A model of growth for an economy with private substitutes for environmental goods

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Abstract

We analyze growth dynamics in an economy where a private good can be consumed as a substitute for a free access environmental good. In this context we show that environmental deterioration may be an engine of economic growth. To protect themselves against environmental deterioration, economic agents are forced to increase their labour supply to increase the production and consumption of the private good. This, in turn, further depletes the environmental good, leading economic agents to further increase their labour supply and private consumption and so on. This substitution process may give rise to self-enforcing growth dynamics characterized by a lack of correlation between accumulation and private consumption levels, on one side, and economic agents' well-being, on the other.

In particular we show that agents' self-protection consumption choices may generate indeterminacy; that is, they may give rise to the existence of a continuum of (Nash) equilibrium orbits leading to the same attracting fixed point or periodic orbit.

JEL Classification: C72, D62, J22, O10, O41, Q20

1 Introduction

In this paper we analyze economic growth dynamics in an economy where there is an infinite number (a continuum) of identical economic agents, whose well-being depends on three goods: leisure, a free access environmental good and a private consumption good. Each agent produces the private good by his own work and (physical) capital. The private good can be consumed as a (perfect) substitute for the environmental good and can be saved and accumulated as capital. The environmental good is deteriorated by the pollution caused by the average consumption activity of the private good in the economy.

At every instant of time each economic agent has to choose the allocation of both his endowment of time between leisure and the production process of the private good and his output between present consumption and accumulation of capital (i.e. future consumption). Since the negative impact on the environmental good of each agent's consumption choice is negligible (agents being a continuum), he doesn't take it into account in his consumption choices.

In this context, by working harder, economic agents can consume more in the present and/or in the future (via accumulation of capital) and consequently can benefit from a better self-protection against environmental deterioration in the present and/or in the future. So, economic agents may react to the deterioration of the environmental good by increasing production and consumption of private goods; by doing so, they cause a further depletion of the environmental resources, which can, in turn, force agents to further increase private consumption and accumulation. Such a substitution mechanism may give rise to a self-enforcing process, by which increasing quantities of private goods are produced and consumed as substitutes for the environmental good becoming increasingly deteriorated. This process can give rise to new consumption patterns based on costly private consumption rather than on the enjoyment of free collective consumption (i.e. the enjoyment of the free access environmental good).

In particular we show that growth dynamics, driven by environmental deterioration via the substitution mechanism described above, may be characterized by a multiplicity of steady states (fixed points) and that there may be no correlation between private consumption and accumulation levels in such states and economic agents' well-being. In fact steady states with high consumption and accumulation levels can be Pareto-dominated by others characterized by lower levels of them.

Furthermore the substitution process between environmental and produced goods may have effects on the stability of fixed points and may generate closed (Nash) equilibrium orbits.

We also show that self-protection choices can produce indeterminacy, that is the existence of an infinite number of equilibrium orbits leading to the same (locally) attractive fixed point or periodic orbit. When indeterminacy occurs, given the initial values of the capital stocks and the environmental good, the economy can reach the attracting fixed point (or periodic orbit) by following an infinite number (a continuum) of growth paths, each one characterized by different consumption (i.e. self-protection) patterns and well-being levels. Consequently, the economy may experience very different well-being stories. Starting from different initial values of the state variables, it can reach different fixed points (characterized by different well-being levels). Furthermore, when indeterminacy occurs, each fixed point can be reached through an infinite number of possible orbits, each of them giving rise to possibly different well-being levels.

2 Related literature

The idea that certain private goods can be consumed as a defence against the damage caused by the choices of other economic agents was introduced into economic literature by Hirsch (1976), who coined the term " defensive consumption" . Subsequently the concept of defensive consumption attracted the attention of many environmental economists, who tackled the problem of identifying, classifying and quantifying defensive consumption implemented as a protection device against environmental deterioration (see Hueting 1980, Leipert 1989a, 1989b, Hugo 1996, Myers 1997, Vincent 2000, Bates 2002).

Possible " classroom" examples of goods which may be consumed to alleviate the damage ensueing from environmental deterioration are: mineral water, which can be drunk in place of spring or tap water, the soundproofing devices used as a defence against acoustic pollution, or holidays in " tropical paradises", partially motivated by the pollution of closer sea. However, a general lesson to be drawn from this literature (see in particular Hueting 1980) is that the variety of goods which can be consumed as a defence against environmental deterioration is extremely extensive. In fact individuals may react to environmental deterioration by radically modifying their consumption patterns, which become progressively hinged on the consumption of costly private goods rather than free access environmental goods.

The mechanism of economic growth we intend to analyze is based on the hypothesis that each economic agent's consumption of the private good contributes to the depletion of the environmental good, and therefore generates a negative externality (in the model the agents do not take into consideration the negative impact of their choices on the environmental good) on the other agents. Consequently, in our model, defensive consumption choices may be classified as self-protective choices " transfering" the negative externalities to other individuals (Bird 1987); that is, each victim of negative environmental externalities defends himself by implementing the defensive consumption which generates further negative externalities for other individuals.

This situation has been analyzed, in a static model, by Shogren and Crocker (1991), who demonstrated that, in a context where individuals do not co-operate (i.e., they do not " internalize" the externalities), the outcome is a degree of self-protection that exceeds socially optimal level. It implies that, if the individuals protect themselves by consuming private goods, the expected outcome is an excess in the consumption of private goods.

Such a result has been extended to the case of economic growth in several works (see e.g. Antoci 1996, Antoci and Bartolini 1999, 2003, Antoci and Borghesi 2002, Bartolini and Bonatti 1999), which have examined the implications for growth dynamics of implementing self-protective choices which transfer negative externalities. More specifically, the papers of Antoci 1996, Antoci and Bartolini 1999, 2003, and Antoci and Borghesi 2002, have emphasized the endogenous dynamics of labour supply, neglecting the accumulation of capital. Bartolini and Bonatti 1999 have analyzed a model of endogenous growth, consequently focusing attention on the role played by defensive consumption in determining the growth rate of the economy[1] [2]. The main result of these papers is that, in different contexts, self-protection consumption may generate economic growth paths Pareto-dominated by other paths with lower aggregate private consumption and accumulation. The present work analyzes a model of capital accumulation, with the aim of highlighting the path-dependence nature of the growth dynamics. In particular it is shown that defensive consumption can generate indeterminacy (for a review of macroeconomic models featuring indeterminacy see Benhabib and Farmer 1999; for an example see e.g. Benhabib and Farmer 1996).

Finally, our model exhibits several features reminding a strand of literature concerned with consequences for economic growth of the abundance of (both exhaustible and renewable) natural resources (e.g. Matsuyama 1992, Rodriguez and Sachs 1999, Sachs and Warner 1999, 2001, Auty 2001). In these articles it is generally argued that abundance of natural resources can inhibit the growth of the economy. Very briefly, the following mechanism is assumed: there exists a sector x which plays an important role in growth; the abundance of resources restrains the development of sector x and,as a consequence, inhibits growth (for a review of this literature see Sachs and Warner 2001, Auty 2001). Our point of view, though, basically differs, in the fact that we take into account the possible negative effects (negative externalities) which sector x (in our model the sector where the private good is produced and accumulated as capital) can produce on the stock and quality of environmental resources. Such an effect may generate undesirable growth dynamics, which, instead, are not present in the above-mentioned literature.

