in policy responses of individual variables amounts to up to 66 percentage points. The average distance lies between 3.6% and 12.4%.

In addition to the big quantitative differences many responses of individual variables differ in sign between RM and FM. These variables include d^x, w, I^x, I^y, consumption expenditures, and real GDP. This result is especially detrimental for FS models as in many cases short run policy results are of greater interest than very long run results.[14]

The table also shows the significant role of the relative price for allocating savings to the sectoral investments in the RM. While the relative price is fully determined by d^x in the FM, it is a *dynamic* variable jumping on the new saddle path upon implementation of the policy in the RM. With the relative price behaving that differently, it is not too surprising that the sectoral variables in the two models respond so differently to the policy shock in the short run.

While the capital intensity determines the type of policy response, it does not delimit the distance in policy results between RM and FM in the short run. In contrast to the steady state results, policy responses are not more different when $(d^x - d^y) < 0$ (or $B < 0$) than when $(d^x - d^y) > 0$ (or $B > 0$).

[14] One example is dynamic CGE studies that analyze the impact of CO_2-taxation in order to comply with the Kyoto Protocol. Many of these studies, including Farmer and Steininger (1999) and Wendner (2001), introduce a sectorally differentiated consumption tax in order to reach a specified CO_2-objective. Within the framework of a model with fixed investment shares, they look at short-run policy results.

Table 2. Short Run Results: First Three Periods (average percentage changes)

	A			B			C			D		
Agg. Variables	RM	FM	d	RM	FM	d	RM	FM	d	RM	FM	d
Real GDP	0.984	-0.223	-1	1.249	0.032	-1	4.792	0.163	-5	0.537	0.075	0
Consumption	-0.317	-0.048	0	-3.897	-0.031	4	5.827	-0.002	-6	15.462	-0.022	-15
Investment	-2.385	-1.656	1	-0.372	0.186	1	-0.662	0.587	1	-0.447	0.323	1
w	-2.385	-1.656	1	-0.372	0.186	1	-0.662	0.587	1	-0.447	0.323	1
p	12.180	1.048	-11	8.082	0.046	-8	11.541	0.219	-11	2.402	0.103	-2
Utility	-1.809	-0.470	1	-8.091	-0.225	8	-3.751	-0.488	3	6.724	-0.280	-7
Sectoral Variables												
x	-3.043	-3.174	0	-2.186	-2.488	0	-4.366	-4.829	0	-2.377	-3.192	-1
y	4.608	10.425	6	6.874	10.063	3	4.051	5.834	2	2.997	5.407	2
d^x	-5.803	-4.086	2	-1.827	0.933	3	-3.202	2.972	6	-2.178	1.624	4
I^x	-6.747	-1.656	5	-2.871	0.186	3	-4.916	0.587	6	-0.187	0.323	1
I^y	2.616	-2.674	-5	1.576	0.140	-1	4.426	0.368	-4	-2.800	0.219	3
k^x (RM)	-4.445	-		-2.563	-		-4.408	-		-0.637	-	-
k^y (RM)	1.496	-		0.558	-		2.679	-		-2.594	-	-
l^x	-0.682	-1.544	-1	-1.823	-2.669	-1	-3.739	-5.384	-2	-1.942	-3.503	-2
l^y	4.608	10.425	6	6.874	10.063	3	4.051	5.834	2	2.997	5.407	2
q (q^x in RM)	0.650	2.539	2	-20.199	-0.740	19	-5.070	-2.314	3	53.235	-1.280	-55
c_1^x	-6.139	-5.438	1	-4.204	-3.667	1	-11.306	-10.190	1	-4.276	-3.536	1
c_1^y	0.751	12.299	12	7.317	15.546	8	-2.967	7.537	11	12.317	15.637	3
c_2^x	-2.748	-2.802	0	-15.885	-4.385	12	11.157	-12.230	-23	52.102	-4.758	-57
c_2^y	4.121	15.430	11	-8.228	14.685	23	15.976	5.093	-11	79.936	14.172	-66
η (FM)	-4.446	0		-2.503	0		-4.267	0		0.270	0	
$d_{\sum}$ ($d_{\varnothing}$)		66 (3.6)			100 (5.5)			98 (5.4)			223 (12.4)	

Table 3 sheds some light on one further question: Is the difference in policy responses between RM and FM dependent on the magnitude of the policy shock? To provide a first answer to this question, τ is raised from zero to 10%, to 20%, to 30%, and to 40%. The steady state impact of an increasing magnitude of the tax shock on aggregate and sectoral variables is compared for RM and FM for parameter set B.[15]

The table displays two results. First, the difference in policy responses between RM and FM (both individual distances as well as the distance sum) increases with increasing strength of the policy shock. An increase of τ by 10 percentage points implies a distance sum of 4, while for an increase of τ by 40 percentage points, $d_{\sum}$ equals 14. Second, in a steady state, an increase of the magnitude of the policy shock affects sectoral variables more strongly than aggregate variables.

5 Sources for Differences in Policy Responses

The comparative policy simulations have shown that generally RM and FM bring about dramatically different policy responses to the same tax shock. In particular, policy responses greatly differ between RM and FM in the short run both in size and in sign. This section investigates the origins of these differences.

5.1 Short Run Impact of Differential Policy

Steady state results of a differential policy are shown to be "similar" between RM and FM, in that the average distance does not exceed 1.5 percentage

[15]PS B was chosen because its distance sum lies in between the two extremes (PS A and PS D). Experimentation with the other parameter sets shows that the qualitative results — concerning the difference in policy responses between RM and FM — hold equally for the other parameter sets.