3 The model

There exists a continuum of identical economic agents. At each instant of time, t, representative agent's well-being depends on three goods:

1. Leisure $1 - l(t)$, where $l(t)$ is representative agent's labour input.

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2. A free access (renewable) environmental good $E(t)$.

3. A private good which can be consumed either as a (perfect) substitute for the environmental good ($c_2(t)$), or in order to satisfy different needs ($c_1(t)$).

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[1] Beltratti (1996) analyses a growth model in which private and environmental goods are substituted; Beltratti does not however consider the labour input as a choice variable; in this context, an increase in the private good for defensive purposes has the effect of reducing accumulation rather than increasing it. In the Bartolini and Bonatti model, and in our own, the individuals can increase present consumption and accumulation at the same time as increasing the labour input.

[2] Antoci, Sacco and Vanin (2003) offer the analysis of a growth model with defensive consumption implemented as a self-protection against the scarcity of social rather than environmental capital.

Consumption of the private good depletes renewable natural resources. To counterbalance such a depletion, agents may increase their labor supply in order to produce and consume higher quantities of the private good. We assume that representative agent's decision problem is

$$\max_{c_1,c_2,l} \int_0^{\infty} (\ln c_1 + a\ln(E + bc_2) + d\ln(1-l))e^{-rt}dt \tag{1}$$

$$\dot{k} = l^{\alpha}k^{1-\alpha}\Omega - c_1 - c_2 \tag{2}$$

$$\dot{E} = \beta E(\overline{E} - E) - \gamma(\overline{c}_1 + \overline{c}_2)E \tag{3}$$

where:

$k(t)$ represents physical capital; $\dot{k}$ and $\dot{E}$ denote the time derivatives of k and E; $\overline{c}_1(t)$, $\overline{c}_2(t)$, $\overline{l}(t)$, $\overline{k}(t)$ are the average values of $c_1(t)$, $c_2(t)$, $l(t)$, $k(t)$, respectively, in the economy.

$$\Omega := \overline{l}^{\delta}\overline{k}^{\varepsilon} \tag{4}$$

represents a positive externality in the production process of the private good.

We assume that $\overline{c}_1(t)$, $\overline{c}_2(t)$, $\overline{l}(t)$, $\overline{k}(t)$ are considered as exogenously given by the representative agent.

The parameters a, b, d, r, α, β, γ, δ, ε, $\overline{E}$ are strictly positive. Furthermore we assume $\alpha < 1$ (i.e., the production function is homogeneous of degree 1 without the effect of Ω) and $\alpha > \varepsilon$ (that is, the sum of the exponents of k and $\overline{k}$ is strictly lower than 1).

Note that the value of the parameter $\overline{E}$ can be interpreted as the endowment of the environmental good in the economy: the state variable E would reach such a value without the negative effect due to the economic activity.

The parameter b represents the constant substitution rate between $E(t)$ and $c_2(t)$.

Since the representative agent doesn't take into account the negative impact of his consumption choice on E, we will study (Nash) equilibrium growth orbits which are not Pareto optimal.

The Hamiltonian function for our problem is

$$\begin{aligned} H(E,k,\lambda,\theta,l,c_1,c_2) = {} & \ln c_1 + a\ln(E + bc_2) + d\ln(1-l) + \\ & + \lambda(l^{\alpha}k^{1-\alpha}\Omega - c_1 - c_2) + \\ & + \theta(\beta E(\overline{E} - E) - \gamma(\overline{c}_1 + \overline{c}_2)E) \end{aligned}$$

where λ and θ are the co-state variables associated to k and E respectively. By applying the maximun principle we obtain that the dynamics of $c_1(t)$, $c_2(t)$,

$l(t)$, $k(t)$, $E(t)$ must satisfy the following conditions:

$$\frac{\partial H}{\partial l} = -\frac{d}{1-l} + \alpha\lambda l^{\alpha-1}k^{1-\alpha}\Omega = 0 \tag{5}$$

$$\frac{\partial H}{\partial c_1} = \frac{1}{c_1} - \lambda = 0 \tag{6}$$

$$\frac{\partial H}{\partial c_2} = \frac{ab}{E+bc_2} - \lambda \leq 0, \quad c_2 \geq 0, \quad c_2\frac{\partial H}{\partial c_2} = 0 \tag{7}$$

$$\dot{k} = \frac{\partial H}{\partial \lambda} = l^{\alpha}k^{1-\alpha}\Omega - c_1 - c_2 \tag{8}$$

$$\dot{\lambda} = r\lambda - \frac{\partial H}{\partial k} = \lambda(r - (1-\alpha)l^{\alpha}k^{-\alpha}\Omega) \tag{9}$$

$$\dot{E} = \frac{\partial H}{\partial \theta} = \beta E(\overline{E} - E) - \gamma(\overline{c}_1 + \overline{c}_2)E \tag{10}$$

where $\overline{c}_1$, $\overline{c}_2$, $\overline{l}$, $\overline{k}$ must be replaced by c_1, c_2, l, k, in expressions (5)-(10) and the control variables c_1, c_2, l are determined by conditions (5)-(7). Notice that, in our model the control variables c_1 and l, differently from c_2, are always strictly positive and $l < 1$.

We omit the dynamics of the co-state variable θ since equations (8)-(10) don't depend on it (precisely because $\overline{c}_1$ and $\overline{c}_2$ are considered exogenous by the representative agent). Furthermore, we assume the usual transversality condition

$$\lim_{t\to\infty} k(t)\lambda(t)e^{-rt} = 0.$$

4 Dynamics with $c_2 = 0$

From (7) it follows that, if the condition

$$\frac{ab}{E} - \lambda \leq 0 \tag{11}$$

is met, then the representative agent chooses $c_2 = 0$, i.e., he doesn't consume the private good as a substitute of the environmental good. Otherwise, he chooses $c_2 > 0$. Condition (11) is satisfied if, given λ, the value of E is high enough.

When $c_2 = 0$, system (8)-(10) is decoupled in the planar system given by (8) and (9) and the non-autonomous differential equation (10). It is easily seen that at most one fixed point, say S', exists, with the following coordinates:

$$k' = \left[\frac{1-\alpha}{r} \left(\frac{\alpha}{\alpha+d} \right)^{\alpha+\delta} \right]^{\frac{1}{\alpha-\varepsilon}}$$

$$\lambda' = \frac{1-\alpha}{rk'}$$

$$E' = \overline{E} - \frac{\gamma r}{\beta(1-\alpha)} k'$$

Such a fixed point exists only if (11) is satisfied, which requires, coeteris paribus, the endowment of the environmental good, $\overline{E}$, to be high enough and the negative impact, γ, of average consumption on the environmental good to be low enough.

The stability of the fixed point is described by

Theorem 1 Let

$$p := \frac{\alpha - \varepsilon}{d(1-\alpha-\delta)+\alpha}$$

$$q := \frac{(1-\alpha)[d(1-\alpha-\delta)+\alpha] + (\alpha+d)\varepsilon}{d(1-\alpha-\delta)+\alpha}$$

Then:

1. *If* $p > 0$, S' *is a saddle with a bi-dimensional stable manifold.*
2. *If* $p < 0$ *and* $q > 0$, S' *is a saddle with a one-dimensional stable manifold.*
3. *If* $p < 0$ *and* $q < 0$, S' *is a sink.*

If case (1) holds, given the initial values of k and E, there exists (at least locally) a unique initial value of λ (determined by the representative agent) from which the economy approaches the fixed point.

Note that condition (1) is satisfied if $\alpha + \delta \leq 1$, where α and δ are the exponents of l and $\overline{l}$, respectively, in the production function.

The fixed point cannot be (generically) reached if (2) holds.

Viceversa, when (3) is satisfied, given the initial values of k and E, there exists a continuum of initial values of λ leading to the fixed point. In other words there exists an infinite number of (Nash) equilibrium orbits that the economy may follow to reach the fixed point. Along each orbit no economic agent has an incentive to change his choices, given other agents' choices.

Observe that the parameters r (discount rate), $\overline{E}$ (endowment of the environmental good), γ (impact of average consumption on the environmental good) play a role for the existence of the fixed point [condition (11)], but don't affect its stability properties.

Theorem 2 When $\alpha > \varepsilon$ and δ crosses the value

$$\overline{\delta} := 1 - \alpha + \frac{\alpha}{d} + \frac{(\alpha+d)}{1-\alpha}$$

an attracting limit cycle (through a Hopf supercritical bifurcation) arises for $\delta > \overline{\delta}$.

Proof. Proofs of the above theorems are given in Antoci, Brugnano and Galeotti (2003). ■

When an attracting orbit exists, by following such an orbit the economy may enter the region of the plane (λ, E) where $c_2 > 0$. However, this is not the case if the periodic orbit is small enough. In Antoci, Brugnano and Galeotti (2003) it is showed, by numerical simulations, that the periodic orbit expands as the bifurcation parameter δ increases.

In the next section we analyze dynamics in the subset of the positive orthant of the space (k, λ, E) where condition (11) is not satisfied, and consequently economic agents consume the private good also as a substitute for the environmental good ($c_2 > 0$).

5 Fixed points in the regime $c_2 > 0$

It is easily checked that, in the regime $c_2 > 0$, there always exists a fixed point at which the environmental good is completely depleted, that is $\widetilde{S} = (k, c_1, E) = \left(\widetilde{k}, \widetilde{c}_1, 0\right)$, $c_1 = \dfrac{1}{\lambda}$, with

$$\widetilde{k} = \left(\frac{r}{1-\alpha}\frac{\alpha(1+a)+d}{\alpha(1+a)}\right)^{\frac{1}{-\alpha+\varepsilon}}$$

$$\widetilde{c}_1 = \frac{\alpha(\widetilde{k})^{1-\alpha+\varepsilon}}{\alpha(1+a)+d}$$

Denote by $S = (k^*, c_1^*, E^*)$ a fixed point satisfying the conditions $E > 0$ and $c_2 > 0$. Then $E^*, k^*, c_1^* > 0$ and $bc_2^* = abc_1^* - E^* > 0$ [see (7)].

As lengthy but straightforward calculations (see Appendix A) show, at S

$$\overline{E} = \psi(k^*) := m(k^*)^{\frac{\varepsilon+\delta}{\alpha+\delta}} - nk^* \tag{12}$$

must hold, with

$$m := \frac{\alpha b(a+1)}{d}\left(\frac{r}{1-\alpha}\right)^{\frac{\alpha+\delta-1}{\alpha+\delta}} \tag{13}$$

$$n := \frac{br(1+a)}{d(1-\alpha)}\left(\alpha + \frac{d(b\beta-\gamma)}{b\beta(a+1)}\right) \tag{14}$$

Furthermore $k^* \in (k_1, k_2)$, where k_1 and k_2 are determined by the intersections of the curve $\overline{E} = \psi(k^*)$ with the lines

$$\overline{E} = m_1 k^* := \frac{(ab\beta + \gamma)r}{(1-\alpha)\beta} k^* \tag{15}$$

$$\overline{E} = m_2 k^* := \frac{\gamma r}{(1-\alpha)\beta} k^* \tag{16}$$

S is unique whenever $\psi'(k^*)$ does not change sign in (k_1, k_2) (see figures 1-2). So, from (12), it follows that there exists at most one fixed point S if $n \leq 0$ [implying $\psi'(k^*) > 0$ for $k^* > 0$], whereas two fixed points (with c_2^*, $E^* > 0$) can exist if $\psi(k^*)$ has a maximun in (k_1, k_2) (see figure 3). Straightforward computations yield that the latter case holds if and only if

$$\gamma < \sigma < ab\beta + \gamma, \tag{17}$$

where

$$\sigma := \frac{[\alpha(a+1)b\beta + (b\beta - \gamma)d](\alpha - \varepsilon)}{d(\varepsilon + \delta)} \tag{18}$$

Then, if (17) is verified, an interval $(\overline{E}_l, \overline{E}_u)$ is given, with

$$\overline{E}_l := \max[\psi^*(k_1), \psi^*(k_2)] \tag{19}$$

$$\overline{E}_u := \psi(k_0), \text{ where } k_0 \text{ satisfies } \psi'(k_0) = 0,\ k_1 < k_0 < k_2, \tag{20}$$

such that for any $\overline{E} \in (\overline{E}_l, \overline{E}_u)$ there exist two fixed points with a strictly positive E in the regime $c_2 > 0$.

Remark that condition (17) is never satisfied if, coeteris paribus, the negative impact, γ, on the environmental good of average consumption is high enough.

Figures 1-3 illustrate the possible configurations of fixed points, in dependence of the parameters. Fixed any point in the positive $\overline{E}$-semiaxis (that is, given the endowment of the environmental good), the intersections between the horizontal straight line passing through it and the continuous lines drawn in each figure give the number of existing fixed points and the corresponding values of k^*.

Figure 1: Case $\sigma \leq \gamma$
Figure 2: Case $\sigma \geq ab\beta + \gamma$
Figure 3: Case $\gamma < \sigma < ab\beta + \gamma$

Note that, in figures 1-3, the fixed point with the lowest level of accumulation is the one with $E > 0$ and $c_2 = 0$ (when existing). Such a fixed point would be unique if the private good could not be consumed as a substitute for the environmental good.