Table 3. Similarity and Magnitude of the Policy Shock

	τ + 10%			τ + 20%			τ + 30%			τ + 40%		
Agg. Variables	RM	FM	d	RM	FM	d	RM	FM	d	RM	FM	d
Real GDP	0.027	0.025	0	0.053	0.050	0	0.080	0.075	0	0.107	0.100	0
Consumption	-0.019	-0.017	0	-0.038	-0.033	0	-0.056	-0.048	0	-0.074	-0.063	0
Investment	0.114	0.105	0	0.227	0.208	0	0.339	0.311	0	0.451	0.412	0
w	0.114	0.105	0	0.227	0.208	0	0.339	0.311	0	0.451	0.412	0
p	0.030	0.026	0	0.059	0.052	0	0.089	0.077	0	0.120	0.103	0
Utility	-0.087	-0.082	0	-0.244	-0.233	0	-0.455	-0.439	0	-0.710	-0.690	0
Sectoral Variables												
x	-1.397	-1.260	0	-2.738	-2.470	0	-4.028	-3.634	0	-5.268	-4.753	1
y	5.689	5.139	-1	11.154	10.077	-1	16.407	14.823	-2	21.460	19.390	-2
d^x	0.572	0.524	0	1.141	1.045	0	1.709	1.563	0	2.275	2.078	0
I^x (k^x in RM)	-0.287	0.105	0	-0.560	0.208	1	-0.819	0.311	1	-1.065	0.412	1
I^y (k^y in RM)	1.689	0.079	-2	3.314	0.157	-3	4.879	0.233	-5	6.385	0.309	-6
l^x	-1.509	-1.363	0	-2.959	-2.673	0	-4.352	-3.932	0	-5.693	-5.144	1
l^y	5.689	5.139	-1	11.154	10.077	-1	16.407	14.823	-2	21.460	19.390	-2
q (q^x in RM)	-0.459	-0.417	0	-0.912	-0.828	0	-1.359	-1.233	0	-1.800	-1.632	0
c_1^x	-1.849	-1.858	0	-3.628	-3.646	0	-5.340	-5.367	0	-6.990	-7.026	0
c_1^y	7.934	7.928	0	15.578	15.565	0	22.948	22.927	0	30.059	30.030	0
c_2^x	-2.300	-2.268	0	-4.507	-4.444	0	-6.627	-6.534	0	-8.664	-8.543	0
c_2^y	7.438	7.478	0	14.524	14.608	0	21.277	21.411	0	27.718	27.908	0
η	-0.401	-		-0.785	-		-1.154	-		-1.509	-	
$d_{\sum}$ ($d_{\varnothing}$)		4 (0.2)			7 (0.4)			10 (0.6)			14 (0.8)	

points. However, it is most important to notice that the policy results in the RM come about in a manner different to those of the FM. Consequently, short run policy responses are dramatically different for RM and FM. In order to gain a better understanding of the source of these differences, the short-run impact of differential policy in both models is discussed below, whereby the differences in the way policy responses come about receive particular emphasis.

Transitional Dynamics of Differential Policy in the FM. There is one equation of motion and one state variable, k, in the FM. The latter is fixed in the tax implementation period. Capital *services* and prices, however, are free to vary.[16] Upon the introduction of the tax program demand for y-goods increases and demand for x-goods diminishes. Thus, the relative price must rise. If $(d^x - d^y) < 0$, an increase in p implies a decline in d^x. If $(d^x - d^y) > 0$, an increase in p implies a rise in d^x.[17]

The labor share $l^x = (k_t - d^y)/(d_t^x - d^y)$. If $d^x \lesseqgtr d^y$ then $k \lesseqgtr d^y$. Thus independent of whether $(d^x - d^y)$ is negative or positive the labor share l^x decreases, l^y rises and y-production increases. In the case that $(d^x - d^y) < 0$ (e.g., PS A), d^x declines and so does the wage. Hence, also x-investment as well as aggregate investment go down. If $(d^x - d^y) > 0$, x-investment as well as aggregate investment improve. As a consequence, savings and growth are enhanced in this case. However, as Table 5 shows, utility always declines as a consequence of the implementation of the tax.[18]

Observe two important general points. First, the relative price is adapting *intra*temporally to a market imbalance. This is a consequence of the homo-

[16] According to the initial condition for d^x, which is given in Appendix A, d_0^x may either increase or decrease upon the introduction of the consumption tax.

[17] $\partial p/\partial d^x = \alpha(1-\alpha)b_0\,(d^x)^{\alpha-2}(d^x - d^y)$

[18] Table 1 shows that utility not only declines for generations alive in the period of policy implementation but also for generations alive in the steady state.

geneity of capital. Second, the investment share, η, is exogenously fixed. Thus, sectoral investment demands vary in proportion to the change in d^x. Both characteristics temper the policy response.

Transitional Dynamics of Differential Policy in the RM. The dynamic system of the RM consists of three equations of motion. Since the steady state is a saddle (see following subsection) there are two state variables, k^x and k^y, and one "jump variable", p. Both capital stocks are fixed in the period of policy implementation. They are determined by investment decisions taken a period before. According to the initial condition for d^x, which is given in Appendix B, d_0^x remains unchanged upon the introduction of the consumption tax. Hence, the allocation of capital services across the two sectors as well as sectoral outputs are also fixed in the period of policy implementation.