Figure 1 illustrates the case $\sigma \leq \gamma$ (that is, the case where the negative impact on the environmental good of average consumption is high enough). In such a case at most three fixed points can exist. If the endowment $\overline{E}$ of the environmental good is high enough, then there exist two fixed points: the one with $E = 0$ and $c_2 > 0$ and the one with $E > 0$ and $c_2 = 0$. As $\overline{E}$ decreases, then three fixed points appear: those with $E = 0$, $c_2 > 0$ and $E > 0$, $c_2 = 0$ and a fixed point where $E > 0$ and $c_2 > 0$. Finally, if $\overline{E}$ is low enough, then only the fixed point with $E = 0$ exists.

Figure 2 can be interpreted in a similar way. Unlike in figure 1, at most two fixed points can coexist.

Figure 3 shows the more interesting case, where the highest number of fixed points can exist, i.e., one with $E > 0$ and $c_2 = 0$, two with $E > 0$ and $c_2 > 0$, one with $E = 0$ and $c_2 > 0$. Such a regime exists if, coeteris paribus, $\overline{E}$ and γ are not “too” high or “too” low.

In figure 4 an example with $\sigma < \gamma$ is given, where three fixed points exist. At the fixed point with the lowest accumulation level (namely A) the (instantaneous) utility function takes the value $U_A = 7.95$, while at the other two fixed points, B and D, the utility values are, respectively, $U_B = 7.69$ and $U_D = 7.81$. Therefore, A Pareto-dominates the others and D Pareto-dominates B. So, no correlation exists between the accumulation level and the well-being level.

In figure 5 an example with $\sigma > ab\beta + \gamma$ is given, where two fixed points, C and D, D Pareto-dominating C, exist, and D is characterized by a higher accumulation level.

In figure 6 an example with $\gamma < \sigma < ab\beta + \gamma$ is considered. There exist four fixed points: A, B, C and D. The fixed point with the lowest accumulation level, A, Pareto dominates the others. In fact $U_A > U_B > U_C > U_D$ holds: therefore an inverse correlation between accumulation and well-being levels exists.

Figure 4: $\beta = 0.5, \epsilon = 0.1, \alpha = 0.3, \gamma = 1.25, \delta = 0.7, \sigma = 0.175, a = 0.5$
$d = 0.2, r = 0.1, b = 1.2, \overline{E} = 580$
Figure 5: $\beta = 0.5, \epsilon = 0.1, \alpha = 0.3, \gamma = 0.03, \delta = 0.7, \sigma = 0.48, a = 0.5$
$d = 0.2, r = 0.1, b = 1.2, \overline{E} = 100$
Figure 6: $\beta = 0.05, \epsilon = 0.75, \alpha = 0.8, \gamma = 0.1, \delta = 0.2, \sigma = 0.1405, a = 2.5$
$d = 0.05, r = 0.1, b = 1, \overline{E} = 75000$

6 Stability analysis

6.1 Stability of the fixed point with $E = 0$

It is easily checked that at $\widetilde{S}$ the E-axis is an eigenspace of the Jacobian matrix, whose associated eigenvalue has the sign of $\overline{E} - \psi(\widetilde{k})$.

The stability of $\widetilde{S}$ as well as the dynamics in the invariant $E = 0$ plane can be reconducted to the projection on the (k, λ) plane of the $c_2 = 0$ regime, by replacing d with $d' := \dfrac{d}{a+1}$ and λ with $\lambda' := \dfrac{\lambda}{a+1}$.

Thus, in particular:

1. If $\delta < 1 - \alpha + \frac{\alpha}{d'}$, $\widetilde{S}$ is a saddle in the invariant plane $E = 0$.
2. If $\delta > 1 - \alpha + \frac{\alpha}{d'} + \frac{(\alpha + d')\varepsilon}{(1-\alpha)d'}$, $\widetilde{S}$ is a source in the invariant plane $E = 0$
3. If $1 - \alpha + \frac{\alpha}{d'} < \delta < 1 - \alpha + \frac{\alpha}{d'} + \frac{(\alpha + d')\varepsilon}{(1-\alpha)d'}$, $\widetilde{S}$ is a sink in the invariant plane $E = 0$.

Therefore, starting from a strictly positive value of E, the fixed point can be (generically) reached by suitably choosing λ if $\overline{E} - \psi(\widetilde{k}) < 0$ is satisfied and 1. or 3. holds. In particular, when 3. holds, there exists a continuum of orbits approaching thefixed point, i.e. indeterminacy occurs.

6.2 Stability of the fixed points with $E > 0$ in the regime $c_2 > 0$

Straightforward calculations allow to write down the Jacobian matrix $J(S)$

$$J(S) = \begin{pmatrix} A & B - (a+1) & \frac{1}{b} \\ \frac{c_1^*}{k^*}((1-\alpha)A - r) & \frac{c_1^*}{k^*}(1-\alpha)B & 0 \\ 0 & -\gamma(a+1)E^* & \frac{\gamma - b\beta}{b}E^* \end{pmatrix} \tag{21}$$

where A, B, and $det(J(S))$ are computed in Appendix B.
We can distinguish the following subcases.

6.2.1 Case $\alpha + \delta \leq 1$

Theorem 3 *If $\alpha + \delta \leq 1$ and $\psi'(k^*) \neq 0$, $J(S)$ has at least one eigenvalue with positive real part. In particular, if $\psi'(k^*) > 0$, S is a saddle with a one-dimensional stable manifold; if $\psi'(k^*) < 0$, S is either a saddle with a bi-dimensional stable manifold or a repellor. In the last case, when k^* approaches k_2, a Hopf bifurcation, generically, takes place: S becomes, from a repellor, a saddle with a bi-dimensional stable manifold.*

Proof. See Appendices C and D ■

Remember that, in the production function of the representative agent, α is the exponent of labour input l and δ is the exponent of average labour input $\overline{l}$. The above theorem says that, if $\alpha + \delta \leq 1$, then the fixed points in the regime $E > 0$ and $c_2 > 0$ cannot be attractive: that is, indeterminacy cannot occur.

Furthermore, if a fixed point satisfies $\psi'(k^*) > 0$, then it cannot be reached (generically) by the economy. If instead the condition $\psi'(k^*) < 0$ holds, then the fixed point has a bi-dimensional stable manifold (and can be reached by the economy) if k^* is near enough to k_2; otherwise, it may be a repellor. In the latter case, such a fixed point might be "surrounded", via a Hopf bifurcation, by a periodic orbit with a bi-dimensional stable manifold and, consequently, approachable by the economy.

Remember that, if $\alpha+\delta\leq 1$, the fixed point S' in the $c_2=0$ regime (when existing) is always a saddle with a bi-dimensional stable manifold. If $\alpha+\delta\leq 1$ and $\sigma<\gamma$ (see figure 1), the fixed point satisfying $E>0$ and $c_2>0$ cannot be (generically) reached by the economy, being a saddle with a one-dimensional stable manifold. The fixed point with $E=0$ cannot be reached (starting from a strictly positive value of E) if $\overline{E}$ is high enough, that is, when $\overline{E}>\psi(k^*)$ and the Jacobian matrix has a strictly positive eigenvalue in the $E-$axis direction.