The equilibrium dynamics requires the relative price to jump onto the new (post-policy) saddle path of the dynamic system. The relative price has two functions here. First, the price adjusts in order to restore market equilibrium. Second, the price adjusts in order to reinstate the intertemporal no-arbitrage condition, (22).

Upon introduction of the consumption tax the relative price jumps upward. Since d_0^x remains constant in the period of policy implementation, x-consumption of the young household decreases for sure. Depending on whether the relative price jumps up or down, as well as on the magnitude of the jump, the responses of y-consumption of the young household and consumption of the older household are ambiguous. Let us assume, for the moment, that there was no change of the relative price in the period of policy implementation. Then, $k_0^x + k_0^y p_{-1}$ does not change either. Moreover, the no-arbitrage condition is satisfied. Accordingly, neither x-investment nor

y-investment changes.[19] Since wealth, as well as p_0, is unchanged under this assumption, $c^y_{0,1}$ and $c^y_{0,2}$ rise, while $c^x_{0,2}$ declines. Accordingly, while both output quantities remain unchanged, the tax shock produces excess supply on the x-market and excess demand on the y-market. An upward jump of the relative price restores equilibrium.

According to wealth equation (14') savings and aggregate investment is constant in the period of policy implementation. According to (31) p_0 and p_{-1} increase. Since d^x_0 is constant, q^x_0 declines. Savings are allocated to sectoral investments in order to ensure the no-arbitrage condition in the period following the time of policy implementation: $q^x_1 = q^y_1/p_0$. An increase in p_0 requires I^x_0 to go down. Whether I^y_0 decreases or increases depends on b_{yx}. The larger b_{yx}, the more a given unit of I^y_0 raises d^x_1 and the more likely it is that I^y_0 has to decline.

In the periods following the policy implementation the relative price converges towards its new steady-state value from above. As a consequence of the increase in p, capital service d^x_1 always falls short of its pre-policy base case value. This short-run response is independent of the value of the generalized capital intensity B. Thereafter, d^x_t converges to its new steady-state value. The new steady state price exceeds its pre-policy value. Therefore, as argued in Section 3, the new steady state value of d^x exceeds its pre-policy value if $B > 0$ (x is more capital intensive than y), and it falls short of its pre-policy value if $B < 0$.

5.2 Dynamic Systems of RM and FM

The dynamic system of the FM consists of the single equation of motion, (20). There is one state variable, k, and the fixed point of (20) is locally,

[19] Remember that the tax policy is income compensated.

asymptotically stable.

In contrast, the dynamic system of the RM consists of three equations of motion, (29) to (31). There are two state variables, k^x, and k^y and one "jump-variable", p. Since capital is heterogeneous in the RM, a no-arbitrage condition that equalizes the rates of return on capital goods holds in the RM. Outside of a steady state this no-arbitrage condition implies a non-trivial dynamics of the relative price. As a consequence, the steady state is always a saddle in the RM. Several implications follow from the differences between RM and FM in the structure of the dynamic systems.

First, the heterogeneity of capital in the RM allows for a much richer menu of dynamical behavior compared to the FM with homogeneous capital. This was demonstrated by the differences in short run policy responses above.

Second, short run policy responses are more moderate in the FM than in the RM. The heterogeneity of capital in the RM implies that the transition from one steady state to another occurs on a saddle-path. In response to the tax program the relative price jumps onto the new saddle-path of the dynamic system in the tax implementation period. However, both capital stocks and, by means of the factor market clearing conditions, the allocation of capital services across the two sectors, are fixed in the period of policy implementation. Thus, the jump in the relative price represents the only possibility to respond to the policy shock in RM. In contrast, the FM responds by adapting the allocation of capital services across sectors even in the period of policy implementation. Therefore, the initial policy responses are seen to be more moderate compared with those in the RM.

Third, the transition path is always monotone and nonoscillatory in the FM. In contrast, the transition path is non-monotone and either nonoscillatory or oscillatory in the RM. Monotonicity of a transition path in FM follows directly from the equation of motion, (20). As shown in (44) (Appendix A), $d\,d^x_{t+1}/d\,d^x_t$ does not change sign and is always positive. Thus, d^x either increases monotonically or decreases monotonically along the transition path, and so do all other variables in the FM.

In the RM, two cases are to be distinguished: $\Delta > 0$ and $\Delta < 0$. Suppose $\Delta > 0$. Then all three eigenvalues of the Jacobian of (29) to (31) are positive and distinct. One of them is larger than and two are less than unity. Thus, the steady state is a saddle and there exists a unique saddle path.[20]

If, however, $\Delta < 0$, two eigenvalues are negative, one larger, one less than unity in absolute value. In this case, we encounter an oscillating saddle path!

Fourth, the RM brings about a capitalization effect, which has no counterpart in the FM. If the policy program raises p_0 as well as p_{-1}, then households experience a gain in their asset values. Consequently, $c^x_{0,2}$, and $c^y_{0,2}$ may rise upon the introduction of the tax program. Old households may encounter a gain in utility upon introduction of the tax program, since their capital goods may be valued more favorably. An example of this kind of behavior is shown in PS D where utility of the household born in period 0 increases upon the introduction of the tax program.

Whenever the transition path is oscillatory in the RM, p_0 jumps up, hence p_{-1} goes down. Consequently, there is a negative capitalization effect for old households. Whether consumption goes down or not depends on both the price effect as well as the tax effect.

5.3 Steady State Impact of Differential Policy

The last point of our discussion is devoted to the following question: Under which conditions can we expect identical steady state effects? The analysis

[20] An appendix that discusses the dynamic system of the RM more deeply is available from the authors upon request.

will allow us to derive two conclusions. First, heterogeneity of capital does not cause differences in steady state policy responses between RM and FM in every case. Steady state responses are necessarily different if the composition of capital services varies among sectors. We will show that steady state policy responses are identical if and only if the composition of capital services is identical across sectors. Second, even if the composition is identical, short run policy responses will still differ because of the different characteristics of the dynamic systems of the RM and the FM. We analyze these below, and then we derive the necessary and sufficient conditions for the equivalence of steady state results.

Differential Policy in the FM. In order to analyze the steady state impact of the tax program, we characterize the FM by two steady state functions in $(d^x,\ \eta\, k)$ space.[21] The first function, the *ww*-function, follows from wealth accumulation, (14).

$$\eta\, k = \eta\, \sigma\, (d^x)^\alpha \tag{33}$$

Equation (33) shows all combinations of $\eta\, k$ and d^x for which the wealth of succeeding young generations remains stationary. It is an increasing function in d^x, since lifetime income and hence savings, which fosters capital accumulation, increase in d^x.

Next, the *xx*-function follows directly from the x-market clearing condition, (18). It shows all combinations of $\eta\, k$ and d^x for which goods and factor markets clear in the steady state.

$$\eta\, k = C^{-1}\, \{\eta\, d^y\, (d^x)^\alpha + A\, \sigma\, [\gamma/(\beta\, \varphi)\, (d^x)^\alpha + \alpha\, \gamma\, (d^x)^{2\alpha-1}/\varphi]\} \tag{34}$$

$$\text{with } A \equiv \eta(d^x - d^y),\ \ C \equiv (d^x)^\alpha - A$$

[21] In the analysis of the FM, $\eta\, k$ $(= I^x)$ is intended to represent the equivalent to the reference model's x-capital k^x $(=I^x)$.

The xx-function is also increasing in d^x. We only show the argument for $d^x > d^y$ (the argument for $d^x < d^y$ is similar). Let's increase k and hold d^x constant. The increase in k raises l^x and thus x-supply for sure, while it reduces l^y and therefore y-supply. For given d^x, all consumption quantities remain unchanged. However, investment demand in both sectors increases due to the rise in k. Thus, y-demand grows and establishes excess demand in the y-sector. In order to resolve the disequilibrium, the relative price must rise. We already showed above that $d\,p/d\,d^x > 0$ if $d^x - d^y > 0$. Hence, a rise in k is associated with an increase in d^x.

Graphically, a steady state can be displayed by the intersection of (33) with (34). By determining the shifts of the ww-function as well as the xx-function due to the policy shock we are able to determine the impact of an increase in τ on $\eta\,k$ and d^x.

Due to intertemporal income compensation a rise of the tax rate on x-consumption does not directly affect savings. Thus, the ww-line does not shift. However, optimal consumption decisions are influenced. Graphically, imagine d^x on the abscissa and $\eta\,k$ on the ordinate.

$$\frac{\partial\,(\eta\,k)}{\partial\,\tau}|_{d^x} = -\frac{A\,\gamma\,(1-\gamma)\sigma}{C\,\varphi^2}[(d^x)^\alpha/\beta + \alpha\,(d^x)^{2\,\alpha-1}] \tag{35}$$

Derivative (35) indicates the vertical shift of the xx-function (for given d^x) in $(d^x,\,\eta\,k)$ space. From (34) it follows that $C > 0$.[22] Thus, the xx-graph shifts down if $A > 0$, i.e., if $d^x > d^y$, and it shifts up if $A < 0$. Consider again the case of $A > 0$. Then derivative (35) is clearly negative. For given d^x and k the tax program reduces both c_1^x and c_2^x and leaves x- and y-investment unaltered. At the same time, y-consumption increases. Thus, the tax shock leads to excess supply in the x-good market and to excess demand in the y-

[22]Suppose $d^x < d^y$, then it follows directly that $C > 0$. Suppose $d^x > d^y$, then $C > 0$. Otherwise, k would be negative.

good market. In order to remove the disequilibrium, the relative price must increase. For $A > 0$, the increase in the price requires d^x to rise. Therefore the xx-function shifts down (and to the right). Since the slope of the xx-line exceeds that of the ww-line, both d^x and $\eta\, k$ rise as a consequence of the tax shock.

The capital intensity condition determines the impact of the tax program on d^x and $\eta\, k$. Both variables rise if the x-sector is more capital intensive than the y-sector. Where this is not the case, both variables decline due to the tax program.

Differential Policy in the RM. We characterize the RM by two steady state functions, the ww-function, and the xx-function, in $(d^x,\, k^x)$ space. The ww-function follows from the wealth accumulation equation, (14'), together with the price equation, (31):

$$k^x = B^{-1} \left[\sigma\, \hat{A}\, (d^x)^\alpha - p\, d^y\, \Delta\, d^x\right], \text{ with } \hat{A} \equiv b_{xx}\, d^x - b_{xy}\, d^y\,. \tag{36}$$

Equation (36) shows all combinations of k^x and d^x for which the wealth of succeeding young generations remains stationary. Its slope depends on the sign of the generalized capital intensity, B. It is an increasing function in d^x if $B < 0$, and it is decreasing if $B > 0$. The function is not defined for $B = 0$. To help gain insight, the slope is explained for the following case: $B < 0$ and $\Delta = 0$. Consider any point on the ww-function. Let d^x rise and hold k^x unchanged for the moment. Then, savings ($= k^x + p\, k^y$) rise. If $\Delta = 0$ then $\partial\, k^y / \partial\, d^x = 0$.[23] Since $B < 0$, the relative price decreases upon a rise in d^x. Thus, in order to restore capital market equilibrium, k^x must rise due to the increase in d^x.[24]

[23]This follows from $k^y = \hat{A}^{-1}[\Delta\, d^x\, d^y + k^x(b_{yx}\, d^x - b_{yy}\, d^y)]$.