Observe that both the fixed point with $E=0$ and the one with $c_2=0$ can be saddles with bi-dimensional stable manifolds. In such a case a bi-stable dynamic regime occurs: the economy can approach either fixed point according to the initial values of E and k.

Analogous considerations can be made about figures 2 and 3.

6.2.2 Case $\alpha+\delta>1$

We aim to show, in particular, that, when $\alpha+\delta>1$, an attractor can exist, with $E>0$, in the $c_2>0$ regime.
Applying formulae (58)-(61) (see Appendix C), it is easily seen that the fixed point S is an asymptotic attractor if and only if

$$trJ(S),\ detJ(S)<0,\ H(S)>0,\ |detJ(S)|<H(S)|trJ(S)|, \tag{22}$$

where $H(S)$ is defined by formula (59) (see appendix C).
In particular conditions (22) imply

$$\frac{r}{1-\alpha}<(\alpha+\delta-1)\frac{d}{\alpha}\frac{c_1^*}{k^*}, \tag{23}$$

and

$$A+\frac{c_1^*}{k^*}(1-\alpha)B<0, \tag{24}$$

where A and B are computed in Appendix B.

From the expression of $det\,(J(S))$ [formula (55) in Appendix B] and from (23) it follows that $\psi'(k^*)<0$ if S is an attractor.
Two cases are then to be examined:

Case A: $\gamma<\sigma<ab\beta+\gamma$, $k^*\in(k_0,k_2)$;

Case B: $ab\beta+\gamma\leq\sigma$, $k^*\in(k_1,k_2)$.

Case A Since $\overline{E}=\psi(k^*)$ and $\frac{c_1^*}{k^*}=\frac{1}{(a+1)b}\frac{\psi(k^*)}{k^*}+\frac{r(b\beta-\gamma)}{(a+1)(1-\alpha)b\beta}$, $\frac{c_1^*}{k^*}$ decreases as $k^*\in(k_0,k_2)$ increases. Therefore (23) holds in a subinterval of (k_0,k_2) if and only if it holds at k_0.

If $\dfrac{d}{\alpha} = \dfrac{1}{\alpha+\delta-1}$, it is easily computed that

$$\frac{r}{1-\alpha} - \frac{c_{1_0}}{k_0} = \frac{r}{(1-\alpha)(a+1)b\beta}(ab\beta + \gamma - \sigma) > 0 \tag{25}$$

Hence (25) implies

$$d > \frac{\alpha}{\alpha+\delta-1} \tag{26}$$

i.e.

$$1 - \alpha + \frac{\alpha}{d} < \delta. \tag{27}$$

Example 4 *Fixed α and δ so that $\alpha + \delta > 1$, the other parameters can be chosen to satisfy*

$$a = b = \beta = 1,\ \gamma = \frac{\alpha(\alpha-\varepsilon)}{d(\varepsilon+\delta)},\ r = \frac{1-\alpha}{2(1+\dfrac{d}{\alpha})^{\alpha+\delta}},\ \frac{(\alpha+\delta-1)d}{2\alpha} = 1 + \frac{\varepsilon}{2(1-\alpha)} \tag{28}$$

Then, if $\alpha - \varepsilon > 0$ is sufficiently small, the conditions

$$trJ_0 < 0, \quad H_0 > 0 \tag{29}$$

are seen to hold.
Hence, when k^* belongs to a suitable right neighborhood of k_0, the corresponding fixed point S is attractive.

Remark Let S be the attractor of the above example. Then, for the same values of the parameters, two more fixed points with a positive E can exist, i.e. $S' = (k', c_1', E')$ in the $c_2 = 0$ regime and $S'' = (k'', c_1'', E'')$ in the $c_2 > 0$ regime, $k'' \in (k_1, k_0)$.
Since (27) holds, it follows:

1. S' is either an attractor or a saddle with a one-dimensional stable manifold. In fact, considering the above example, S' is an attractor when, for instance, both α and ε are sufficiently close to 1, while it can be a saddle when α and ε are themselves “small”.

2. S'' is a saddle with a bi-dimensional stable manifold. Such a manifold is locally a separatrix.

Note that, when $\alpha + \delta > 1$, there is the possibility of three reachable fixed points; this case occurs if, for example, the fixed point in $c_2 = 0$ is attractive, the one with $E > 0$, $c_2 > 0$, $\psi'(k^*) > 0$ is a saddle with a bi-dimensional stable manifold and the fixed point satisfying $E > 0$, $c_2 > 0$, $\psi'(k^*) < 0$ is also attractive.

Case B In such a case $\sigma \geq ab\beta + \gamma$ and $\dfrac{c_1^*}{k^*}$ decreases along (k_1, k_2). Hence (23) holds in a subinterval of (k_1, k_2) if and only if it holds at k_1.
Denote by S_1 the fixed point (k_1, c_1, E_1). We have

$$\overline{E}_1 = \psi^*(k_1) = \frac{r(ab\beta + \gamma)}{(1-\alpha)\beta} k_1, \tag{30}$$

$$\frac{c_1}{k_1} = \frac{1}{(a+1)b} \frac{\overline{E}_1}{k_1} + \frac{r(ab\beta - \gamma)}{(a+1)(1-\alpha)b\beta} = \frac{r}{1-\alpha} \tag{31}$$

Therefore, again, (23) implies (27).

Let us check, next, condition (24). Exploiting (23), through easy calculations, (24) implies

$$\frac{r}{1-\alpha}(1-\alpha+\varepsilon) - [(1-\alpha)(\alpha+\delta-1) - \varepsilon] \frac{dc_1}{\alpha k_1} > 0 \tag{32}$$

i.e., because of (31),

$$(1-\alpha)(\alpha+\delta)\frac{d}{\alpha} < (1-\alpha+\varepsilon)(1+\frac{d}{\alpha}), \tag{33}$$

or

$$\delta < 1 - \alpha + \frac{\alpha}{d} + \frac{\varepsilon(\alpha+d)}{d(1-\alpha)}. \tag{34}$$

Letting $J_1 = J(S_1)$ and writing the characteristic polynomial $P_1(\lambda)$ of J_1, it follows that S is an attractor for k^* belonging to a suitable right neighborhood of k_1 if and only if

$$detJ_1 < 0,\ trJ_1 < 0,\ \ H_1 > 0,\ |detJ_1| < |trJ_1|H_1. \tag{35}$$

Example 5 *Let $\alpha + \delta > 1$, d satisfying (27) and (34), $\alpha - \varepsilon > 0$ sufficiently small.*
Furthermore we set

$$b = \beta = 1,\ a = \frac{\alpha - \varepsilon}{2(\varepsilon + \delta)},\ \gamma = (\alpha - \varepsilon)^2,\ r = \frac{1-\alpha}{2}\left(\frac{\frac{\alpha}{d}}{\frac{\alpha}{d}+1}\right)^{\alpha+\delta} \tag{36}$$

Then it can be checked that, when $\alpha - \varepsilon$ is small enough, for k^* belonging to a suitable right neighborhood of k_1, the corresponding fixed point S is an attractor.