[24]If $\Delta < 0$, k^y decreases and k^x has to rise by more than when $\Delta = 0$. If $\Delta > 0$, k^y

Next, the xx-function follows from the x-market clearing condition. It shows all combinations of k^x and d^x for which goods and factor markets clear in the steady state.

$$k^x = \hat{C}^{-1} \left\{ b_{xy}\, d^y\, (d^x)^\alpha + \hat{A}\, \sigma \left[\frac{\gamma\,(d^x)^\alpha}{\beta\,\varphi} + \frac{\alpha\,\gamma\,(d^x)^{2\alpha-1}}{\varphi\,(b_{xx} + b_{yx}\, p)} \right] \right\} \tag{37}$$

with $\hat{C} \equiv (d^x)^\alpha - \hat{A}$

The xx-function is always increasing in d^x. Suppose, $\hat{A} > 0$. Hold, for the moment, d^x constant and increase k^x. Then, since $l^x = (k^x - b_{xy}\, d^y)/\hat{A}$, the labor share l^x rises. The increase in l^x, however, raises x-production by more than the supposed increase in x-investment (k^x). Hence d^x and accordingly x-consumption demand has to rise in order to restore market equilibrium. A similar argumentation holds for the case where $\hat{A} < 0$.

Again, once we know how the policy shifts the ww-function as well as the xx-function we can infer the impact of the tax shock on the dynamic variables of the RM. Due to intertemporal income compensation a rise in τ does not *directly* affect savings and, thus, does not affect the ww-line. This is also shown by equation (14') where the tax term does not enter directly. However, a rise in τ does affect optimal consumption decisions and hence makes the xx-line shift. From (37) it follows:

$$\frac{\partial\, k^x}{\partial\, \tau}|_{d^x} = -\frac{\hat{A}\,\gamma\,(1-\gamma)\sigma}{\hat{C}\,\varphi^2} \left[\frac{(d^x)^\alpha}{\beta} + \frac{\alpha\,(d^x)^{2\,\alpha-1}}{b_{xx} + b_{yx}\, p} \right] . \tag{38}$$

Derivative (38) indicates the vertical shift of the xx-function (for given d^x) in (d^x, k^x) space. The sign of (38) depends on the sign of $\hat{A}$. If, e.g., $\hat{A}$ is positive, then the xx-function shifts down as a result of taxation. If $\hat{A}$ is not positive, it shifts up.

An economically intuitive explanation for the case $\hat{A} > 0$ and $B > 0$ is now given. Here, we expect the xx-line to shift downwards. Since the

rises and k^x rises by less than when $\Delta = 0$.

ww-function has a negative slope, d^x increases, while k^x decreases. Hold, for the moment, both d^x and k^x constant. Then an increase in τ reduces x-consumption and induces excess supply of x and excess demand for y. To restore equilibria the relative price must rise. An increase in p requires d^x to increase as well, since $B > 0$. Moreover, as argued above, a rise in d^x requires k^x to decline, since the ww-function is decreasing in d^x if $B > 0$. Therefore, the tax shock results in an increase in d^x as well as a decline in k^x.

Equivalence of steady state results. Equivalence of steady state results occurs if both of the following hold: The xx-functions and ww-functions of both models coincide, and the shifts of the xx-functions of both models are identical. These requirements are satisfied if the following two conditions hold:

$$b_{xx} = b_{xy} = \eta \,, \tag{39}$$

$$b_{yx} = b_{yy} \,. \tag{40}$$

Because of (39) $A = \hat{A}$ and $C = \hat{C}$. Moreover, due to calibration we know that $(b_{xx} + b_{yx}\, p) = 1$.[25] Thus, the xx-functions of both the RM and the FM coincide. From (40) it follows that $\Delta = 0$. Moreover, calibration implies that $B = d^x - d^y$. Therefore, $\hat{A}/B = \eta$. Consequently, the ww-functions of both the RM and the FM are identical. Moreover, since $A = \hat{A}$, $C = \hat{C}$, and $(b_{xx} + b_{yx}\, p) = 1$ it is easy to see from (35) and (38) that the shifts of the xx-lines are equal.

The two conditions (39) and (40) can be understood in the following way. In the FM, capital as well as the capital service are homogeneous.

[25] Observe that the RM was calibrated such that all common parameters and variables are identical to those of the FM in a base case. It follows that for each base case solution it is true that $(b_{xx} + b_{yx}\, p) = (b_{xy} + b_{yy}\, p) = 1$. This is the case which is argued here.

The composition of capital is fully determined by η. In terms of x-units, the percentage of x to y-goods in the composition of k equals $\eta/(1-\eta)$. Equivalently, k consists of $\eta\, k$ units of the x-good and $(1-\eta)\, k/p$ units of the y-good. In the RM one unit of x-capital services consists of b_{xx} units of the x-good and of $b_{yx}\, p$ units of the y-good. Similarly, one unit of y-capital services consists of b_{xy} units of the x-good and of $b_{yy}\, p$ units of the y-good. First, the two conditions imply that both capital services in the RM are composed of I^x and I^y in exactly the same proportions. Second, the two conditions imply that the capital services are composed in the same proportions as the capital service of the FM.

If (39) and (40) hold, the composition of both capital services in the RM corresponds exactly to that of the capital service of the FM. In this case, the steady state responses to the tax shock are identical for FM and RM. It follows that heterogeneity of capital is not per se the reason for differences in steady state results. However, whenever capital services are composed differently across sectors a model with heterogeneous capital (RM) implies different steady state policy results from those implied by a model with a capital composite (FM).

Even in the case where the equivalence conditions (39) and (40) hold, the short run policy results are different for the RM and the FM because of the different characteristics of the two models' dynamic systems. A model with capital heterogeneity — and consequently no-arbitrage conditions — requires different short-run responses than a model with one composite capital good and homogeneous capital service.

6 Conclusions

Dynamic, multi-sector CGE models are powerful tools in modern applied economics. Given the great complexity of a dynamic multi-sector CGE model it is not surprising to find that various kinds of simplifications are used in the literature. One simplification, nowadays commonplace, involves the aggregation of sectoral investment according to fixed shares and thus, the use of one composite capital aggregate.

The present paper has tried to answer the question: "Does the framework with a capital composite and fixed investment shares represent a 'sufficiently close' approximation to an optimization model that determines investment shares in a heterogeneous capital context by household optimization?" Several responses to this question were found.

First, the theoretical analyses and numerical simulations demonstrated that the way investment aggregation is modeled in dynamic, multi-sector CGE models dramatically influences policy results. Short run policy responses for a model with one capital aggregate and fixed investment shares (FM) and a reference model (RM) with optimal shares and heterogeneous capital are shown to be strikingly different. Steady state policy responses are generally more similar than short run responses. However, the differences in the steady state policy results increase with the magnitude of the tax shock. The main reason is shown to be the different dynamic characteristics of RM and FM.[26]

In many important circumstances, a model with a capital aggregate and

[26] It needs to be pointed out that these results are independent of the specific parameter sets that were used for the numerical simulations as they are independent of the assumption of fixed coefficients b_{ij}. In a working paper that is available from the authors upon request it is shown that the results hold for a broad range of parameter sets as well as for the case that the coefficients b_{ij} are determined optimally, following a Cobb-Douglas production function of capital services.

fixed investment shares implies very different policy responses from those obtained by a model with heterogeneous capital and optimal investment shares. This observation is important because, as discussed in the introduction, many papers make use of the former framework while the other papers make use of the latter framework. However, the implication of the assumption regarding investment aggregation is in no case clearly provided in the literature dealing with CGE analysis.

Second, because of the result stated above, we strongly recommend employing a framework with optimal investment shares and heterogeneous capital for multi-sector analysis. We discourage the use of fixed shares models because in addition to big quantitative differences, many responses of individual variables to a tax shock differ in sign for the RM and the FM.

Third, our interpretation of the frequent use of FM for dynamic, multi-sector CGE analyses is as follows. We have shown that with respect to many characteristics, a FM is easier to handle than a model with heterogeneous capital and optimal shares. There are three reasons for the great complexity of the RM. First, the dimension of the dynamic system of the RM is $(2\,n-1)$ when n sectors are considered. The dynamic system of a FM is always equal to 1. Second, while the steady state of the RM is (in the best case) a saddle, the steady state of a FM is generally asymptotically stable. Third, data regarding the composition of capital (capital composition matrix) are sometimes hardly available. The informational requirement in a FM is low compared to that of the RM. Each of these complications makes it tempting to make use of the FM rather than a framework with optimal shares and heterogeneous capital — especially, if it is assumed that the policy response is approximately the same. In the present paper, however, we demonstrated that the latter assumption is not always correct.

Fourth, in the light of the present analysis there are some circumstances under which the use of FM appears to be more justifiable. These include steady state policy analysis, the assessment of small policy shocks, the analysis of tax shocks that are *not* sector-specific and policy analysis in the case where the composition of capital services is identical (similar) across sectors.

One word of caution is needed before concluding this paper. The equivalence conditions point out that only if capital services are composed in exactly the same way across sectors, do both models imply the same steady state policy results. But this case, obviously, is the case of one capital aggregate! Whenever capital services are not composed in exactly the same way across sectors, steady state policy results do differ. The careful policy analyst might therefore prefer to use the RM framework even for steady state analyses.

Appendices

Appendix A The Model with FS

The equation of motion, (20), can be derived in the following way. From the capital services market clearing condition, (17), it follows that

$$l_t^x = \frac{k_t - d^y}{d_t^x - d^y}\,, \quad d_t^x \neq d^y\,. \tag{41}$$

Since, because of (14), $k_t = \sigma\,(d_{t-1}^x)^\alpha$, the supply of x becomes:

$$x_t = l_t^x\,(d_t^x)^\alpha = \frac{\sigma\,(d_{t-1}^x)^\alpha - d^y}{d_t^x - d^y}\,(d_t^x)^\alpha\,. \tag{42}$$