7 Bifurcations in the regime $c_2 > 0$

Our interest about the existence of periodic orbits is motivated by the fact that oscillations of the state variables k and E produce a well-being reduction with respect to a state of the economy where the values of k and E are equal to the time averages of k and E along the periodic orbit, if economic agents are risk-averse (see e.g. Benhabib and Farmer 1999).

The existence of periodic orbits in the $c_2 = 0$ regime was analyzed in Antoci, Brugnano and Galeotti (2003), where a supercritical Hopf bifurcation was shown to give rise to a locally attracting periodic orbit (i.e. with a three-dimensional stable manifold). Let us now investigate, in the $c_2 > 0$ regime, local bifurcations taking place at the equilibrium $S = (k^*, c_1^*, E^*)$, with $E^* > 0$, when k^* varies in (k_1, k_2).

1. Assume $\alpha + \delta \leq 1$ and S is a repellor for k^* belonging to a sub-interval I of (k_1, k_2).
   Then $\frac{c_1^*}{k^*}$ is decreasing and $\psi'(k^*) < 0$ in I. It follows that, when k^* approaches k_2, generically a Hopf bifurcation occurs, as the real part of two complex conjugate eigenvalues turns from positive into negative. In other words, for k^* belonging to a suitable left neighborhood of k_2, S has a bi-dimensional stable and a one- dimensional unstable manifold.

2. Assume $\alpha + \delta > 1$ and S is an attractor for k^* belonging to a sub-interval $I = (k_3, k_4)$ of (k_1, k_2).
   Then $\psi'(k^*) < 0$ and $\frac{c_1^*}{k^*}$ is decreasing in I.
   Furthermore
   $$1 - \alpha + \frac{\alpha}{d} < \delta < 1 - \alpha + \frac{\alpha}{d'} + \frac{(\alpha + d')\varepsilon}{(1-\alpha)d'}, \tag{37}$$
   as conditions (23) and (24) are checked to imply.
   It is easily seen that $k_4 = k_2$ if
   $$1 - \alpha + \frac{\alpha}{d'} < \delta < 1 - \alpha + \frac{\alpha}{d'} + \frac{(\alpha + d')\varepsilon}{(1-\alpha)d'}, \tag{38}$$
   while $k_4 < k_2$ if
   $$1 - \alpha + \frac{\alpha}{d} < \delta < 1 - \alpha + \frac{\alpha}{d'} \tag{39}$$
   In the latter case, when k^* crosses k_4, one real negative eigenvalue becomes positive, passing through ∞, and S has a bi-dimensional stable and a one-dimensional unstable manifold as $k^* \in (k_4, k_2)$.

Furthermore it may happen that $k_3 > k_m$, where $k_m = k_0$ or $k_m = k_1$, respectively in cases A and B.

If this occurs, then, generically, a Hopf bifurcation takes place when k^* crosses k_3, as S becomes an attractor from a saddle with a one-dimensional stable manifold.

3. Consider the case

$$\delta > 1 - \alpha + \frac{\alpha}{d'} + \frac{(\alpha + d')\varepsilon}{(1-\alpha)d'} \tag{40}$$

Then no bifurcation occurs in the possible interval $J \subseteq (k_1, k_2)$, where $\psi'(k^*) < 0$.
In such an interval S has a one-dimensional stable and a bi-dimensional unstable manifold.
If, furthermore, $\gamma < \sigma < ab\beta + \gamma$, then, passing through k_0 [recall $\psi'(k_0) = 0$], one real eigenvalue changes sign: it may turn either from positive into negative or viceversa.
4. Finally a generic Hopf bifurcation can take place in the possible interval $H \subseteq (k_1, k_2)$, where $\psi'(k^*) > 0$.

Example 6 *In the following numerical example (17) and (40) hold and* (k_1, k_2) *is divided into three sub-intervals:* (k_1, k_h), (k_h, k_0), (k_0, k_2). *As* $k^* \in (k_1, k_h)$, *S is a repellor. Then at* k_h *a Hopf bifurcation occurs and S has a bi-dimensional stable and a one-dimensional unstable manifold for* $k^* \in (k_h, k_0)$.
Finally, when k^* *crosses* k_0, *a real negative eigenvalue becomes positive and S has a one-dimensional stable and a bi-dimensional unstable manifold as* $k^* \in (k_0, k_2)$.
The example is

$$\begin{aligned} &\alpha = \frac{1}{2},\ a = b = \beta = 1,\ d = 4,\ \delta = 2,\ r = \frac{1}{2\sqrt{(6^5)}}, \\ &\alpha - \varepsilon \quad \textit{sufficiently small,} \\ &\gamma = (\alpha - \varepsilon)^2. \end{aligned} \tag{41}$$

In this section we have shown all the Hopf bifurcations which can occur in our model. In figure 7, a locally attractive periodic orbit and some orbits approaching it are plotted . In such a case, given the initial values of k and E [near enough to the projection of the orbit on the (k, E) plane], there exists a continuum of initial values of λ (or, alternatively, of c_1 or c_2) by which the economy can reach the periodic orbit. Consequently, an indeterminacy problem occurs.

Figure 7: $\beta = 0.5, \epsilon = 0.0789, \alpha = 0.5, \gamma = 0.175, \delta = 1.32, \sigma = 0.3077$
$a = 0.56, d = 1.3, r = 0.067, b = 1.5, \overline{E} = 0.088$

8 Behaviour of orbits for high values of $\overline{E}$

We aim to prove that, when $\overline{E}$ is sufficiently high, orbits starting in the region $c_2 > 0$ *enter and remain*, after a finite time, in the regime $c_2 = 0$.

To this end, let us replace, first, the variables (k, λ, E) in system (8)-(10) by (l, c_1, E), where $l \in (0, 1)$ is defined by (5) and $c_1 = \frac{1}{\lambda}$. It follows from (4) and (5) that

$$k = \left(\frac{dl^{1-\alpha-\delta}}{\alpha(1-l)} c_1 \right)^{\frac{1}{1-\alpha+\varepsilon}} \tag{42}$$

Hence we get, after simple steps,

$$\dot{c_1} = c_1 \left[(1-\alpha) \left(\frac{\alpha}{d} \right)^{\frac{\alpha-\varepsilon}{1-\alpha+\varepsilon}} \frac{l^{\frac{\varepsilon+\delta}{1-\alpha+\varepsilon}} (1-l)^{\frac{\alpha-\varepsilon}{1-\alpha+\varepsilon}}}{c_1^{\frac{\alpha-\varepsilon}{1-\alpha+\varepsilon}}} - r \right] \tag{43}$$

This implies $\dot{c_1} < 0$ if

$$\begin{aligned} c_1 \quad &> \quad c_1^M := \left(\frac{1-\alpha}{r} \right)^{\frac{1-\alpha+\varepsilon}{\alpha-\varepsilon}} \max_{l \in (0,1)} l^{\frac{\varepsilon+\delta}{\alpha-\varepsilon}} (1-l) = \\ &= \quad \frac{\alpha(\alpha-\varepsilon)}{d(\alpha+\delta)} \left(\frac{1-\alpha}{r} \right)^{\frac{1-\alpha+\varepsilon}{\alpha-\varepsilon}} \left(\frac{\varepsilon+\delta}{\alpha+\delta} \right)^{\frac{\varepsilon+\delta}{\alpha-\varepsilon}} \end{aligned} \tag{44}$$

Assume now

$$\overline{E} > b(a+1)c_1^M \quad if \quad b\beta > \gamma \tag{45}$$

or

$$\overline{E} > \frac{\gamma}{\beta}(a+1)c_1^M \quad if \quad b\beta \leq \gamma \tag{46}$$

It follows, in particular, as it is easily checked, that no equilibrium exists in the $c_2 > 0$ regime.