Consumption and investment demand is given by (11), (12), and (15):

$$\begin{aligned} x_t = {} & \frac{\gamma}{(1+\beta)\varphi}\,(1-\alpha)(d_t^x)^\alpha \\ & + \frac{\beta\,\gamma}{(1+\beta)\varphi}\,\alpha\,(d_t^x)^{\alpha-1}\,(1-\alpha)(d_{t-1}^x)^\alpha + \eta\,\sigma\,(d_t^x)^\alpha\,. \end{aligned} \tag{43}$$

By setting (42) equal to (43), the equation of motion of the model with FS, (20), follows:

$$\begin{aligned}
&\frac{\alpha\,\gamma\,\sigma\,d^y}{\varphi}(d^x_{t+1})^{-1}(d^x_t)^{\alpha} - \sigma\left(\frac{\gamma}{\beta\,\varphi} + \eta\right) d^x_{t+1} \\
&= \sigma\left(\frac{\alpha\,\gamma}{\varphi} - 1\right)(d^x_t)^{\alpha} + \left[1 - \sigma\left(\frac{\gamma}{\beta\,\varphi} + \eta\right)\right] d^y\,.
\end{aligned}$$

There is the following boundary condition on d^x:

$$\begin{aligned}
&d^x_0 = \frac{\psi_0 + \sqrt{4\,\alpha\,\beta\,\gamma(\gamma + \beta\,\eta\,\varphi)\sigma\,d^y\,k_0 + \psi_0^2}}{2(\gamma + \beta\,\eta\,\varphi)\sigma}\,, \\
&0 < k_0 \text{ given}\,, \\
&\text{with } \psi_0 \equiv -\alpha\,\beta\,\gamma\,k_0 + \gamma\,\sigma\,d^y + \beta\,\varphi\,(k_0 - (1 - \eta\,\sigma)d^y)\,.
\end{aligned}$$

A fixed point of the equation of motion is locally asymptotically stable if

$$\frac{d\,d^x_{t+1}}{d\,d^x_t} = \frac{\alpha^2\,\gamma\,d^y + \alpha(\varphi - \alpha\,\gamma)d^x}{\alpha\,\gamma\,d^y + (\gamma/\beta + \eta\,\varphi)(d^x)^{2-\alpha}} < 1\,. \tag{44}$$

Moreover, $(\varphi - \alpha\,\gamma) > 0$. Thus $d\,d^x_{t+1}/d\,d^x_t > 0$, and the transition path is monotonic.

Appendix B The Reference Model

The dynamic system of the RM can be derived in the following way. Start with equation (29). From both the capital service market clearing condition (27) and the labor market clearing condition (16), it follows that

$$l^x_t = \frac{k^x_t - b_{xy}\,d^y}{b_{xx}\,d^x_t - b_{xy}\,d^y}\,, \quad b_{xx}\,d^x_t \neq b_{xy}\,d^y\,. \tag{45}$$

Taking into account the other capital service market clearing condition together with (45), an intratemporal relationship between k^x and k^y can be established.

$$k^y_t = \frac{\Delta\,d^y\,d^x_t + (b_{yx}\,d^x_t - b_{yy}\,d^y)k^x_t}{b_{xx}\,d^x_t - b_{xy}\,d^y} \tag{46}$$

Savings equal $k^x_{t+1} + p_t\, k^y_{t+1} = \sigma\,(d^x_t)^\alpha$. Shifting (46) forward by one period and solving for d^x_{t+1} yields the first equation of motion:

$$d^x_{t+1} = \frac{d^y\, k^x_{t+1}(b_{xy} + p_t\, b_{yy}) - b_{xy}\,\sigma\, d^y (d^x_t)^\alpha}{k^x_{t+1}(b_{xx} + p_t\, b_{yx}) + p_t\,\Delta\, d^y - \sigma\, b_{xx}(d^x_t)^\alpha}\,. \tag{29}$$

The second equation of motion follows from equalizing supply of the x-commodity, (42) with demand and taking (45) into account:

$$k^x_{t+1} = \frac{k^x_t - b_{xy}\, d^y}{b_{xx}\, d^x_t - b_{xy}\, d^y}(d^x_t)^\alpha - \frac{\gamma\,\sigma}{\beta\,\varphi}(d^x_t)^\alpha - \frac{\alpha\,\gamma\,\sigma}{\varphi}(d^x_t)^{\alpha-1}\,\frac{(d^x_{t-1})^\alpha}{b_{xx} + b_{yx}\, p_{t-1}}\,. \tag{47}$$

Equation (14') allows us to express d^x_{t-1} in terms of k^i_t and p_{t-1}. Moreover, (5') allows us to express p_{t-1} as function of p_t. From these two relationships the second equation of motion follows:

$$\begin{aligned} k^x_{t+1} =& \frac{(\varphi - \alpha\,\gamma)(d^x_t)^\alpha(k^x_t - b_{xy}\, d^y)}{(b_{xx}\, d^x_t - b_{xy}\, d^y)\,\varphi} - \frac{\gamma\,\sigma(d^x_t)^\alpha}{\beta\,\varphi} \\ &+ \frac{\gamma(k^x_t - b_{xx} d^x_t)[p_t - (1-\alpha)\, b_0\,(d^x_t)^\alpha]}{b_0(b_{xx}\, d^x_t - b_{xy}\, d^y)\varphi}\,. \end{aligned}$$

The third equation of motion simply follows from the price equation together with the definition of price of x-capital services, (3').

$$p_{t+1} = (1-\alpha)\, b_0\,(d^x_{t+1})^\alpha + \alpha\, b_1 (d^x_{t+1})^{\alpha-1}\frac{b_{xy} + b_{yy}\, p_t}{b_{xx} + b_{yx}\, p_t}$$

There are the following boundary conditions:[27]

$$\begin{aligned} &0 < d^x_0 = \frac{(b_{xy}\, k^y_0 - b_{yy}\, k^x_0) d^y}{b_{xx}\, k^y_0 - b_{yx}\, k^x_0 - \Delta\, d^y}\,, \quad b_{xx}\, k^y_0 \neq b_{yx}\, k^x_0 + \Delta\, d^y\,, \\ &0 < k^x_0 \text{ given}, \quad 0 < k^y_0 \text{ given}\,. \end{aligned}$$

[27]Moreover, it holds that $\lim_{t\to\infty} p_t = p_{t-1}$.