Furthermore, consider an orbit starting in the region $c_2 > 0$, i.e., such that $ac_1(0) > \frac{E(0)}{b}$.

Due to (43) and (44), and since no equilibrium exists when $c_2 > 0$, there is a time t_1 such that either

$$c_2(t_1) = 0$$

or

$$c_1(t) \leq c_1^M \text{ when } t \geq t_1$$

If $c_2(t_1) > 0$, then $\dot{E}(t_1) > 0$, because of assumptions (45) and (46). Hence E increases, by a speed bounded away from zero, whereas $c_1 \leq c_1^M$. Consequently there exists a time $t_2 > t_1$ such that

$$c_2(t_2) = 0$$

Finally, we can choose $\bar{t}$, $\bar{t} \geq t_i$, $i = 1, 2$, in such a way that $c_1(t) \leq c_1^M$[3] for $t \geq \bar{t}$ and

$$c_2(t) = 0$$

for t being in some right neighborhood of $\bar{t}$. Otherwise, in fact, E would keep increasing, because of our assumptions, in the region $c_2 > 0$, until we would obtain

$$E > abc_1^M \geq abc_1,$$

thus contradicting the condition $c_2 > 0$.

Now it is easily checked that assumptions (45) and (46) imply that, in the $c_2 = 0$ regime, $\dot{E}(t) \leq 0$, $t \geq \bar{t}$, requires $E(t)$ to be larger than abc_1^M. However, in this case, the orbit would never cross back the plane $ac_1 = \frac{E}{b}$. Therefore, should the orbit keep crossing forth and back such a plane, i.e., moving indefinitely from the regime $c_2 > 0$ to the regime $c_2 = 0$ and viceversa, $\dot{E}(t)$ would be positive and bounded away from zero as $t \geq \bar{t}$, until at some time, in the $c_2 > 0$ regime,

$$E > abc_1^M \geq abc_1,$$

yielding a contradiction.

Hence we can choose the previous $\bar{t}$ as the time at which the orbit *enters* into and then *remains* in the $c_2 = 0$ regime.

9 References

Antoci, A., 1996, “Negative externalities and growth of the activity level”, Quaderno n. 9308 del Progetto Nazionale M.U.R.S.T “Dinamiche non lineari ed applicazioni alle scienze economiche e sociali”, Università di Firenze.

Antoci, A., Bartolini, S., 1999, “Negative externalities as the engine of growth in an evolutionary context”, Nota di lavoro 83.99, Fondazione ENI Enrico Mattei, Milan.

Antoci, A., Bartolini, S., 2003, “Negative externalities, defensive expenditures and labour supply in an evolutionary game”, forthcoming in Environment and Development Economics.

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[3] In the particular case $\frac{\alpha}{\alpha+d} = \frac{\varepsilon+\delta}{\alpha+\delta}$, the (44) value of c_1^M can be replaced by any *slightly* larger value.

A. Antoci, L. Brugnano e M. Galeotti, 2004, "Oscillations, Sustainability and Indeterminacy in a Growth Model with Environmental Assets", forthcoming in Nonlinear Dynamics: Real World Applications.

A. Antoci, P. L. Sacco e P. Vanin, 2004, "On the Possible Conflict Between Economic Growth and Social Development", in B. Gui e R. Sugden (Eds), Economics and Social Interaction: Accounting for Interpersonal Relations, Cambridge University Press (forthcoming).

Antoci, A., Borghesi, S., 2002, "Working too much in a polluted world: a North-South evolutionary model", Nota di Lavoro 44.2002, Fondazione ENI Enrico Mattei, Milan.

Auty, R. M., 2001, "The political economy of the resource-driven growth", European Economic Review, 45, pp. 839-846.

Bartolini, S., Bonatti, L., 1999, "Endogenous growth and negative externalities", Nota di Lavoro 91.99, FEEM (Fondazione Eni Enrico Mattei), Milan, forthcoming in Environment and Development Economics.

Bates, D., 2002, "Environmental refugees? Classifying human migrations caused by environmental change", Population and Environment, vol.23, No.5, pp.465-477, Human Sciences Press.

Beltratti, A., 1996, "Models of economic growth with environmental assets", Fondazione ENI Enrico Mattei series on Economics, Energy and Environment, Kluwer Academic Publishers, Dordrecht, The Netherlands.

Benhabib, J., Farmer, R. E. A., 1996, "Indeterminacy and sector-specific externalities", Journal of Monetary Economics, 37.

Benhabib, J., Farmer, R. E. A., 1999, "Indeterminacy and sunspots in macroeconomics", Handbook of Macroeconomics, North Holland.

Bird, J., 1987, "The transferability and depletability of externalities", Journal of Environmental Economics and Management, 14, pp. 54-57.

Hirsch, F., 1976, "Social limits to growth", Harvard University Press, Cambridge, USA.

Hueting, R., 1980, "New scarcity and economic growth. More welfare through less production?", North Holland.

Hugo, G., 1996, "Environmental concerns and international migration", International Migration Review, 30, pp.105-131.

Leipert, C., 1989, "National income and economic growth: the conceptual side of defensive expenditures", Journal of Economic Issues, pp.843-856.

Leipert, C., Simonis U. E., 1989, "Environmental protection expenditures: the German example", Rivista Internazionale di Scienze Economiche e Commerciali, vol. 36, pp.255-270.

Matsuyama, K., 1992, "Agricultural productivity, comparative advantage, and economic growth", Journal of Economic Theory, 58, pp. 317-334.

Rodriguez, F., Sachs, J. D., 1999, "Why do resource-abundant economies grow more slowly?", Journal of Economic Growth, 4, pp. 277-303.

Sachs, J. D., Warner, A. M., 1999, "The big push, natural resource booms and growth", Journal of Development Economics, 59, pp. 43-76.

Sachs, J. D., Warner, A. M., 2001, "The curse of natural resources", European Economic Review, 45, pp. 827-838.

Myers, N., 1997, "Environmental refugees", Population and Environment, 19, pp.167-182.

Shogren, J.F., Crocker, T.D., 1991, "Cooperative and non-cooperative protection against transferable and filterable externalities", Environmental and Resource Economics, 1, pp.195-213.

Vincent, J.R., 2000, "Green accounting: from theory to practice", Environment and Development Economics, 5, pp.13-24, Cambridge University Press.

Appendix A

From $\dot{c}_1 = 0$ and $\dfrac{\dot{E}}{E} - \gamma \dot{k} = 0$ it follows

$$E^* = \overline{E} - \frac{\gamma r}{(1-\alpha)\beta} k^* \tag{47}$$

Then, from $\dot{E} = 0$,

$$c_1^* = c_1^* = \frac{\overline{E}}{(a+1)b} + \frac{(b\beta - \gamma)r}{(a+1)(1-\alpha)b\beta} k^* \tag{48}$$

Since $E^*, c_2^* > 0$, (47) and (48) imply

$$\frac{(1-\alpha)\beta}{(ab\beta+\gamma)r}\overline{E} < k^* < \frac{(1-\alpha)\beta}{\gamma r}\overline{E} \tag{49}$$

Furthermore $\dot{c}_1 = 0$ and $\dfrac{\partial H}{\partial l} = 0$ imply

$$\frac{dl^*}{1-l^*} = \frac{\alpha r}{1-\alpha}\frac{k^*}{c_1^*} \tag{50}$$

and, putting $A = l^\delta k^\epsilon$ in $\dot{c}_1 = 0$,

$$(l^*)^{\alpha+\delta} = \frac{r}{1-\alpha}(k^*)^{\alpha-\epsilon} \tag{51}$$

From (50) and (51) it follows

$$\frac{(1-\alpha)d}{\alpha r} c_1^* + k^* = \left(\frac{1-\alpha}{r}\right)^{\frac{1}{\alpha+\delta}} (k^*)^{\frac{\varepsilon+\delta}{\alpha+\delta}} \tag{52}$$

and, finally, from (48) we get (12).

Appendix B

It is easily computed that

$$A = \frac{(1-\alpha+\varepsilon)(\alpha+\delta)\dfrac{dr}{\alpha(1-\alpha)}\dfrac{c_1^*}{k^*}}{\dfrac{r}{1-\alpha} - (\alpha+\delta-1)\dfrac{d}{\alpha}\dfrac{c_1^*}{k^*}} + (1-\alpha+\varepsilon)\frac{r}{1-\alpha} \tag{53}$$

$$B = \frac{-(\alpha+\delta)\dfrac{dr}{\alpha(1-\alpha)}}{\dfrac{r}{1-\alpha} - (\alpha+\delta-1)\dfrac{d}{\alpha}\dfrac{c_1^*}{k^*}} \tag{54}$$

Recalling the form of $\psi(k^*))$ in (12)-(55), one obtains

$$det(J(S)) = \frac{-(\alpha+\delta)\beta drc_1^* E^* \psi'(k^*)}{\alpha\beta k^*(\dfrac{r}{1-\alpha} - (\alpha+\delta-1)\dfrac{d}{\alpha}\dfrac{c_1^*}{k^*})}. \tag{55}$$

Appendix C

Letting $\alpha + \delta \leq 1$, (55) implies

$$(det(J(S))\psi'(k^*) < 0 \quad \text{when } \psi'(k^*) \neq 0 \tag{56}$$

Hence, if $\psi'(k^*) < 0$, $detJ(S) > 0$ and the proposition follows.
Then, let $\psi'(k^*) > 0$ and consequently $detJ(S) < 0$.
Denote by g_{ik}, $i, k = 1, 2, 3$, the entries of $J(S)$.
Observe, firstly, that

$$g_{11} + g_{22} = \frac{\varepsilon(\alpha+\delta)\dfrac{dr}{\alpha(1-\alpha)}\dfrac{c_1^*}{k^*}}{\dfrac{r}{1-\alpha} + (1-\alpha-\delta)\dfrac{d}{\alpha}\dfrac{c_1^*}{k^*}} + (1-\alpha+\varepsilon)\frac{r}{1-\alpha} > 0 \tag{57}$$

The characteristic polynomial of $J(S)$ is

$$P(\lambda) = \lambda^3 - (tr(J))\lambda^2 + H\lambda - detJ, \tag{58}$$

where

$$H = g_{11}g_{22} + g_{33}(g_{11} + g_{22}) - g_{12}g_{21} \tag{59}$$

From elementary algebra a cubic poynomial

$$\lambda^3 + a\lambda^2 + b\lambda + c \tag{60}$$

has all non-negative real part roots if and only if

$$a,\, b,\, c,\, ab - c \geq 0. \tag{61}$$

It follows from (57) that

$$tr(J) = g_{11} + g_{22} + g_{33} \leq 0 \text{ implies } g_{33} < 0, \tag{62}$$

while $H \geq 0$, being $g_{11}g_{22}$, $g_{33}(g_{11} + g_{22})$, $g_{12} < 0$, requires

$$g_{21} > 0 \tag{63}$$

Finally the condition $ab - c \geq 0$ means

$$|detJ| \leq H|trJ| \tag{64}$$

Through simple calculations,though,

$$\begin{aligned} |detJ| = |g_{33}|(g_{11}g_{22} - g_{12}g_{21}) > \\ (|g_{33}| - (g_{11} + g_{22}))(g_{11}g_{22} + g_{33}(g_{11} + g_{22}) - g_{12}g_{21}) = |trJ|H \end{aligned} \tag{65}$$

Hence we get a contradiction.
We can conclude that, when $\psi'(k^*) > 0$, $J(S)$ has only one eigevalue with negative real part, and therefore negative.

Appendix D

Let us show that for any value $\alpha + \delta \leq 1$ it is possible to have a repellor S, in the $c_2 > 0$ regime, with $E^* > 0$ and $\psi'(k^*) < 0$.

Let $\gamma < \sigma < ab\beta + \gamma$, so that there exist $k_0 \in (k_1, k_2)$ satisfying $\psi'(k_0) = 0$. Call S_0 the corresponding equilibrium.

In order that S be a repelling equilibrium for k^* lying in a suitable right neighborhood of k_0 , it suffices that the two non-zero eigenvalues of $J(S_0)$ have positive real part.

Pose $J_0 = J(S_0)$ and denote by $P_0(\lambda)$ its characteristic polynomial, i.e.

$$P_0(\lambda) = \lambda^3 - (trJ_0)\lambda^2 + H_0\lambda \tag{66}$$

The non-zero roots of $P_0(\lambda)$ have positive real part if and only if

$$trJ_0,\, H_0 > 0 \tag{67}$$

It can be easily checked that (67) is verified when

$$\alpha - \varepsilon > 0 \tag{68}$$

is small enough and, furtherly,

$$\gamma = 2(\alpha - \varepsilon),\, \beta = d = \alpha - \varepsilon,\, b = 1,\, a > 2\frac{(\epsilon + \delta)}{\alpha},\, \frac{(1-\alpha)(\varepsilon + \delta)}{r(\alpha + \delta)} > 1. \tag{69}$$