Appendix C Parameter Sets, Base Cases and First Period Policy Effects

Table 4: Parameter Sets and Base Cases

	A	B	C	D
Aggregate Variables				
Real GDP	0.38323	0.76635	0.77532	0.77114
Consumption	0.29341	0.50243	0.50049	0.50100
Investment ($= k$)	0.08982	0.26392	0.27482	0.27015
w	0.21119	0.62057	0.64621	0.63521
p	0.59429	0.73105	0.74017	0.73584
Sectoral Variables				
x	0.30658	0.61308	0.42006	0.48184
y	0.12897	0.20966	0.47997	0.39315
d^x	0.07350	0.28088	0.34389	0.31559
d^y	0.20000	0.20000	0.20000	0.20000
I^x ($= k^x$ in RM)	0.07185	0.21114	0.21986	0.08104
I^y ($= k^y$ in RM)	0.03023	0.07220	0.07426	0.25699
l^x	0.87103	0.79034	0.52003	0.60685
l^y	0.12897	0.20966	0.47997	0.39315
q^x	1.91552	0.55235	0.46978	0.50319
q^y	1.13837	0.40379	0.34772	0.37027
c_1^x	0.09710	0.28532	0.14855	0.29205
c_1^y	0.04085	0.09757	0.30105	0.09922
c_2^x	0.13763	0.11662	0.05164	0.10875
c_2^y	0.05790	0.03988	0.10466	0.03695
Parameters				
α	0.40000	0.20000	0.20000	0.20000
b_0	1.00000	1.00000	1.00000	1.00000
b_1	0.20000	0.20000	0.20000	0.20000
b_{xx}	0.92087	0.85667	0.96104	0.21788
b_{xy}	0.50000	0.50000	0.50000	0.50000
b_{yx}	0.13315	0.19607	0.05264	1.06288
b_{yy}	0.84134	0.68395	0.67552	0.67949
β	0.74000	0.74000	0.74000	0.74000
γ	0.80000	0.80000	0.40000	0.80000
η	0.80000	0.80000	0.80000	0.30000

Table 5. First Period Policy Effects (percentage changes)

	A			B			C			D		
Agg. Variables	RM	FM	d	RM	FM	d	RM	FM	d	RM	FM	d
Real GDP	4.589	0.155	-4	4.638	0.008	-5	15.194	0.082	-15	2.956	0.032	-3
Consumption	5.994	0.560	-5	7.074	-0.070	-7	23.537	-0.130	-24	4.550	-0.092	-5
Investment	0	-1.170	-1	0	0.157	0	0	0.470	0	0	0.263	0
w	0	-1.170	-1	0	0.157	0	0	0.470	0	0	0.263	0
p	22.947	0.732	-22	23.188	0.039	-23	33.159	0.174	-33	7.880	0.084	-8
Utility	-2.673	-0.373	2	-23.544	-0.213	23	-10.992	-0.435	11	21.161	-0.256	-21
Sect. Variables												
x	0	-2.807	-3	0	-2.512	-3	0	-4.916	-5	0	-3.229	-3
y	0	11.186	11	0	10.045	10	0	5.808	6	0	5.375	5
d^x	0	-2.899	-3	0	0.788	1	0	2.370	2	0	1.321	1
I^x	-6.264	-1.170	5	-5.627	0.157	6	-9.380	0.470	10	-2.616	0.263	3
I^y	1.716	-1.888	-4	-0.554	0.118	1	3.274	0.295	-3	-6.265	0.179	6
k^x (RM)	0	-		0	-		0	-		0	-	
k^y (RM)	0	-		0	-		0	-		0	-	
l^x	0	-1.656	-2	0	-2.665	-3	0	-5.361	-5	0	-3.482	-3
l^y	0	11.186	11	0	10.045	10	0	5.808	6	0	5.375	5
q (q^x in RM)	-6.693	1.780	8	-61.667	-0.626	61	-22.075	-1.856	20	159.730	-1.045	-161
c_1^x	-3.846	-4.971	-1	-3.846	-3.695	0	-10.714	-10.295	0	-3.846	-3.593	0
c_1^y	-6.151	13.206	19	-6.335	15.521	22	-19.538	7.459	27	12.279	15.591	3
c_2^x	5.984	-2.134	-8	19.596	-4.448	-24	70.751	-12.372	-83	6.957	-4.851	-12
c_2^y	3.443	16.585	13	16.501	14.618	-2	53.878	4.971	-49	24.893	14.083	-11
η (FM)	-6.264	0		-5.627	0		-9.380	0		-2.616	0	
$d_{\sum}$ ($d_{\varnothing}$)		125 (7.0)			200 (11.1)			300 (16.7)			252 (14.0)	

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